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HIGHER ORDER NORMALITY AND CASTELNUOVO REGULARITY OF SMOOTH THREEFOLDS

SIJONG KWAK

School of Mathematics
Korea Institute for Advanced Study

§0. Introduction

Let X be a nondegenerate projective variety of dimension n and of degree d in $\mathbb{P}^r$. We say that X is *k-normal* if the homomorphism $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective, i.e., hypersurfaces of degree k cut out a complete linear system on X. According to [EG], [Mu1], X is m-regular iff one of the following conditions holds:

(1) $H^i(\mathbb{P}^r, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$;
(2) $H^i(\mathbb{P}^r, \mathcal{I}_X(j)) = 0$ for $i \geq 1$, $i + j \geq m$;
(3) For all $k \geq 0$ the degrees of minimal generators of the k-th syzygy modules of the homogeneous saturated ideal I_X of X are bounded by $k + m$.

Attempts to bound the regularity of a projective variety $X \subset \mathbb{P}^r$ are motivated in part by the desire to bound the complexity of computing the syzygies in terms of related invariants of X.

The importance of m-regularity stems from the above equivalences and the following well-known results [Mu1]; If X is m-regular then X is cut out by hypersurfaces of degree m set-theoretically and scheme-theoretically. Furthermore, Hilbert polynomial and the Hilbert function of X have the same values for all $k \geq m - 1$.

It is not hard to show that $\operatorname{reg} X \geq 2$ and X is 2-regular if and only if X is of minimal degree. There is a well-known conjecture concerning the k-normality and k-regularity of X:

Regularity conjecture [EG], [GLP]. *Let X be a nondegenerate integral projective scheme of dimension n and degree d in $\mathbb{P}^r$ which is defined over algebraically closed field of characteristic zero.*

(1) *X is m-normal for all $m \geq d - \operatorname{codim}(X)$.*

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(2) *X is m-regular for all $m \geq d - \operatorname{codim}(X) + 1$, i.e.,* $\operatorname{reg}(X) = \min\{m \in \mathbb{Z}\colon X \text{ is } m\text{-regular}\} \leq d - (r-n) + 1$.
(3) *Classification of all extremal examples with geometric interpretations which make the bound best possible.*

This conjecture including the classification of borderline examples was verified for integral curves ([C],[GLP]) and an optimal bound was also obtained for smooth surfaces ([P], [L]). For a historical remark and further results for higher dimensional smooth varieties, see [K2]. Roughly speaking, the varieties on the boundary of regularity conjecture are characterized by the property of having a $(d-(r-n)+1)$-secant line. (Clearly, $d-(r-n)+1$ is the largest possible number of intersections of a line with a nondegenerate variety of degree d by the generalized Bezout Theorem.) Note that if X has $(m+1)$-secant line, X can not be m-regular.

In addition, for smooth projective varieties, as we will see in this note, the locus of multisecant lines of a smooth variety X plays an important role in bounding regularity of X. So, geometry of multisecant lines of a given smooth variety and vector bundle techniques (developed by Lazarsfeld, [L]) link the behavior of graded Betti numbers of the minimal graded free resolution of homogeneous coordinate ring of X to vanishing of cohomology groups of certain vector bundles on a smooth variety and the structure sheaf on a smooth variety (see Lemma 2.1, Lemma 2.3 and Theorem 2.6). This is a connection between syzygies and geometry of a smooth variety. In particular, Kodaira-Kawamata-Viehweg vanishing theorem is very useful for Castelnuovo regularity in the case of smooth varieties, see [BEL] where we can see some important and interesting results and various applications which bound Castelnuovo regularity in terms of degrees of defining equations of a given smooth variety.

On the other hand, there are various approaches in the categories of monomial ideals, toric varieties, and locally Cohen-Macaulay Buchsbaum varieties, see [HM] [PS], [SV].

§1. Basic background

In this section we recall basic results which will be used in subsequent sections. We work over an algebraically closed field of characteristic zero.

Definition 1.1. *For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$, $\mathcal{F}$ is m-regular if $H^i(\mathbb{P}^r, \mathcal{F}(m-i)) = 0$ for all $i > 0$ and $\operatorname{reg}(\mathcal{F})$ is defined by $\inf\{m \in \mathbb{Z}\colon \mathcal{F} \text{ is } m\text{-regular}\}$.*

Proposition 1.2.

(a) *Let $\mathcal{F}$ be a p-regular vector bundle and $\mathcal{G}$ be a q-regular vector bundle on $\mathbb{P}^r$ which is defined over an algebraically closed field of characteristic zero. Then $\mathcal{F} \otimes \mathcal{G}$ is $(p+q)$ regular and $S^k(\mathcal{F})$ and $\Lambda^k(\mathcal{F})$ are (kp)-regular.*

(b) *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$ and let $\cdots \to \mathcal{F}_i \to \cdots \to \mathcal{F}_0 \to \mathcal{F} \to 0$ be an exact sequence of coherent sheaves on $\mathbb{P}^r$ such that $\mathcal{F}_i$ is $(p+i)$–regular. Then $\mathcal{F}$ is p–regular.*

Proof. See [L], 428p. $\square$

Definition 1.3. A scheme X is called *punctual* if $\operatorname{Supp} X = x$, where $x \in X$ is a point. A punctual scheme X is called *curvilinear* if $\mathcal{O}_x$ is isomorphic to $\mathbb{C}[x]/(x^k)$ for some $k \geq 1$.

It is clear that a punctual scheme is curvilinear if and only if it admits an embedding into a smooth curve.

Lemma 1.4. *Let X be a n-dimensional smooth projective variety in $\mathbb{P}^r$, and suppose that $n = \dim X \leq 5$. Let Λ^{r-n-2} be a general linear subspace of dimension $(r-n-2)$, so that, in particular, Λ is disjoint from X, and let π_Λ be the projection with center Λ, and put $Y = \pi_\Lambda(X) \subset \mathbb{P}^{n+1}$. Then all fibers of $\pi_\Lambda\colon X \to Y$ are curvilinear.*

Proof. It is proved by J. Mather that the variety $X_q = \{x \in \mathrm{X} \mid \dim T_x(X) \cap \Lambda = q-1\}$ has codimension $q(q+1)$ in X(see [AO] Theorem 2). So, if $\dim X \leq 5$ then $X_q = \varnothing$ for $q \geq 2$. Therefore, for $x \in \operatorname{Supp} \pi_\Lambda^{-1}(y)$, $T_x(X) \cap \Lambda = \varnothing$(so, x is a reduced point) or one point (in this case, $\mathcal{O}_{\pi_\Lambda^{-1}(y),x}$ is isomorphic to $\mathbb{C}[x]/(x^k)$ for some $k \geq 1$.) $\square$

Theorem 1.5 (J. Mather). *Let $X \subset \mathbb{P}^r$ be a smooth nondegenerate n-dimensional variety, let $\Lambda^{r-n-2} \subset \mathbb{P}^N$ be a generic linear subspace, and let $\pi_\Lambda\colon \mathbb{P}^r \dashrightarrow \mathbb{P}^{n+1}$, $Y = \pi_\Lambda(X) \subset \mathbb{P}^{n+1}$. Let $Y_k = \{y \in Y \mid \operatorname{length} \pi_\Lambda^{-1}(y) \geq k\}$, and put $X_k = \pi_\Lambda^{-1}(Y_k)$, so that $X_1 \supset \cdots \supset X_k \supset X_{k+1} \cdots$ is a decreasing filtration. Assume that $n \leq 14$, so that we are in Mather's "nice" range. Then $X_{n+2} = \varnothing$ and $\dim X_k \leq n+1-k$. If $\dim X_k = \dim Y_k = n+1-k$, then there exists a dense open subset of Y_k over which all the fibers of π_Λ are reduced.*

Proof. This follows from the main theorem of [Ma1] and the discussion in §5 of [Ma2]. A key ingredient is the inequality

$$\sum_{x \in \pi^{-1}(y)} (\delta_x + \gamma_x) \leq n+1, \qquad y \in Y, \tag{1.0}$$

where $\delta_x = \operatorname{length} \mathcal{O}_{\pi^{-1}(y),x}$ and γ_x is another non-negative invariant introduced by J. Mather for all stable germs in the "nice" range (cf. [Ma2]); in particular, $\gamma_x = k-1$ if $\mathcal{O}_x \simeq \mathbb{C}[x]/(x^k)$ for some $k \geq 1$), which is always the case for $n \leq 5$. $\square$

Remark 1.6. Let $X \subset \mathbb{P}^r$ be a smooth nondegenerate n-dimensional subvariety and let $S_m(X)$ be the locus of m-secant lines of X in $\mathbb{P}^r$. Assume that $n \leq 14$. Then by Theorem 1.6 one has $\dim S_{n+2-m} \leq n+1+m$, which gives us some information on "collinear" fibers of a generic linear projection of X to a hypersurface.

Theorem 1.7 (The dimension+2-secant lemma). *Let $X \subset \mathbb{P}^r$ be a smooth n-dimensional subvariety and let Y be an irreducible variety parametrizing a family of lines in $\mathbb{P}^r$. Assume that, for a general L_y, the length of the scheme-theoretic intersection $L_y \cap X$ is at least $n+2$. Then we have*

$$dim(\cup_{y \in Y} L_y) \leq n+1$$

Proof. See [R2]. □

We remark that the importance of Theorem 1.7 is that it is true in all dimensions. In the nice range it is an immediate corollary of Theorem 1.5.

Let X be a nondegenerate zero-dimensional subscheme of length d, not necessarily reduced, and let $r = \dim \langle X \rangle$, where $\langle X \rangle = \mathbb{P}^r$ is the span of X. Let's put $t = \max\{k \mid \dim \langle X' \rangle = \text{length}\, X' - 1 \quad \forall X' \subset X, \text{length}\, X' \leq k+1\}$. It is clear that $1 \leq t \leq r$, and that $t = 1$ iff X has a trisecant line.

The following Proposition 1.8 was communicated to me by F. L. Zak. However, for lack of suitable references we give brief proofs here.

Proposition 1.8. *In the above situation,*

(a) *X is k-normal for all $k \geq \lceil \frac{d-r-1}{t} \rceil + 1$, where $\lceil a \rceil$ is the smallest integer that is not less than a.*
(b) *assume that $d \geq r+3$. X is $(d-r)$-normal but fails to be $(d-r-1)$-normal if and only if X has a $(d-r+1)$-secant line;*

Proof. We proceed by induction on r. If $r = t$, i.e. X is a "general position scheme" then our assertion is proved in [Pe,Theorem 28.8]. Let us fix an integer r_0 and suppose Proposition 1.2 holds for $t \leq N \leq r_0 - 1$. For $r = r_0$, we may also assume that Proposition 1.2 is true for finite schemes of degree smaller than d. Let A be a graded homogeneous ring of X. Equivalently, we show the surjectivity of the natural morphism

$$A_k \to H^0(X, \mathcal{O}_X(k))$$

for all k such that $d \leq tk + (r-t) + 1$. Choose a hyperplane H such that $\deg Y \geq r$ and $\langle Y \rangle = H$, where $Y = X \cap H$. As in the proof of [Pe, Theorem 28.8], we consider the diagram

$$
(*)\qquad
\begin{array}{ccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
 & [A/(0:H)]_k & \xrightarrow{\alpha_k} & H^0(Z, \mathcal{O}_Z(k)) \simeq \mathbb{C}^{d_1} \\
 & H\downarrow & & \downarrow \\
 & A_{k+1} & \xrightarrow{\rho_{k+1}} & H^0(X, \mathcal{O}_X(k+1)) \simeq \mathbb{C}^{d}, \\
 & \downarrow & & \downarrow \\
 & [A/HA]_{k+1} & \xrightarrow{\beta_{k+1}} & H^0(Y, \mathcal{O}_Y(k+1)) \simeq \mathbb{C}^{d_2} \\
 & \downarrow & & \downarrow \\
 & 0 & & 0
\end{array}
$$

where $d_2 = \deg Y$ and Z is the subscheme of X of degree $d_1 \geq 1$ corresponding to the graded ring $A/(0:H)$. Clearly, any closed subscheme of degree $(t+1)$ in either Y or Z spans $\mathbb{P}^t$. So, by the induction hypothesis, α_k is surjective for all k such that $d_1 \leq tk + (n-t) + 1$, $n = \dim \langle Z \rangle$ and β_{k+1} is surjective for all k such that $d_2 \leq t(k+1) + (r-1-t) + 1$. By the snake lemma, to show that ρ_{k+1} is surjective it suffices to verify the surjectivity of α_k and β_{k+1}. If $n < t$, then Z is a "general position scheme" and α_k is surjective for all $k \geq 1$. Furthermore, if $d = d_1 + d_2 \leq t(k+1) + (r-t) + 1$, then $d_2 \leq d_1 + d_2 - 1 \leq t(k+1) + (r-1-t) + 1$ and either $n < t$ or $d_1 \leq d_1 + d_2 - N \leq t(k+1) + (r-t) + 1 - r = tk + 1 \leq tk + (n-t) + 1$. Thus α_k and β_{k+1} are surjective, and we are done.

(b) The fact that X is $(d-r)$-normal is an immediate consequence of Proposition 1.2 for $t = 1$. If X has a $(d-r+1)$-secant line, then it is clear that it fails to be $(d-r-1)$-normal. To prove the converse, we argue as in the proof of Proposition 1.2 and proceed by induction on N. Assertion (b) is clear for $r = 1$. Suppose that it is true in the case when $\dim \langle X \rangle < r$, and let $X \subset \mathbb{P}^r$, $\deg X = d$, $\langle X \rangle = \mathbb{P}^r$ be a scheme without $(d-r+1)$-secant lines. We may also assume that (b) holds for finite schemes in $\mathbb{P}^r$ of degree smaller than d. First of all, if X fails to be $(d-r-1)$-normal, then from Proposition 1.2 we get $(d-r-1) < \lceil \frac{d-r-1}{t} \rceil + 1$ and $d \geq r+3$ implies $t = 1$, i.e. X has a trisecant. Choose a hyperplane H passing through this trisecant such that $r + 1 \leq \deg (Y = H \cap X) \leq d - 1$ and $\langle Y \rangle = H$. Since Y has no $(d-r+1)$-secant line, from the induction hypothesis it follows that Y is $(d-r-1)$-normal. The scheme Z introduced in the proof of Proposition 1.2 satisfies the condition $1 \leq \deg Z \leq d-r-1$, and so Z is $(d-r-2)$-normal. This means that the morphisms α_{d-r-2} and β_{d-r-1} in the commutative diagram $(*)$ are surjective. By the snake lemma, ρ_{d-r-1} is also surjective, i.e. X is $(d-r-1)$-normal, which contradicts our assumptiom that X fails to be $(d-r-1)$-normal. So X should be $(d-r-1)$-normal as required (it should be mentioned that from the proof it is clear that for $d \geq r+3$ the $(d-r+1)$-secant line is unique). □

§2. Monoidal construction and its applications for smooth varieties of dimension 3

A useful tool for study of regularity of smooth projective varieties of small dimension and small codimension is provided by well-known monoidal constructions via many generic projection theorems (cf. [BM], [K1], [K2], [L], [Ma1], [Ma2], [Pi], and [R1]). Applications of this method in Castelnuovo regularity so far depend on vanishing theorems for cohomology of vector bundles (e.g., Kodaira-Kawamata-Viehweg vanishing theorem and vanishing theorems for positive vector bundles) and information about the fibers of generic projections from X to a hypersurface of the same dimension. There are good bounds for regularity of smooth projective varieties of $\dim X \leq 6$ (see Remark 2.7.(c)). More precisely, $\operatorname{reg} X \leq d-e+1$ for integral curves and smooth surfaces (see [GLP], [L]). In this section, we deal with higher normality and regularity of smooth threefolds by using well-known monoidal construction via a generic projection.

Let X be a smooth threefold of degree d and codimension e in $\mathbb{P}^r$ defined over the complex number field $\mathbb{C}$. We will use a general construction considered in [L], [G], and [K2]. Let $\Lambda = \mathbb{P}^{r-5} \subset \mathbb{P}^r$, $\Lambda \cap X = \varnothing$, $\Lambda = \mathbb{P}(V)$ be a general linear subspace, and let $\pi_\Lambda : X \to Y$ be the projection with center at Λ, so that $Y \subset \mathbb{P}^4$ is a hypersurface of degree d. Let $\mathcal{V}$ be a collection of linear subspaces $V_j \subset S^j(V)$ such that $V_1 = V$ and $V_2 = S^2(V)$. Consider the natural restriction morphism $\tilde{\omega}_{3,k,\mathcal{V}}$. If $\tilde{\omega}_{3,k,\mathcal{V}}$ is surjective, then we get the following exact sequence:

$$(2.0) \quad 0 \to E_{3,k,\mathcal{V}} \to V_k \otimes \mathcal{O}_{\mathbb{P}^4}(-k) \oplus \cdots \oplus V_1 \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \xrightarrow{\tilde{\omega}_{3,k,\mathcal{V}}} \pi_{\Lambda *}\mathcal{O}_X \to 0,$$

where $E_{3,k,\mathcal{V}}$ is locally free of the same rank as the middle.

Lemma 2.1. *Suppose that $\tilde{\omega}_{3,k,\mathcal{V}}$ is surjective. Then*

(a) $\operatorname{reg}(E^*_{3,k,\mathcal{V}}) \leq -2$;

(b) $E^*_{3,k,\mathcal{V}}$ *is (-3)-regular if and only if X is linearly normal and* $H^0(\mathcal{I}_{X/\mathbb{P}^r}(2)) = H^1(\mathcal{O}_X) = 0$.

Proof. (a) is proved in [L, Lemma 2.1]. (b) Our argument is similar to that in [Al]. By definition, $E^*_{3,k,\mathcal{V}}$ is (-3)-regular iff $H^i(\mathbb{P}^4, E^*_{3,k,\mathcal{V}}(-3-i)) = 0$ for $i > 0$. By Serre duality, this is equivalent to $H^j(\mathbb{P}^4, E_{3,k,\mathcal{V}}(2-j)) = 0$ for $0 \leq j \leq 3$. For $j = 3$, this vanishing follows from Kodaira vanishing theorem. ¿From the exact cohomology sequence corresponding to (2.0), it follows that

$$j = 0, H^0(\mathbb{P}^4, E_{3,k,\mathcal{V}}(2)) = 0 \text{ if and only if } H^0(\mathcal{I}_{X/\mathbb{P}^r}(2)) = 0,$$
$$j = 1, H^1(\mathbb{P}^4, E_{3,k,\mathcal{V}}(1)) = 0 \text{ if and only if } X \text{ is linearly normal},$$
$$j = 2, H^2(\mathbb{P}^4, E_{n,k,\mathcal{V}}) = 0 \text{ if and only if } H^1(\mathcal{O}_X) = 0.$$

This completes the proof of (b). □

Remark 2.2. For a smooth threefold X in $\mathbb{P}^5$, conditions of Lemma 2.1 can be verified using Zak's linear normality theorem (i.e., X is linearly normal if $\dim(X) \gneqq \frac{2}{3}(N-1)$) and Barth's Lefschetz theorem ($H^1(\mathcal{O}_X) = 0$ if $N < 2n$). Furthermore, for a locally Cohen-Macaulay threefold $X \subset \mathbb{P}^5$ which is contained in a hyperquadric, it is shown in [K1] that if X has an even degree $2m$ then it is a complete intersection of hyperquadric and a hypersurface of degree m and if X has an odd degree $2m+1$ then it is projectively Cohen-Macaulay and linked to a $\mathbb{P}^3$ via a complete intersection of a hyperquadric and a hypersurface of degree $m+1$. This means such a variety is very simple.

Lemma 2.3. *Suppose that $\tilde{\omega}_{3,k,\mathcal{V}}$ is surjective. Then*

(a) $\operatorname{reg} E_{3,k,\mathcal{V}} = \operatorname{reg} X \leq (d-e+1) + \sum_{j=3}^{k}(j-2)\dim V_j$;
(b) *If $E^*_{3,k,\mathcal{V}}$ is (-3)-regular, then*

$$\operatorname{reg} E_{3,k,\mathcal{V}} = \operatorname{reg} X \leq (d-e+1) - \dim V_1 - \dim V_2 + \sum_{j=4}^{k}(j-3)\dim V_j.$$

Proof. This is an easy consequence of Lemma 2.1, (2) in [K2],(cf. see also [L]). □

For a smooth codimension two subvariety $X^n \subset \mathbb{P}^{n+2}, n \geq 4$, by Serre's construction([OSS], Ch.I, Theorem 5.1.1) it is defined scheme-theoretically as the zero locus of a section of a rank two vector bundle (corresponding to X) and such a section induces an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+2}} \to \mathcal{E} \to \mathcal{I}_X(k) \to 0$$

where $\mathcal{E}$ is a rank two vector bundle with $c_1(\mathcal{E}) = k$ and $c_2(\mathcal{E}) = \deg X$. Note that $\mathcal{E}$ is uniquely determined up to isomorphism.

In this case, it is also known that X is a complete intersection if and only if its corresponding rank two vector bundle splits as a sum of trivial line bundles([OSS], Ch.I, Lemma 5.2.1.) if and only if it is projectively normal (due to Gherardelli, Gaeta, Peskine, and Grifiths and Evans) and in particular, by Hartshorne's conjecture $X^n \subset \mathbb{P}^{n+2}$ should be a complete intersection for $n \geq 4$ [H1]. There is no known smooth projective variety of codimension two other than complete intersections if $\dim(X) \geq 4$.

However, there are many non-projectively normal threefolds in $\mathbb{P}^5$ [Rao] (of course, this case is not in the Hartshorne's conjecture range), and so it would be interesting to get an optimal bound for these varieties on higher order normality and regularity in terms of degree and codimension as mentioned before.

Theorem 2.4. *Let X be a smooth threefold of degree d in $\mathbb{P}^5$.*

(a) *X is k-normal for all $k \geq d-4$ which is sharp as the Palatini scroll of degree 7 shows.*

(b) *$\operatorname{reg} X \leq d-1$ and $\operatorname{reg} X = d-1$ if and only if it is a complete intersection of two quadrics or a Segre threefold.*

Proof. We give a sketch of proof here for simplicity. Suppose X is contained in a hyperquadric in $\mathbb{P}^5$. By Remark 2.2, X is k-normal for all $k \geq 0$. In this case, it is easy to compute $\operatorname{reg} X$ and $\operatorname{reg} X = d-1$ if and only if it is a complete intersection of two quadrics or a Segre threefold [K1]. On the other hands, for a smooth threefold with $h^0(\mathcal{I}_X(2)) = 0$, by Theorem 1.7 and Proposition 1.8, we have an exact sequence

$$0 \to E_{3,3} \to \mathcal{O}_{\mathbb{P}^4}(-3) \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \xrightarrow{[T_5^3, T_5^2, T_5, 1]} \pi_{p_*}\mathcal{O}_X \to 0$$

where $p = (0,0,0,0,0,1) = Z(T_0,T_1,T_2,T_3,T_4) \notin X$ and $\pi_p : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ is a generic projection [L], [K2]. Then, by Lemma 2.1, E^* is (-3)-regular and consequently, by Lemma 2.3, $\operatorname{reg} X \leq d-3$. So, there is no boundary examples with $\operatorname{reg} X = d-1$ if $h^0(\mathcal{I}_X(2)) = 0$. □

Remark 2.5. *As for the Palatini scroll of degree 7, consider an exact sequence*

$$0 \to \Omega^2_{\mathbb{P}^5}(2) \to \Lambda^2 V \otimes \mathcal{O}_{\mathbb{P}^5} \to \Omega_{\mathbb{P}^5}(2) \to 0$$

Therefore, $\Omega_{\mathbb{P}^5}(2)$ is globally generated. Choose four generic sections s_1, s_2, s_3, s_4 of $H^0(\mathbb{P}^5, \Omega_{\mathbb{P}^5}(2))$ which induce an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \xrightarrow{\varphi=(s_1,s_2,s_3,s_4)} \Omega_{\mathbb{P}^5}(2) \xrightarrow{(\Lambda^4\varphi)^*} \mathcal{I}_X(c_1(\Omega_{\mathbb{P}^5}(2))) \to 0$$

where X is the dependency locus of φ and it is a smooth 3-fold in $\mathbb{P}^5$ by Kleiman's Bertini-type theorem. Since $X \sim c_2(\Omega_{\mathbb{P}^5}(2))$ up to rational equivalence and by the formula

[see OSS, p.16]

$$c_k(E \otimes \mathcal{L}) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) \cdot c_1(\mathcal{L})^{k-i},\ r = rank(E),\ \mathcal{L} \text{ a line bundle},$$

we get $c_2(\Omega_{\mathbb{P}^5}(2)) = 7$. *Note that for a smooth* 3*-fold* X *in* $\mathbb{P}^5$, $\deg X = c_2(\mathcal{I}_X(5)) = 7$. *¿From the exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^5}^{\oplus 4} \to \Omega_{\mathbb{P}^5}(2) \to \mathcal{I}_X(4) \to 0,$$

we can determine the module structure of $H^i_*(\mathcal{I}_X) = \bigoplus_{n\in\mathbb{Z}} H^i(\mathcal{I}_X(n))$. *We get*

$$h^1(\mathbb{P}^5, \mathcal{I}_X(k)) = \begin{cases} 1 & \textit{when } k = 2 \\ 0 & \textit{when } k \neq 2 \end{cases}$$

In other words, X *is* k*-normal for* $k \neq 2$. *Furthermore,* $\operatorname{reg} X = 4$. *We see that* $h^0(\mathbb{P}^5, \mathcal{I}_X(k)) = 0, k = 2, 3$ *and* $h^0(\mathbb{P}^5, \mathcal{I}_X(4)) \neq 0$.

Let X be a smooth projective n-fold and codimension e. Let $S_i(X)$ be the locus of i-secant lines of X. Note that $\dim S_{n+2}(X) \leq n+1$ by Theorem 1.7.

Theorem 2.6. *Let* X *be a smooth threefold of degree* d *and codimension* e *in* $\mathbb{P}^r$.

(a) $\operatorname{reg} X \leq (d - e + 1) + 1$
(b) *If* $\dim S_4(X) \leq 4$, *then* $\operatorname{reg} X \leq (d - e + 1)$

Proof. For a proof of (a), the bad fibers under the generic projection are collinear 4 distict points which can only be occurred in a finite number of points in $\pi_p(X)$ by Theorem 1.5. So, there is an exact sequence

$$0 \to E \to H^3 \otimes \mathcal{O}_{\mathbb{P}^4}(-3) \oplus S^2(V) \otimes \mathcal{O}_{\mathbb{P}^4}(-2) \oplus V \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \to \pi_{\Lambda_*}\mathcal{O}_X \to 0$$

, where $\Lambda = \mathbb{P}^{r-4} \subset \mathbb{P}^r$ is a general linear subspace such that $\Lambda \cap X = \varnothing$, H is a linear form in Λ which does not meet with any line of 4 aligned points $\pi_\Lambda^{-1}(y)$ and $\pi_\Lambda : X \to Y$ is the projection with center at Λ. Then, by Lemma 2.1, E^* is (-2)-regular and thus by Lemma 2.3, $\operatorname{reg} X \leq (d - e + 1) + 1$. Furthermore, if we assume $\dim S_4(X) \leq 4$ then $\operatorname{reg} X \leq (d - e + 1)$ because all fibers are 2-normal and consequently, we have an exact sequence

$$0 \to E \to S^2(V) \otimes \mathcal{O}_{\mathbb{P}^4}(-2) \oplus V \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4} \to \pi_{\Lambda_*}\mathcal{O}_X \to 0$$

□

Remark 2.7.

(a) $\dim S_4(X) \leq 4$ *is equivalent to the fact that the generic projection of* X *into* $\mathbb{P}^4$ *has no collinear fiber of length* 4. *In fact Z.Ran showed in* [R1] *that for any* *smooth threefold in* $\mathbb{P}^n, n \geq 9$, *we get* $\dim S_4(X) \leq 4$ *and as a result, regularity conjecture is true in this case.*

(b) *It would be natural and interesting to describe all threefolds which have no collinear fibers of length* 4 *under the generic projection even in the case of smooth threefolds in* $\mathbb{P}^5$.

(c) *The methods used in the proof of Theorem 2.6 can be extended to higher dimensional smooth varieties only if we know the length of fibers and their positions under the generic projection of such a smooth variety. It can be shown that* $\operatorname{reg} X \leq (d - e + 1) + \epsilon_n$, *where* $\epsilon_n = 4, 10, 20$ *as* $\dim X = 4, 5, 6$ *respectively. In particular, when* $\dim X \leq 5$, *multiple points of fibers are always curvilinear from Lemma 1.4 which makes the analysis of all fibers possible in a sense of k-normality of fibers under generic projection. For details, see* [K3].

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School of Mathematics, Korea Institute for Advanced Study, 207-43 Chungryangri-dong, Dongdaemoon-gu, Seoul 130-010, Korea

E-mail address: sjkwak@kias.re.kr