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An Equivariant Torelli Theorem for K3 Surfaces with Finite Group Action and Its Applications

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This is a very brief summary of my talk given in the workshop held in July 1998 at RISM. First of all, I would like to express the gratitude to the organiser, Professor Miyazaki.

In the talk, I first formulated an equivariant version of Torelli Theorem for K3 surface with prescribed finite group action (Theorem (1.1)), which describes the automorphisms of a K3 surface commuting with a presciribed group action in terms of cohomology, and then explained how one can apply this to prove the following two results:

(1) Finiteness of $c_2 = 0$ fiber space structures on a Calabi-Yau threefold up to isomorphisms as fiber spaces (Theorem (2.1));

(2) Classification of K3 surfaces admitting order 11 automorphisms (Theorem (3.1)).

The purpose of this short note is to recall the results presented in my talk precisely, while detailed explanation of proofs will be omitted. I hope that details will be found in the preliminary version of my preprint, "An Equivariant Torelli Theorem for K3 surfaces with finite group action and its application to fibered Calabi-Yau threefolds" (the final version inculding non-trivial examples are now in preparations) and also in the preliminary version of my preprint written jointly with D. Q. Zhang, "K3 surfaces with order 11 automorphisms", which will be included in our joint paper, "Classification of finite group actions on K3 surfaces" (under progress) in future.

§1. An equivariant Torelli Theorem for K3 surfaces with finite group actions

Let X be a projective complex K3 surface and $G \subset \mathrm{Aut}(X)$ a finite automorphism group. Throughout this section these are fixed. We consider the second cohomology group $H^2(X, \mathbf{Z})$ as a lattice by the non-degenerate symmetric bilinear form $(*.*)$ induced by the cup product on this space. We will employ the following notation:

S: the Néron Severi lattice of X;
$N := \{[E] \in S | E \subset X, E \simeq \mathbf{P}^1\}$, the set of nodal classes;
$M := (S)^G = \{x \in S | g^*(x) = x \text{ for all } g \in G\}$;
$(C)^\circ :=$ the connected component of $\{x \in S \otimes \mathbf{R} | (x.x) > 0\}$ containing the ample classes of X;
$C :=$ the union of $(C)^\circ$ and all $\mathbf{Q}$-rational rays in the closure of $(C)^\circ$ in $S \otimes \mathbf{R}$;
$C_M := C \cap (M \otimes \mathbf{R})$;
$A :=$ the intersection of the nef cone $\overline{Amp}(X)$ and C;
$A_M := A \cap (M \otimes \mathbf{R})$;
$Q := \{f \in \mathrm{Aut}(X) | f \circ g = g \circ f \text{ for all } g \in G\}$;
$O(M) :=$ the orthogonal group of the lattice M preserving C_M;
$O(M)^\circ := \{\sigma \in O(M) | \sigma = \tilde{\sigma}|M \text{ for some Hodge isometry } \tilde{\sigma} \text{ with } \tilde{\sigma} \circ g^* = g^* \circ \tilde{\sigma} \text{ for all } g \in G\}$;

$P(M) := \{\sigma \in O(M) | \sigma(A_M) = A_M\}$;
$P(M)^\circ := \{\sigma \in O(M) | \sigma = f^*|M \text{ for some } f \in Q\} (\subset P(M))$.

My idea for the formulation is a quite simple one, namely, to modify everything in S appeared in the abstract version of the Torelli Theorem for K3 surface to put into the invariant lattice M in an $G-$equivariant way. For this, the most important part is the formulation of the $G-$equivariant reflection group on M, which I will first present.

Let us identify $[b] \in N$ and the unique smooth rational curve b which represents $[b]$ and set for $b \in N$, $B := (\sum_{g \in G} g^*(b))_{red}$, the reduction of the divisor $\sum_{g \in G} g^*(b)$, and denote by $B = \coprod_{k=1}^{n(b)} B_k$ the decomposition of B into the connected components. Note that the value $(B_k.B_k)$ is independent of k and that $(B_k.B_k) = -2$ if $(B_k.B_k) < 0$. Set $N_M := \{b \in N | (B_k.B_k) = -2\}$ and define for $b \in N_M$ and $k \in \{1, 2, ..., n(b)\}$ the reflection r_{B_k} on $H^2(X, \mathbf{Z})$ by $r_{B_k}(x) = x + (x.B_k)B_k$. It is well known and is easily checked that r_{B_k} are Hodge isometries and satisfy $r_{B_k}(M) = M$ and $r_{B_k}(C_M) = C_M$. Using $(B_k.B_l) = -2\delta_{kl}$, we easily calculate that $r_{B_k} \circ r_{B_l} = r_{B_l} \circ r_{B_k}$ and $r_{B_k}^2 = id$. These equalities readily imply the following formulas:

(1) $(\prod_{k=1}^{n(b)} r_{B_k})(x) = x + \sum_{k=1}^{n(b)} (x.B_k)B_k$.
(2) $(\prod_{k=1}^{n(b)} r_{B_k})^2 = id$.

Set $R_b := \prod_{k=1}^{n(b)} r_{B_k}$ for $b \in N_M$ and $\Gamma_M := < R_b \,|\, b \in N_M > \ (\subset O(M))$. This is the reflection group which we want to formulate. I can now state our Equivariant Torelli Theorem:

Theorem (1.1)
(1) Γ_M is a normal subgroup of $O(M)^\circ$ and fits in the semi-direct decomposition $O(M)^\circ = \Gamma_M \cdot P(M)^\circ$.
(2) There exists a finite rational polyhedral fundamental domain Δ for the action $P(M)^\circ$ on A_M.

For application, the next three Corollaries will be useful.
Corollary (1.2).
The set of $G-$stable fiber space structures on X is finite up to $G-$equivariant isomorphism.
Corollary (1.3).
Let Z be a normal K3 surface and G_Z a finite automorphism group of Z. Then Z admits only finitely many G_Z-stable fiber space structures up to G_Z-equivariant isomorphism.
Corollary (1.4).
(1) Assume that M represents 0. Then X admits a $G-$stable elliptic fibration. In particular, if the rank of M is greater than or equal to 5, then X admits a $G-$stable elliptic fibration.
(2) Assume that M contains the even unimodular hyperbolic lattice U of rank 2. Then X admits a $G-$stable Jacobian fibration.

§2. Finiteness of $c_2 = 0$ fibrations on a Calabi-Yau threefold

In this section, I will explain the first application, which was indeed my motivation to reformulate the Torelli Theorem for K3 surface in an equivariant setting. Thoughout this section, a Calabi-Yau threefold means a smooth projective complex threefold X which satisfies that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $\pi_1(X) = \{1\}$. A fiber space means a surjective morphism $\varphi : X \to W$ between positive dimensional normal projective varieties with connected fibers. Two fiber spaces $\Phi : X \to W$ and $\Phi' : X' \to W'$ are said to be isomorphic if there exist isomorphisms $F : X \to X'$ and $f : W \to W'$ such that $\Phi' \circ F = f \circ \Phi$.

Let X be a Calabi-Yau threefold. A fiber space $\Phi : X \to W$ is called a $c_2 = 0$ fibration if $(D.c_2(X)) = 0$ for a divisor D such that $\Phi = \Phi_{|D|}$. This notion does not depend on the choice of representatives $|D|$ of the morphism Φ.

The first application of the Equivariant Torelli Theorem is the following:

Theorem (2.1).
Each Calabi-Yau threefold X admits only finitely many $c_2 = 0$ fibrations up to isomorphism. In particular, each Calabi-Yau threefold admits only finitely many abelian fiber space structures up to isomorphism.

I will omit to repeat the detailed proof here. However, I hope that readers will recognaise that there certainly exist close relations between Equivariant Torelli Theorem (1.1) and Theorem (2.1) through (1.3) and the following Classification Theorems, particularly (2.3)(b), due to the present author, which themselves might have their own interests.

Theorem (2.2).
Let $\varphi : X \to W$ be a $c_2 = 0$ fibration. Assume that $\dim W = 3$. Then $\varphi : X \to W$ is isomorphic to either
(a) the unique crepant resolution $\Phi_3 : X_3 \to \overline{X_3} := (E_{\zeta_3})^3 / < g_3 >$, where E_{ζ_3} is the elliptic curve of period ζ_3 and $g_3 = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3)$, or
(b) the unique crepant resolution $\Phi_7 : X_7 \to \overline{X_7} := A_7 / < g_7 >$, where A_7 is the Jacobian threefold
of the Klein quartic curve $C := (x_0 x_1^3 + x_1 x_2^3 + x_2 x_0^3 = 0) \subset \mathbf{P}^2$ and $g_7 \in \mathrm{Aut}(A_7)$ is the Gorenstein automorphism of order 7 induced by the automorphism g_C of C defined by $g_C^*([x_0 : x_1 : x_2]) = [\zeta_7 x_0 : \zeta_7^2 x_1 : \zeta_7^4 x_2]$.

Theorem (2.3)
Let $\varphi : X \to W$ be a $c_2 = 0$ fibration. Assume that $\dim W = 2$. Then $\varphi : X \to W$ is isomorphic to either
(a) a minimal smooth birational model, over the base $E_{\zeta_3}^2 / < \mathrm{diag}(\zeta_3, \zeta_3) >$, of the composite of the crepant resolution of $((E_{\zeta_3})^2 \times E_{\zeta_3}) / < \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3) >$ and the natural projection $p_1 : ((E_{\zeta_3})^2 \times E_{\zeta_3}) / < \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3) > \to (E_{\zeta_3})^2 / < \mathrm{diag}(\zeta_3, \zeta_3) >$, or
(b) a minimal smooth birational model, over the base Z/G, of the composite of a crepant resolution of $(Z \times E)/G$ and the natural projection $p_1 : (Z \times E)/G \to Z/G$, where Z is a normal K3 surface, E is an elliptic curve and G is a finite Gorenstein automorphism group of $Z \times E$ whose elements $g \in G$ are of the forms $g = (g_Z, g_E) \in \mathrm{Aut}\,(Z) \times \mathrm{Aut}\,(E)$.

In three dimensional birational geometry, there is a particular phenomenon that minimal models of a given threefold are no more unique. This makes three dimensional birational geometry more rich. It will be also worth recalling here that another important material of the proof of Theorem (2.1) is Kawamata's Theorem of the finiteness of the minimal models of Calabi-Yau fiber space $X \to B$ over the base B up to $\mathrm{Aut}(X/B)$, [Y. Kawamata, On the cone of divisors of Calabi-Yau fiber spaces, Internat. J. Math. 8 (1997), especially Theorem 3.6].

§3. K3 surfaces with order 11 automorphisms (joint work with D. Q. Zhang)

In this section, I would like to explain our second application, the classification of complex K3 surface with automorphism of order 11, or slightly more generally, describe all the families of pairs (X, G) consisting of a complex projective K3 surface X and a finite group G of automorphisms on X which fits in the exact sequence:

$$1 \to G_N \to G \to \mu_{11n} = \langle \zeta_{11n} \rangle \to 1,$$

where the last map ρ is the natural representaion of G on the space $H^{2,0}(X) = \mathbb{C}\omega_X$ and n is some positive integer. Recalling Nikulin's result that $\mathrm{ord}(a) \leq 8$ for $a \in G_N$, we choose and fix an element $g \in G$ with $\rho(g) = \zeta_{11}$, and set $M := H^2(X, \mathbb{Z})^{<g>} = S^{<g>}$. This invariant lattice M appeared also in section 1 will plays an important role for our classification. For simplicity of description, G is also assumed to be maximal in the sense that if (X, G') also satisfies the same conditon as above for some n' and $G \subset G'$ then $G = G'$.

In order to state the classification, I first recall three kinds of examples of such pairs. In these examples, U denotes the lattice defined by the Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $U(m)$ the lattice defined by $\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$ and A_*, D_*, E_* the negative definite lattices given by the Dynkin diagrams of the indicated types.

Example 1.

Let S_{66} be the K3 surface given by the Weierstrass equation $y^2 = x^3 + (t^{11} - 1)$, and σ_{66} the automorphism of S_{66} given by $\sigma_{66}^*(x, y, t) = (\zeta_{66}^{22}x, \zeta_{66}^{33}y, \zeta_{66}^{12}t)$. Then the pair $(S_{66}, \langle\sigma_{66}\rangle)$ gives an example of (X, G) such that $n = 6$ (and $G_N = \{1\}$), i.e., $G \simeq \mu_{66}$ and that $M \simeq U$.

Example 2.

Let us consider the rational, fibered threefold $\varphi : \mathcal{X} \to \mathbb{C}$ defined by $y^2 = x^3 + x + (t^{11} - s)$, and its order 22 automomrphism σ given by $\sigma^*(x, y, t, s) = (x, -y, \zeta_{11}t, s)$, where s is the coordinate of $\mathbb{C}$. φ is a smooth morphism over $s \neq \pm\sqrt{-4/27}$ and $\mathcal{X}_{\sqrt{-4/27}}$ ($\simeq \mathcal{X}_{-\sqrt{-4/27}}$) has a unique singular point of type A_{10}.

The pair $(\mathcal{X}_0, \langle\sigma_{44}\rangle)$, where $\sigma_{44}^*(x, y, t) = (\zeta_{44}^{22}x, \zeta_{44}^{11}y, \zeta_{44}^{34}t)$, gives an example of (X, G) such that $n = 4$ (and $G_N = \{1\}$), i.e., $G \simeq \mu_{44}$ and that $M = U$. (The minimal resolution of) $(\mathcal{X}_s, \langle\sigma\rangle)$ $(s \neq 0)$ gives an example of (X, G) such that $n = 2$ (and $G_N = \{1\}$), i.e., $G \simeq \mu_{22}$ and that $M = U$ (resp. $U \oplus A_{10}$) if $s \neq 0, \pm\sqrt{-4/27}$ (resp. if $s = \pm\sqrt{-4/27}$).

Example 3.

Let us consider the following three series of rational Jacobian elliptic surfaces:

(1) $j^{(1)} : J^{(1)} \to \mathbb{P}^1$, defined by the Weierstrass equation $y^2 = x^3 + (t - 1)$, whose singular fibers are $J_1^{(1)}$ of Kodaira's type II and $J_\infty^{(1)}$ of Kodaira's type II^*,

(2) $j^{(2)} : J^{(2)} \to \mathbb{P}^1$, defined by the Weierstrass equation $y^2 = x^3 + x + (t - s)$ with $s \neq \pm\sqrt{-4/27}$, whose singular fibers are $J^{(2)}_\alpha$, $J^{(2)}_\beta$ (where $t = \alpha, \beta$ are two distinct non-zero roots of the discriminant $\Delta(t) = 4 + 27(t-s)^2$) of Kodaira's type I_1, and $J^{(2)}_\infty$ of Kodaira's type II^*, and
(3) $j^{(3)} : J^{(3)} \to \mathbb{P}^1$, defined by the Weierstrass equation $y^2 = x^3 + x + (t - s)$ with $s = \sqrt{-4/27}$, whose singular fibers are $J^{(3)}_0$, $J^{(3)}_{2s}$ of Kodaira's type I_1, and $J^{(3)}_\infty$ of Kodaira's type II^*.
Let $p^{(i,e)} : P^{(i,e)} \to \mathbb{P}^1$ be a non-trivial principal homogeneous space of $j^{(i)} : J^{(i)} \to \mathbb{P}^1$ given by an element e of order 11 in $(J^{(i)})_0$. Then $p^{(i,e)} : P^{(i,e)} \to \mathbb{P}^1$ is a rational elliptic surface with a multiple fiber of multiplicity 11 over 0 (of type I_0 in the cases $i = 1, 2$ and of type I_1 in the case $i = 3$).
Let $Z^{(i,e)}$ be the log Enriques surface of index 11 obtained by the composite of the blow up at the intersection of the components of multiplicities 5 and 6 in $(P^{(i,e)})_\infty$, which is of Kodaira's type II^*, and the blow down of the proper transform of $(P^{(i,e)})_\infty$. Let $X^{(i,e)}$ be the global canonical cover of $Z^{(i,e)}$ and $G^{(i,e)}$ the Galois group of this covering. Then, each of these pairs $(X^{(i,e)}, G^{(i,e)})$ gives an example of (X, G) such that $n = 1$ (and $G_N = \{1\}$), i.e., $G \simeq \mu_{11}$ and that $M = U(11)$.

Our result states that these are all:

Theorem (3.1).
Under the notation above, the following are true:
(1) $G_N = \{1\}$ so that $G \simeq \mu_{11n}$ and g is unique and of order 11.
(2) M is isomorphic to either one of U, $U \oplus A_{10}$ or $U(11)$.
(3) In the case where $M \simeq U$ or $U \oplus A_{10}$, (X, G) is isomorphic to either $(S_{66}, \langle\sigma_{66}\rangle)$, $(\mathcal{X}_0, \langle\sigma_{44}\rangle)$, or $(\mathcal{X}_s, \langle\sigma\rangle)$ $(s \neq 0)$ in Examples 1 and 2.
Moreover, $M \simeq U \oplus A_{10}$ if and only if $(X, \langle g\rangle)$ is isomorphic to $(\mathcal{X}_{\sqrt{-4/27}}, \langle\sigma^2\rangle)$ $(\simeq (\mathcal{X}_{-\sqrt{-4/27}}, \langle\sigma^2\rangle))$.
(4) In the case where $M \simeq U(11)$, (X, G) is isomorphic to one of $(X^{(i,e)}, G^{(i,e)})$ in Example 3.

The key of the proof is to determine all the possibilities of M and then apply (1.4) to find a $< g >$ $-$stable elliptic fiber space structure on $(X, < g >)$ to get concrete descriptions of such pairs as in Examples 1-3.