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A note on Hilbert-Kunz multiplicity

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1 Introduction

This is a joint work with Prof. Kei-ichi Watanabe in Nihon University; see [WY].

Throughout this talk, let $(A, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic $p > 0$. Put $d := \dim A \geq 1$. Let $\widehat{A}$ denote the $\mathfrak{m}$-adic completion of A, and let $\mathrm{Ass}(A)$ (resp. $\mathrm{Min}(A)$) denote the associated prime ideals (resp. minimal prime ideals) of A. Moreover, unless specified, let I denote an $\mathfrak{m}$-primary ideal of A and M a finite A-module.

First, we recall the notion of Hilbert-Kunz multiplicity which was defined by Kunz [Ku1]; see also Monsky [Mo], Huneke [Hu].

Definition 1.1 The Hilbert-Kunz multiplicity $e_{HK}(I, M)$ of M with respect to I is defined as follows:

$$e_{HK}(I, M) := \lim_{e\to\infty} \frac{\lambda_A(M/I^{[q]}M)}{q^d},$$

where $q = p^e$ and $I^{[q]} = (a^q \mid a \in I)A$. For simplicity, we put $e_{HK}(I) := e_{HK}(I, A)$ and $e_{HK}(A) := e_{HK}(\mathfrak{m})$.

The following question is fundamental but still open.

Question 1.2 Is $e_{HK}(I)$ always a rational number?

• Known Results.

(1.3.1) Let $e(I)$ be the multiplicity of A with respect to I. Then we have the following inequalities:

$$\frac{e(I)}{d!} \leq e_{HK}(I) \leq e(I).$$

(1.3.2) $e_{HK}(I) \geq e_{HK}(A) \geq 1$.

(1.3.3) Put $\mathrm{Assh}(A) = \{P \in \mathrm{Spec}(A) \mid \dim A/P = d\}$. Then

$$e_{HK}(I, M) = \sum_{P\in \mathrm{Assh}(A)} e_{HK}(I, A/P) \cdot l_{A_P}(M_P).$$

For example, if A is a local domain and B is a torsion free A-module of rank r, then $e_{HK}(I, B) = r \cdot e_{HK}(A)$.

(1.3.4) (Kunz [Ku2]) For any prime ideal $P \in \operatorname{Spec}(A)$ such that height $P + \dim A/P = \dim A$, we have

$$e_{HK}(A_P) \leq e_{HK}(A).$$

(1.3.5) If A is a regular local ring, then $e_{HK}(I) = \lambda_A(A/I)$.

(1.3.6) If I is a parameter ideal, then $e_{HK}(I) = e(I)$.

(1.3.7) We recall the notion of tight closure. An element $x \in A$ is said to be in the *tight closure* I^* of I if there exists an element $c \in A^0$ such that for all large $q = p^e$, $cx^q \in I^{[q]}$, where $A^0 := A \setminus \bigcup\{P \mid P \in \operatorname{Min}(A)\}$.

Let I, J be $\mathfrak{m}$-primary ideals such that $I \subseteq J$. Then if $I^* = J^*$, then $e_{HK}(I) = e_{HK}(J)$. Furthermore, if, in addition, $\widehat{A}$ is equidimensional and reduced, then the converse is also true.

(1.3.8) ([WY] or [BCP]) Let $(A, \mathfrak{m}) \subseteq (B, \mathfrak{n})$ be a module-finite extension of local domains. Then

$$e_{HK}(I, A) = \frac{[B/\mathfrak{n} : A/\mathfrak{m}]}{[Q(B) : Q(A)]} \cdot e_{HK}(IB, B),$$

where $Q(A)$ denotes the fraction field of A.

Question 1.4 If $\operatorname{pd}_A A/I < \infty$, then does the same formula as that in (1.3.5) hold?

• Background and Questions.

In general, there is an example such that $e_{HK}(I) = e(I)$; for instance, let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}$. If $\mathfrak{q}^* = \mathfrak{m}$, then we have $e_{HK}(\mathfrak{m}) = e_{HK}(\mathfrak{q}) = e(\mathfrak{q}) = e(\mathfrak{m})$. However, we have no example such that $\frac{e(I)}{d!} = e_{HK}(I)$. On the other hand, if $A = k[[X_1, \ldots, X_d]]^{(r)}$, then

$$e_{HK}(A) = \frac{1}{r}\binom{d+r-1}{r-1} \quad \text{and} \quad e(A) = r^{d-1}.$$

Thus if we tend r to ∞, then the limit $\frac{e_{HK}(A)}{e(A)}$ tends to $\frac{1}{d!}$. So we consider the following question.

Question 1.5 *Is there a constant number $\alpha > 0$ depending on $d = \dim A$ alone such that*

$$e_{HK}(I) \geq \frac{e(I)}{d!} + \alpha?$$

On the other hand, in [WY], we proved the following theorem.

Theorem 1.6 [WY, Theorem (1.5)] *If A is an unmixed (i.e. $\operatorname{Ass}(\widehat{A}) = \operatorname{Assh}(\widehat{A})$) local ring with $e_{HK}(A) = 1$, then it is regular.*

In the above theorem, we cannot remove the assumption that A is "unmixed". For instance, if $e(A) = 1$, then $e_{HK}(A) = 1$. We now consider the case of Cohen-Macaulay local rings. Then the following question is a natural extension of the above theorem.

Question 1.7 If A is a Cohen-Macaulay local ring with $e_{HK}(A) < 2$, then is it F-regular ?

The following conjecture is related to the above questions.

Conjecture 1.8 *Let A be a quasi-unmixed (i.e.* $\mathrm{Min}(\widehat{A}) = \mathrm{Assh}(\widehat{A})$*) local ring. Then $e_{HK}(I) \geq \lambda(A/I^*)$ for any $\mathfrak{m}$-primary ideal I.*

Further, if A is a Cohen-Macaulay local ring then $e_{HK}(I) \geq \lambda(A/I)$ for any $\mathfrak{m}$-primary ideal I.

2 A positive answer to Question 1

Throughout this section, let A be a Noetherian local ring with $\dim A = 2$ and suppose that $k = A/\mathfrak{m}$ is infinite. The following theorem is a main result in this section.

Theorem 2.1 (cf. [WY, Section 5]) Suppose $\dim A = 2$. Then for any $\mathfrak{m}$-primary ideal I, we have

$$e_{HK}(I) \geq \frac{e(I)+1}{2} \left(> \frac{e(I)}{2} \right).$$

First, we consider the case of Cohen-Macaulay local rings. Now suppose that A is Cohen-Macaulay. Let I be an $\mathfrak{m}$-primary ideal and J its minimal reduction, that is, $J = (a, b)$ is a parameter ideal of A and $I^{n+1} = JI^n$ for some $n \geq 1$.

Lemma 2.2 *Suppose that A is Cohen-Macaulay, $1 \leq s < 2$ and $q = p^e$. We define $I^x = I^{\lfloor x \rfloor}$ for any positive real number x. Then we have*

(1) $\lambda_A(A/I^{(s-1)q}) = \dfrac{e(I)}{2}(s-1)^2q^2 + o(q^2)$, *where $f(q) = o(q^2)$ means* $\displaystyle\lim_{e\to\infty} \frac{f(q)}{q^2} = 0$.

(2) $\lambda_A\left(\dfrac{I^{sq}+J^{[q]}}{J^{[q]}}\right) = \dfrac{e(I)}{2}(2-s)^2q^2 + o(q^2)$.

Proof. Put $n = \lfloor (s-1)q \rfloor$ and $\epsilon = (s-1)q - n$.

(1) $\lambda_A(A/I^{(s-1)q}) = \lambda_A(A/I^n) = \dfrac{e(I)}{2}n^2 + f(n)$, where $\displaystyle\lim_{e\to\infty}\frac{f(n)}{n^2} = 0$.
Thus we get

$$\lambda_A(A/I^{(s-1)q}) = \frac{e(I)}{2}\left((s-1)q-\epsilon\right)^2 + o(q^2) = \frac{e(I)}{2}(s-1)^2q^2 + o(q^2).$$

(2) $\lambda_A\left(\dfrac{I^{sq}+J^{[q]}}{J^{[q]}}\right) \leq \lambda_A\left(\dfrac{J^{sq}+J^{[q]}}{J^{[q]}}\right) + \lambda_A\left(\dfrac{I^{sq}}{J^{sq}}\right).$

First, we estimate the second term. Since $e(I) = e(J)$, we have

$$\lambda_A(I^{sq}/J^{sq}) = \lambda_A(A/J^{sq}) - \lambda_A(A/I^{sq}) = o(q^2).$$

Next, we estimate the first term.

$$\begin{aligned} \lambda_A\left(\frac{J^{sq}+J^{[q]}}{J^{[q]}}\right) &\leq \sum_{l=n}^{2q}\left\{(x,\,y)\in\mathbb{Z}^2 \mid 0\leq x,\,y\leq q-1,\ x+y=l\right\}\times\lambda_A(A/J)+o(q^2) \\ &= \frac{1}{2}(2q-sq)^2\cdot e(I)+o(q^2). \qquad \textbf{Q.E.D.} \end{aligned}$$

Lemma 2.3 *Suppose that A is Cohen-Macaulay. Let I be an $\mathfrak{m}$-primary ideal of A and J a minimal reduction of I. If I/J is generated by r elements (i.e. $r \geq \mu_A(I) - 2$), then we have*

$$\lambda_A(I^{[q]}/J^{[q]}) \leq \frac{r}{2(r+1)}e(I)\cdot q^2 + o(q^2).$$

Moreover, if $J^* \subseteq I$ and I/J^* is generated by r elements, the same result holds.

Proof. Let s be any real number such that $1 \leq s < 2$. Then

$$\lambda_A\left(\frac{I^{[q]}}{J^{[q]}}\right) \leq \lambda_A\left(\frac{I^{[q]}+I^{sq}}{J^{[q]}+I^{sq}}\right)+\lambda_A\left(\frac{J^{[q]}+I^{sq}}{J^{[q]}}\right) =: (E1)+(E2).$$

Since we can write as $I = Au_1 + \cdots Au_r + J$, we get

$$\begin{aligned} (E1) &\leq \sum_{i=1}^{r}\lambda_A\left(\frac{u_i^qA+J^{[q]}+I^{sq}}{J^{[q]}+I^{sq}}\right)=\sum_{i=1}^{r}\lambda_A\left(\frac{A}{(J^{[q]}+I^{sq}):u_i^q}\right) \\ &\leq r\cdot\lambda_A\left(\frac{A}{I^{(s-1)q}}\right)=r\cdot\frac{e(I)}{2}(s-1)^2q^2+o(q^2) \qquad \text{by (2.2).} \end{aligned}$$

On the other hand, by (2.2) again, $(E2) = \dfrac{e(I)}{2}(2-s)^2q^2 + o(q^2)$. Thus

$$\lambda_A\left(\frac{I^{[q]}}{J^{[q]}}\right) \leq \frac{e(I)}{2}q^2\left\{(r+1)s^2-2(r+2)s+(r+4)\right\}+o(q^2).$$

Put $s = \dfrac{r+2}{r+1}$, and we get the required inequality.

Further, the last statement follows from the fact $\lambda_A(A/J^{[q]}) = \lambda_A(A/(J^*)^{[q]}) + o(q^2)$. **Q.E.D.**

Next proposition easily follows from the above lemma.

Proposition 2.4 *Suppose that A is Cohen-Macaulay. Let I be an $\mathfrak{m}$-primary ideal of A and J a minimal reduction of I. If I/J is generated by r elements then we have*

$$e_{HK}(I) \geq \frac{r+2}{2(r+1)}\cdot e(I).$$

Moreover, if $J^* \subseteq I$ and I/J^* is generated by r elements (i.e. $r \geq \mu_A(I/J^*) = \lambda_A(I/J^* + I\mathfrak{m})$), the same result holds.

We now give a proof of Theorem (2.1). First, we suppose that A is Cohen-Macaulay and let J be a minimal reduction of $\mathfrak{m}$. Since

$$e(I)-1=\lambda_A(\mathfrak{m}/J)=\lambda_A(I/J)+\lambda_A(\mathfrak{m}/I)\geq \lambda_A(I/J+I\mathfrak{m})+\lambda_A(\mathfrak{m}/I),$$

we have $e(I)-1\geq e(I)-1-\lambda_A(\mathfrak{m}/I)\geq \mu_A(I/J)$. By virtue of Proposition (2.4), we get

$$e_{HK}(I)\geq \frac{r+2}{2(r+1)}\cdot e(I)\geq \frac{e(I)+1}{2e(I)}\cdot e(I)=\frac{e(I)+1}{2},\quad \text{where } r=e(I)-1-\lambda_A(\mathfrak{m}/I).$$

We remark that Equality $e_{HK}(I)=(e(I)+1)/2$ implies $I=\mathfrak{m}$.

Next, we consider about general local rings. Since $e_{HK}(I)=e_{HK}(I\widehat{A})$ and $e(I)=e(I\widehat{A})$, we may assume that A is complete. Moreover, since

$$\begin{aligned} e_{HK}(I) &= \sum_{P\in \mathrm{Assh}(A)} e_{HK}(I,\, A/P)\cdot \lambda_{A_P}(A_P) \\ e(I) &= \sum_{P\in \mathrm{Assh}(A)} e(I,\, A/P)\cdot \lambda_{A_P}(A_P), \end{aligned}$$

we may assume that A is a complete local domain. Let B be the integral closure of A in its fraction field. Then B is a complete normal local domain and a finite A-module; thus it is a two-dimensional Cohen-Macaulay local ring. Let $\mathfrak{n}$ be an unique maximal ideal of B and put $t=[B/\mathfrak{n}:A/\mathfrak{m}]$. Then we have

$$e_{HK}(I)=t\cdot e_{HK}(IB,B),\qquad e(I)=t\cdot e_{HK}(IB,B).$$

Thus by the argument in the Cohen-Macaulay case, we get

$$e_{HK}(I)=t\cdot e_{HK}(IB,B)\geq t\cdot \frac{e_{HK}(IB,B)+1}{2}\geq \frac{e_{HK}(I)+1}{2}.$$

Corollary 2.5 *If A is a non-Cohen-Macaulay, unmixed local ring (with $\dim A=2$), then*

$$e_{HK}(I,A)>\frac{e(I)+1}{2}$$

for any $\mathfrak{m}$-primary ideal I of A.

Proof. By the above proof, we may assume that A is a complete local domain. With the same notation as in the proof of Theorem, B is a torsion free A-module. If $\mu_A(B)=1$, then $B\cong A$; this contradicts the assumption that A is not Cohen-Macaulay. Thus $\lambda_A(B/\mathfrak{m}B)=\mu_A(B)\geq 2$.

When $t:=[B/\mathfrak{n}:A/\mathfrak{m}]=1$, since $\lambda_B(B/\mathfrak{m}B)=\lambda_A(B/\mathfrak{m}B)\geq 2$, we have $IB\subseteq \mathfrak{m}B\subsetneq \mathfrak{n}$. Hence

$$e_{HK}(I)=e_{HK}(IB,B)>\frac{e(IB)+1}{2}=\frac{e(I)+1}{2}.$$

On the other hand, when $t\geq 2$, we have

$$e_{HK}(I)\geq \frac{e(I)+t}{2}>\frac{e(I)+1}{2}.\qquad \textbf{Q.E.D.}$$

Corollary 2.6 *Let A be a local ring with* $\dim A = 2$. *Then*

(1) *When $e(A) = 1$, we have $e_{HK}(A) = 1$.*

(2) *When $e(A) \geq 2$, we have $e_{HK}(A) \geq \dfrac{3}{2}$.*

3 Local rings with small Hilbert-Kunz multiplicity

In this section, we consider Question (1.7) in case of local rings with $\dim A = 2$. In order to state the main theorem, we recall the notion of F-regular rings. A local ring A is said to be *F-regular*(resp. *F-rational*) if $I^* = I$ for every ideal (resp. parameter ideal) I of A. We are now ready to state the main theorem, which is a slight generalization of Theorem (5.4) in [WY].

Theorem 3.1 (cf. [WY, Theorem (5.4)]) Let A be an unmixed local ring with $\dim A = 2$ and suppose $k = \overline{k}$. Then

(1) $1 < e_{HK}(A) < 2$ if and only if $\widehat{A}$ is an F-rational double point, that is, $\widehat{A} \cong k[[X, Y, Z]]/(f)$, where f is given by the list below (3.2).

(2) $e_{HK}(A) = 2$ if and only if A satisfies either one of the following conditions:

(a) A is not F-regular with $e(A) = 2$.

(b) $\widehat{A} \cong k[[X^3, X^2Y, XY^2, Y^3]]$.

Corollary 3.2 *Let A be an unmixed local ring with* $\dim A = 2$. *If $e_{HK}(A) < 2$, then $\widehat{A}$ is isomorphic to the completion of the ring $k[X,Y]^G$ where G is a finite subgroup of $SL_2(k)$. In particular, A is a module-finite subring of $k[[X, Y]]$ and $e_{HK}(A) = 2 - \dfrac{1}{|G|}$.*

In fact, $|G|$ is given by the following table.

type	f	$\lvert G\rvert$	
(A_n)	$f = xy + z^{n+1}$	$n+1$	$n \geq 1$
(D_n)	$f = x^2 + yz^2 + y^{n-1}$	$4(n-2)$	$n \geq 4,\ p \geq 3$
(E_6)	$f = x^2 + y^3 + z^4$	24	$p \geq 3$
(E_7)	$f = x^2 + y^3 + yz^3$	48	$p \geq 5$
(E_8)	$f = x^2 + y^3 + z^5$	120	$p \geq 7$

From now on, let A be an unmixed local ring with $\dim A = 2$. In order to prove the above theorem, we give several lemmas.

Lemma 3.3 *If $1 < e_{HK}(A) < 2$, then $\widehat{A}$ is an integral domain with $e(\widehat{A}) = 2$ and $\widehat{A}_P$ is regular for any prime ideal $P \neq \mathfrak{m}\widehat{A}$.*

Proof. We may assume that A is complete. First, we observe that $e(A) = 2$. Actually, it follows from Theorem (2.1).

Next, we show that A is a local domain with isolated singularity. For any prime ideal $P \neq \mathfrak{m}$, we have $e_{HK}(A_P) \leq e_{HK}(A) < 2$. Since $e_{HK}(A_P)$ must be a positive integer, we have $e_{HK}(A_P) = 1$. Hence A_P is regular.

On the other hand, $\#\operatorname{Ass}(A) = \#\operatorname{Assh}(A) = 1$. Actually, if $\#\operatorname{Assh}(A) \geq 2$,

$$2 > e_{HK}(A) = \sum_{P \in \operatorname{Assh}(A)} e_{HK}(A_P) \cdot \lambda_{A_P}(A_P) \geq \#\operatorname{Assh}(A) \geq 2$$

gives a contradiction. Hence $\#\operatorname{Ass}(A) = 1$. Therefore A is a local domain. **Q.E.D.**

Corollary 3.4 *Let A be a Cohen-Macaulay local ring with $e(A) = 2$ and suppose that $\widehat{A}$ is reduced. Then*

(1) *If A is F-regular, then $e_{HK}(A) < 2$.*

(2) *If A is not F-regular, then $e_{HK}(A) = 2$.*

Proof. Let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}$. Since A is Cohen-Macaulay, we have $\lambda_A(A/\mathfrak{q}) = e(A) = 2$; thus $\mathfrak{q}^* = \mathfrak{q}$ or $\mathfrak{q}^* = \mathfrak{m}$, because $\mathfrak{q} \subseteq \mathfrak{q}^* \subseteq \mathfrak{m}$.

When $\mathfrak{q}^* = \mathfrak{q}$, since A is Gorenstein, A must be F-regular. Moreover, since $\mathfrak{m} \neq \mathfrak{q}^*$ and $\widehat{A}$ is reduced, we get

$$e_{HK}(A) := e_{HK}(\mathfrak{m}) < e_{HK}(\mathfrak{q}^*) = e_{HK}(\mathfrak{q}) = e(\mathfrak{q}) = 2.$$

On the other hand, when $\mathfrak{q}^* = \mathfrak{m}$, A is not F-regular and $e_{HK}(A) = e_{HK}(\mathfrak{q}) = 2$. **Q.E.D.**

We now give an outline of the proof of Theorem (3.1). Let A be an unmixed local ring with $\dim A = 2$ and suppose $k = \overline{k}$.

Step 1. When A is a complete Cohen-Macaulay local ring with $e_{HK}(A) < 2$, it is an F-rational double point.

Proof. In fact, by Lemma (3.3), A is a complete local domain with $e(A) = 2$. Thus Corollary (3.4) implies that A is F-regular. Then A is given by the list in Corollary (3.2).

Step 2. If A is unmixed local ring with $e_{HK}(A) < 2$, then $\widehat{A}$ is F-regular.

Proof. We may assume that A is complete. By Lemma (3.3), A is a complete local domain with $e(A) = 2$. Let B the integral closure of A in its fraction field. Then $\lambda_A(B/A) < \infty$ and B is a local domain and is a module-finite extension of A. Let $\mathfrak{n}$ be an unique maximal ideal of B. In order to show that A is F-regular it is enough to show $A = B$, for B is Cohen-Macaulay. As $A/\mathfrak{m} \cong B/\mathfrak{n}$, we get

$$2 > e_{HK}(A) = e_{HK}(\mathfrak{m}, B) \geq e_{HK}(\mathfrak{n}, B) =: e_{HK}(B).$$

According to Step 1, B is F-regular with $e_{HK}(B) = 2 - \dfrac{1}{|G|}$ and is a module-finite subring of $C = k[[X, Y]]$ such that $|G| = [Q(C) : Q(B)]$.

Now suppose $A \neq B$. Then $\mathrm{H}^1_{\mathfrak{m}}(A) \cong B/A \neq 0$ and thus A is not Cohen-Macaulay. Further, as $\mu_A(B) \geq 2$, we have $\mathfrak{m}.B \subsetneq \mathfrak{n}$. Moreover, since both B and C are F-regular rings, we obtain that $I.C \cap B = I$ for any ideal I of B. In particular, we have $\mathfrak{m}.C \subsetneq \mathfrak{n}.C$. Hence we get

$$\begin{aligned} e_{HK}(A) - e_{HK}(B) &= \frac{1}{|G|}\lambda_A(C/\mathfrak{m}.C) - \frac{1}{|G|}\lambda_A(C/\mathfrak{n}.C) \\ &= \frac{1}{|G|}\lambda_A(\mathfrak{n}.C/\mathfrak{m}.C) \geq \frac{1}{|G|}. \end{aligned}$$

Thus

$$e_{HK}(A) \geq e_{HK}(B) + \frac{1}{|G|} = \left(2 - \frac{1}{|G|}\right) + \frac{1}{|G|} = 2.$$

Thus we conclude that $A = B$ as required. $\square$

Step 3. Let A be a complete Cohen-Macaulay local ring. Then $e_{HK}(A) = 2$ if and only if A is not F-regular with $e(A) = 2$ or $A \cong k[[X^3, X^2Y, XY^2, Y^3]]$.

Proof. If part is easy. But only if part is hard. See [WY, Section5] for details. $\square$

Step 4. Suppose that A is unmixed but not Cohen-Macaulay. Then $e_{HK}(A) = 2$ if and only if $e(A) = 2$.

Proof. If part: If $e(A) = 2$, then $e_{HK}(A) \leq 2$. If $e_{HK}(A) < 2$, then A is Cohen-Macaulay by Step 2. However, this contradicts the assumption. Hence $e_{HK}(A) = 2$.
Only if part follows from Corollary (2.5). **Q.E.D.**

In the final of this section, we give the following problem.

Problem 3.5 *Let A be an unmixed local ring with* $\dim A = 2$. *Characterize the ring A which satisfies* $e_{HK}(A) = \dfrac{e(A)+1}{2}$.

In fact, if $A = k[[X,Y]]^{(e)}$ then $e(A) = e$ and $e_{HK}(A) = \dfrac{e+1}{2}$. Further, the proof of the above theorem implies that if $e_{HK}(A) = \dfrac{e(A)+1}{2}$ and $e(A) \leq 3$ then $A \cong k[[X,Y]]^{e(A)}$. Moreover, the following proposition gives a partial answer to this problem.

Proposition 3.6 *If A is an unmixed local ring with $e_{HK}(A) = \dfrac{e(A)+1}{2}$, then it is F-rational.*

Proof. By Cor (2.5), A is Cohen-Macaulay. Then we show that A has a minimal multiplicity, that is, $\mathrm{emb}(A) = e(A) + \dim A - 1$. Let $\mathfrak{q}$ be a minimal redcution of $\mathfrak{m}$. Then since

$$e(A) - 1 = \lambda_A(\mathfrak{m}/\mathfrak{q}) \geq \lambda_A(\mathfrak{m}/\mathfrak{q} + \mathfrak{m}^2) = \mu_A(\mathfrak{m}/\mathfrak{q}).$$

If $e(A) - 1 > \mu_A(\mathfrak{m}/\mathfrak{q}) =: r_0$, then

$$e_{HK}(A) \geq \frac{r_0 + 2}{2(r_0+1)} \cdot e(A) > \frac{e(A)+1}{2};$$

see the proof of Theorem (2.1) for detail. Thus $e(A) - 1 = \mu_A(\mathfrak{m}/\mathfrak{q})$. It follows that $\mathfrak{m}^2 \subseteq \mathfrak{q}$; thus A has a minimal multiplicity.

We will show that A is F-rational. Suppose not. Then $\mathfrak{q}^* \neq \mathfrak{q}$. Since $\mathfrak{m}^2 \subseteq \mathfrak{q} \subseteq \mathfrak{q}^*$, we have $r_1 := \mu_A(\mathfrak{m}/\mathfrak{q}^*) < \mu_A(\mathfrak{m}/\mathfrak{q}) = r_0$. Thus by virtue of (2.4), we get

$$e_{HK}(A) \geq \frac{r_1+2}{2(r_1+1)} \cdot e(A) > \frac{r_0+2}{2(r_0+1)} \cdot e(A) = \frac{e(A)+1}{2}.$$

This contradicts the assumption. Hence we conclude that A is F-rational. **Q.E.D.**

4 Extended Rees Rings.

In this section, we consider the following question.

Question 4.1 *Let A be a local ring and $F = \{F_n\}$ a filtration of A. Then does $e_{HK}(A) \leq e_{HK}(G_F(A))$ always hold ? Further, when does equality hold?*

In order to state our result, we recall the definition of Rees ring, extended Rees ring and the associated graded ring.

Let A be a local ring of A with $d := \dim A \geq 1$. Then $F = \{F_n\}_{n\in\mathbb{Z}}$ is said to be a filtration of A if the following conditions are satisfied:

(a) F_i is an ideal of A such that $F_i \supseteq F_{i+1}$ for each i.

(b) $F_i = A$ for each $i \leq 0$ and $\mathfrak{m} \supseteq F_1$.

(c) $F_iF_j \subseteq F_{i+j}$ for each i, j.

For a given filtration $F = \{F_n\}_{n\in\mathbb{Z}}$ of A, we define

$$\begin{aligned} R := R_F(A) &:= \bigoplus_{n=0}^{\infty} F_n t^n. \\ S := R'_F(A) &:= \bigoplus_{n\in\mathbb{Z}} F_n t^n. \\ G := G_F(A) &:= \bigoplus_{n=0}^{\infty} F_n/F_{n+1} \cong S/t^{-1}S \cong R/R(1). \end{aligned}$$

$R_F(A)$ (resp. $R'_F(A)$, $G_F(A)$) is said to be the Rees (resp. the exteded Rees, the associated graded) ring with respect to a filtration F of A.

Then our main result in this section is the following theorem.

Theorem 4.2 *Let A be any local ring with $d := \dim A > 0$ and let $F = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of A. Suppose that $R_F(A)$ is a Noetherian ring with $\dim R_F(A) = d+1$. Then for any $\mathfrak{m}$-primary ideal I of A such that $F_1 \subseteq I \subseteq \mathfrak{m}$, we have*

(1) *$e_{HK}(I, A) \leq e_{HK}(N, S)$, where $N = (t^{-1}, I, S_+)$.*

(2) *If F_1 is an $\mathfrak{m}$-primary ideal, then $e_{HK}(N, S) \leq e_{HK}(G)$.*

In particular, if F_1 is an $\mathfrak{m}$-primary ideal, then

$$e_{HK}(A) \leq e_{HK}(S) \leq e_{HK}(G).$$

Question 4.3 *In the above theorem, when does equality hold? How about $e_{HK}(A) \leq e_{HK}(R_F(A))$?*

Example 4.4 *Let $A = k[[X, Y]]$ and $I = (X^m, Y^n)$, where $m \geq n \geq 1$. Then*

(1) $e(R(I)) = n + 1$.

(2) $e_{HK}(R(I)) = n + 1 - \dfrac{n(3m-1)}{3m^2}$.

(3) $e(R'(I)) = n + 2$ (*if* $n \geq 2$), $= 2$ (*otherwise*).

(4) $e_{HK}(R'(I)) = n + 2 - \dfrac{n}{m} - \dfrac{1}{n}$.

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