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On nef values of determinants of ample vector bundles

Masahiro Ohno
大野 真裕*†

0 Introduction

Let M be an n-dimensional complex projective manifold and $\mathcal{E}$ an ample vector bundle of rank r on M. The nefness of the adjoint bundle $K_M + \det \mathcal{E}$ has been studied by several authors in the case where $r \geq n-2$. In this note, we investigate the nef value $\tau(M, \det \mathcal{E})$ of the polarized manifold $(M, \det \mathcal{E})$, and show the following results.

Proposition 0.1. *$\tau(M, \det \mathcal{E}) \leq (n+1)/r$ and equality holds if and only if $(M, \mathcal{E}) \cong (\mathbf{P}^n, \mathcal{O}(1)^{\oplus r})$.*

If we put $r = n+1$, this proposition implies [YZ, Theorem 1] and [P1, Theorem]. This proposition can be strengthened as follows.

Proposition 0.2. *If $r \leq n$, then $\tau(M, \det \mathcal{E}) \leq n/r$ unless $(M, \mathcal{E}) \cong (\mathbf{P}^n, \mathcal{O}(1)^{\oplus r})$.*

Proposition 0.3. *If $r \geq n$, $\tau(M, \det \mathcal{E}) \leq (n+1)/(r+1)$ unless $(M, \mathcal{E}) \cong (\mathbf{P}^n, \mathcal{O}(1)^{\oplus r})$.*

If we put $r = n$, these propositions are the same proposition of Ye and Zhang [YZ, Theorem 2]. The main theorems of this note are the following:

Theorem 0.4. *If $r \leq n$, then $\tau(M, \det \mathcal{E}) = n/r$ if and only if $(M, \mathcal{E})$ is one of the following;*
1) $(\mathbf{P}^n, T_{\mathbf{P}^n})$
2) $(\mathbf{P}^n, \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2))$
3) $(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(1)^{\oplus r})$*, where $\mathbf{Q}$ is a hyperquadric in $\mathbf{P}^{n+1}$*
4) $(\mathbf{P}(\mathcal{F}), H(\mathcal{F}) \otimes \psi^*\mathcal{G})$ *where $\mathcal{F}$ is a vector bundle of rank n on a smooth proper curve C, $\psi : \mathbf{P}(\mathcal{F}) \to C$ is the projection, and $\mathcal{G}$ is a vector bundle of rank r on C.*

Note that if $r = n$ then Theorem 0.4 implies Peternell's theorem [P2, Theorem 2] and if $r \geq n$ then Theorem 0.4 and Proposition 0.3 (or 0.1) lead Fujita's theorem [F4, Main Theorem].

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*Research Fellow of the Japan Society for the Promotion of Science from April 1st to September 30th 1998.
†The author has moved to the University of Electro-Communications since October 1st 1998. Current E-mail address: ohno@e-one.uec.ac.jp

Theorem 0.5. *Suppose that* $\tau(M, \det \mathcal{E}) < n/r$. *If* $r \leq n-1$, *then* $\tau(M, \det \mathcal{E}) \leq (n-1)/r$ *unless* $(M, \mathcal{E}) \cong (\mathbf{P}^n, \mathcal{O}(1)^{\oplus(r-1)} \oplus \mathcal{O}(2))$ *and* $r > (n-1)/2$.

Note also that if $r = n-1$ then Theorem 0.5 combined with Proposition 0.2 leads [YZ, Theorem 3].

Theorem 0.6. *Suppose that* $2 \leq r \leq n-2$. *If* $\tau(M, \det \mathcal{E}) = (n-1)/r$, *then* $(M, \mathcal{E})$ *is one of the following;*
0) $(\mathbf{P}^n, \mathcal{O}(1)^{\oplus(r-1)} \oplus \mathcal{O}(2))$ *where* $r = (n-1)/2$ *and* n *is odd.*
1) M *is a Del Pezzo manifold with* $\operatorname{Pic} M \cong \mathbf{Z}$, *and* $\mathcal{E} \cong L^{\oplus r}$ *where* L *is the ample generator of* $\operatorname{Pic} M$.
2) *There exist a hyperquadric fibration* $\psi : M \to C$ *over a smooth curve* C, *a* ψ*-ample line bundle* $\mathcal{O}_M(1)$ *on* M *and an ample vector bundle* $\mathcal{G}$ *of rank* r *on* C *such that* $\mathcal{E} \cong \mathcal{O}_M(1) \otimes \psi^*\mathcal{G}$ *where* $\mathcal{O}_M(1)|_F \cong \mathcal{O}_Q(1)$ *for any fiber* $F \cong Q$ *of* ψ.
3) *There exists a* $\mathbf{P}^{n-2}$*-fibration* $\psi : M \to S$, *locally trivial in the étale (or complex) topology, over a smooth surface* S *such that* $\mathcal{E}|_F \cong \mathcal{O}_{\mathbf{P}^{n-2}}(1)^{\oplus r}$ *for every fiber* F *of* ψ.
4) M *is the blowing-up* $\psi : M \to M'$ *of a projective manifold* M' *at finite points, and there exists an ample vector bundle* $\mathcal{E}'$ *of rank* r *on* M' *such that* $\tau(M', \det \mathcal{E}') < (n-1)/r$ *and* $\mathcal{E} \cong \psi^*\mathcal{E}' \otimes \mathcal{O}_M(-E)$ *where* E *is the exceptional divisor of* ψ.

Theorem 0.6 could be seen as a natural continuation of [ABW, Theorem], [PSW, Main Theorem(0.3)] and [F1, Theorem 3'] from the view point of nef value.

Notation and conventions

In this note we work over the complex number field $\mathbf{C}$. Basically we follow the standard notation and terminology in algebraic geometry. We use the word *manifold* to mean a smooth variety. For a manifold M, we denote by K_M or simply by K the canonical divisor of M. We use the word *line* to mean a smooth rational curve of degree 1. We also use the words "locally free sheaf" and "vector bundle" interchangeably. For a vector bundle $\mathcal{E}$ on a variety X, we denote also by $H(\mathcal{E})$ the tautological line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ on $\mathbf{P}(\mathcal{E})$. We are going to use the terminology in the Minimal Model Program. For our terminology, we fully refer to [KMM] and [M2]. For an extremal ray R of $\overline{\mathrm{NE}}(M)$, we denote by $l(R)$ the length of the ray R.

1 Preliminaries and proofs of propositions

We first recall the nef value $\tau(M, L)$ of a polarized manifold (M, L): $\tau(M, L)$ is defined to be the minimum of the set of real numbers t such that $K_M + tL$ is nef.

We also recall, for convenience of the reader, the following theorem [CM, Main Theorem] due to Koji Cho and Yoichi Miyaoka.

Theorem 1.1. *Let* M *be a Fano manifold of dimension* n *over the complex numbers. If* $(C, -K_M) \geq n+1$ *for every effective rational curve* $C \subset M$, *then* M *is isomorphic to* $\mathbf{P}^n$.

Now we begin with the proof of Proposition 0.1

Proof of Proposition 0.1. Let τ be the nef value $\tau(M, \det\mathcal{E})$ of the polarized manifold $(M, \det\mathcal{E})$. We may assume that τ is positive. Then there exists an extremal rational curve C on M such that $(K + \tau\det\mathcal{E}).C = 0$. Thus $\tau \leq (n+1)/r$ since $-K.C \leq n+1$ and $\det\mathcal{E}.C \geq r$. If equality holds, then M is a Fano manifold of Picard number one by [I, Theorem (0.4)]. Hence M is isomorphic to $\mathbf{P}^n$ by Theorem 1.1. Since $\mathcal{E}$ turns out to be a uniform vector bundle of type $(1, \dots, 1)$, $\mathcal{E}$ is isomorphic to $\mathcal{O}(1)^{\oplus r}$. □

Proof of Proposition 0.2. Assume that $K + (n/r)\det\mathcal{E}$ is not nef. Let R be an extremal ray of $\overline{\mathrm{NE}}(M)$ such that $(K + (n/r)\det\mathcal{E}).R < 0$ and let C be an extremal rational curve which belongs to R. Then $n \leq (n/r)\det\mathcal{E}.C < -K.C \leq n+1$. Thus $-K.C = n+1$ and therefore the length $l(R)$ of R is $n+1$. Hence M is a Fano manifold of Picard number one by [I, Theorem (0.4)] and M is isomorphic to $\mathbf{P}^n$ by Theorem 1.1. Moreover $\det\mathcal{E}.C < r(n+1)/n = r + (r/n)$. Since $r \leq n$, this implies that $\det\mathcal{E}.C = r$. Therefore $\mathcal{E}$ is a uniform vector bundle of type $(1, \dots, 1)$ and isomorphic to $\mathcal{O}(1)^{\oplus r}$. □

Remark 1.2. *We can give another proofs of Propositions 0.1 and 0.2 without using Theorem 1.1.*

Proof of Proposition 0.3. Assume that $K + (n+1/r+1)\det\mathcal{E}$ is not nef. Let R be an extremal ray of $\overline{\mathrm{NE}}(M)$ such that $(K+(n+1/r+1)\det\mathcal{E}).R < 0$ and let C be an extremal rational curve which belongs to R. Then $r \leq \det\mathcal{E}.C < -(r+1)/(n+1)K.C \leq r+1$ and so $\det\mathcal{E}.C = r$. Hence $n \leq (n+1)r/(r+1) = (n+1)/(r+1)\det\mathcal{E}.C < -K.C \leq n+1$. Thus $-K.C = n+1$ and the length $l(R)$ of R is $n+1$. Hence M is a Fano manifold of Picard number one by [I, Theorem (0.4)]. Therefore M is isomorphic to $\mathbf{P}^n$ by Theorem 1.1 and $\mathcal{E}$ is a uniform vector bundle of type $(1, \dots, 1)$, so that $\mathcal{E}$ is isomorphic to $\mathcal{O}(1)^{\oplus r}$. □

2 Proofs of Theorems 0.4 and 0.5

First we give a proof of Theorem 0.4.

Proof of Theorem 0.4. Let P be the projective space bundle $\mathbf{P}(\mathcal{E})$ over M, $\pi : P \to M$ the projection, and L the tautological line bundle $H(\mathcal{E})$. Let R be an extremal ray of $\overline{\mathrm{NE}}(M)$ such that $(K_M + (n/r)\det\mathcal{E}).R = 0$ and let $\psi : M \to C$ be the contraction morphism of R. Since $r \leq n$, we have $(K_M + \det\mathcal{E}).R \leq 0$ so that $-\pi^*(K_M + \det\mathcal{E})$ is $\psi\circ\pi$-nef. Thus $-K_P$ is $\psi\circ\pi$-ample because $-K_P = rL - \pi^*(K_M + \det\mathcal{E})$. This implies that $\psi\circ\pi$ is the contraction morphism of an extremal face. Let R_π be the extremal ray corresponding to $\pi : P \to M$ and H an ample Cartier divisor on C. Then the extremal face $((\psi\circ\pi)^*H)^\perp \cap \overline{\mathrm{NE}}(P)$ corresponding to $\psi\circ\pi$ can be expressed as $R_\pi + R_1$, where R_1 is an extremal ray of $\overline{\mathrm{NE}}(P)$ different from R_π. Let $\varphi : P \to N$ be the contraction morphism of R_1. Then there exists a unique morphism $\pi' : N \to C$ such that $\pi'\circ\varphi = \psi\circ\pi$, and we have the following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & N \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\psi} & C. \end{array}$$

Let $z \in N$ be a point such that $\dim \varphi^{-1}(z) > 0$ and put $d = \dim \varphi^{-1}(z)$. Let A_z be a d-dimensional irreducible component of $\varphi^{-1}(z)$. Since $\pi|_{A_z} : A_z \to M$ is finite, we have $d \le n$. Hence we have $l(R_1) \le n+1$ by Wiśniewski's theorem [W, Theorem (1.1)]. Let $C_1 \subset P$ be a rational curve which belongs to R_1 and which attains the length $l(R_1)$ of R_1. Since $\psi(\pi(C_1))$ is a point, $\pi(C_1)$ belongs to R, and therefore $(K_M + (n/r)\det \mathcal{E}).\pi(C_1) = 0$. Hence we have

$$\begin{aligned} n+1 \ge -K_P.C_1 &= rL.C_1 - \pi^*(K_M + \det \mathcal{E}).C_1 \\ &= rL.C_1 + ((n/r)-1)\det \mathcal{E}.\pi_*(C_1) \\ &\ge r + n - r = n. \end{aligned}$$

If $L.C_1 \ge 2$, then we have $r = 1$ by these inequalities. Thus

$$n+1 \ge rL.C_1 + ((n/r)-1)\det \mathcal{E}.\pi_*(C_1) = nL.C_1 \ge 2n,$$

and we have $n = 1$. The theorem is obvious when $n = 1$. Therefore we may assume that $L.C_1 = 1$.

Since $L.C_1 = 1$, we know that $C_1 \to \pi(C_1)$ is birational. Let $f : W \to A_z$ be the normalization, $\tilde{W} \to W$ a desingularization, and $g : \tilde{W} \to W \to A_z$ the composite of these two morphisms.

Assume that $-K_P.C_1 = n+1$. Then we have $1 \le -n - K_P.C_1 = -nL.C_1 - K_P.C_1$. It follows from the argument in [Ma, (2.3)] that $h^d(\tilde{W}, -tg^*(L|_{A_z})) = 0$ for all $t \le n$. Since $d \le n$, this implies that $(W, f^*(L|_{A_z})) \cong (\mathbf{P}^d, \mathcal{O}(1))$ by [F2, (2.2) Theorem]. If $d \le n-1$, then $h^d(\tilde{W}, -ng^*(L|_{A_z})) = h^d(\mathbf{P}^d, \mathcal{O}(-n)) \ne 0$, which is a contradiction. Hence we have $d = n$. Therefore Lazarsfeld's theorem [L, §4] implies that $M \cong \mathbf{P}^n$. Let D be a line in $\mathbf{P}^n$. Since $\det \mathcal{E}.D = (r/n)(-K_M).D = r(1+(1/n))$ and $r \le n$ and $\det \mathcal{E}.D$ is an integer, we have $r = n$. Thus $\mathcal{E}$ is a uniform vector bundle of type $(1, \dots, 1, 2)$ and so $\mathcal{E} \cong T_{\mathbf{P}^n}$ or $\mathcal{E} \cong \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)$ (see, e.g., [OSS]). Since φ has n-dimensional fibers, we know that $\mathcal{E} \cong \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)$. This is the case 2) of the theorem.

Assume that $-K_P.C_1 = n$. The theorem is true for $r = n$ by [F4, Main Theorem] or [P2, Theorem 2], and so we may assume that $r \le n-1$ in the following. Then we have $\det \mathcal{E}.\pi(C_1) = r$ and $-K_M.\pi(C_1) = n$. On the other hand, for every rational curve $D \subset M$ belonging to R, we have $-K_M.D = n/r \det \mathcal{E}.D \ge n$. Therefore the length $l(R)$ of R is n. It follows from Wiśniewski's theorem [W, Theorem (1.1)] that $\dim C \le 1$.

Suppose that $\dim C = 1$. Let U denote the largest open subset of C such that $\psi^{-1}(U) \to U$ is smooth. Let F be any fiber of the morphism $\psi^{-1}(U) \to U$. Then $K_F + n/r \det \mathcal{E}|_F = 0$, i.e., $\tau(F, \det \mathcal{E}|_F) = ((n-1)+1)/r$. Hence Proposition 0.1 shows that $(F, \mathcal{E}|_F) \cong (\mathbf{P}^{n-1}, \mathcal{O}(1)^{\oplus r})$. Since $H^2(U, \mathcal{O}_U^\times) = 0$ by Tsen's theorem, where we consider U with metric (or étale) topology, $\psi^{-1}(U)$ is isomorphic to $\mathbf{P}(\mathcal{F}_U)$ over U for some vector bundle $\mathcal{F}_U$ on U. Let H denote the tautological line bundle $H(\mathcal{F}_U)$ on $\psi^{-1}(U)$. We can extend H to a line bundle on M, which we also denote by H by abuse of notation. Let F' be an arbitrary fiber of ψ. Then F' is irreducible and reduced because ψ is the contraction morphism of an extremal ray and $\dim C = 1$. Since the polarized variety $(F, H|_F)$ has Fujita's delta genus $\Delta(F, H|_F) = 0$ and degree $H|_F^{n-1} = 1$, $(F', H|_{F'})$ also has the same delta genus and degree, so that $(F', H|_{F'}) \cong (\mathbf{P}^{n-1}, \mathcal{O}(1)^{\oplus r})$. Thus $\det \mathcal{E}|_{F'} = \mathcal{O}(r)$. Therefore $\mathcal{E}|_{F'} \cong \mathcal{O}(1)^{\oplus r}$. This is the case 4) of the theorem.

Suppose that $\dim C = 0$. Then M is a Fano manifold of Picard number one and $K_M + n/r \det \mathcal{E} \equiv 0$. Let A be the ample generator of $\operatorname{Pic} M$: $\operatorname{Pic} M = \mathbf{Z} \cdot A$. Since $0 = -n - K_P.C_1 = -nL.C_1 - K_P.C_1$, we get $h^d(\tilde{W}, -tg^*(L|_{A_z})) = 0$ for all $t \leq n-1$ by the argument in [Ma, (2.3)]. Thus we obtain $d \geq n-1$ by the same reason as before.

If φ is birational, then $h^d(\tilde{W}, -ng^*(L|_{A_z})) = 0$ by [F3, (11.4) Lemma]. Therefore we know that $d = n$ and $(W, f^*(L|_{A_z})) \cong (\mathbf{P}^n, \mathcal{O}(1))$ by [F2, (2.2) Theorem]. Hence it follows from Lazarsfeld's theorem [L, §4] that $M \cong \mathbf{P}^n$, which contradicts the assumption that $l(R) = n$. Thus φ is of fiber type.

If $d = n-1$, then $(W, f^*(L|_{A_z})) \cong (\mathbf{P}^{n-1}, \mathcal{O}(1))$ by [F2, (2.2) Theorem]. We claim that φ has equidimensional fibers. Suppose, to the contrary, that φ has an n-dimensional fiber $\varphi^{-1}(z')$ over a point $z' \in N$. Let $A_{z'}$ denote an n-dimensional irreducible component of $\varphi^{-1}(z')$. Let $f' : W' \to A_{z'}$ be the normalization, $\tilde{W}' \to W'$ a desingularization, and $g' : \tilde{W}' \to W' \to A_{z'}$ the composite of these two morphisms. Since $0 = -nL.C_1 - K_P.C_1$, we have $h^n(\tilde{W}', -tg'^*(L|_{A_{z'}})) = 0$ for all $t \leq n$ by [YZ, Lemma 4]. Thus Fujita's theorem [F2, (2.2) Theorem] again implies that $(W', f'^*(L|_{A_{z'}})) \cong (\mathbf{P}^n, \mathcal{O}(1))$. Hence $M \cong \mathbf{P}^n$ as before, which contradicts the assumption that $l(R) = n$. Therefore φ has equidimensional fibers. This implies that φ is a $\mathbf{P}^{n-1}$-bundle over a projective manifold N by [F1, (2.12) Lemma]. Note that $\dim N = r$. Let $\mathcal{F}$ denote $\varphi_* L$. Then $\mathcal{F}$ is a vector bundle of rank n. Moreover $\mathcal{F}$ is ample because $H(\mathcal{F}) = L$.

We have $\operatorname{Pic} N \cong \mathbf{Z}$: let B denote the ample generator of $\operatorname{Pic} N$. Since

$$-rL + \pi^*(K_M + \det \mathcal{E}) = K_P = -nL + \varphi^*(K_N + \det \mathcal{F}),$$

we have $n - r = \varphi^*(K_N + \det \mathcal{F}).l = (K_N + \det \mathcal{F}).\varphi_*(l)$, where l denote a line in a fiber of π. Note that $l \to \varphi(l)$ is birational because $L.l = 1$. Thus $-K_N.\varphi(l) = \det \mathcal{F}.\varphi(l) + r - n \geq r$.

We claim here that $-K_N.\varphi(l) \leq r+1$. Assume, to the contrary, that $-K_N.\varphi(l) \geq r+2$. Then $\varphi(l)$ can be deformed to a sum $\sum_{i=1}^{\delta} l_i$ of at least two rational curves l_i's (some of which may be equal) $(i = 1, \ldots, \delta, \delta \geq 2)$ such that $-K_N.l_i \leq r+1$ by [M1, Theorem 4]. Thus $n - r = (K_N + \det \mathcal{F}).\varphi(l) = \sum_{i=1}^{\delta}(K_N + \det \mathcal{F}).l_i \geq \delta(-r-1+n)$. Hence $(\delta - 1)(n-r) \leq \delta$. Since $r \leq n-1$ by the preceding assumption, we have $1 \leq n - r \leq 1 + (1/(\delta-1)) \leq 2$. If $n - r = 1$, then $1 = (K_N + \det \mathcal{F}).\varphi(l) = \sum_{i=1}^{\delta}(K_N + \det \mathcal{F}).l_i$, which is a contradiction because $\operatorname{Pic} N \cong \mathbf{Z}$ and so $K_N + \det \mathcal{F}$ is ample. Hence $n - r = 2$, $\delta = 2$, and $(K_N + \det \mathcal{F}).l_i = 1$. Since $n \leq \det \mathcal{F}.l_i = 1 - K_N.l_i \leq r+2 = n$, we obtain $n = \det \mathcal{F}.l_i$ and $-K_N.l_i = r+1$. This implies that $K_N + (r+1)(K_N + \det \mathcal{F}) = 0$. Applying Kobayashi and Ochiai's theorem [KO], we infer that $(N, K_N + \det \mathcal{F}) \cong (\mathbf{P}^r, \mathcal{O}(1))$. Therefore $\det \mathcal{F} \cong \mathcal{O}(r+2) = \mathcal{O}(n)$ and $\mathcal{F} \cong \mathcal{O}(1)^{\oplus n}$. This means that π is $\mathbf{P}^r$-bundle, which is a contradiction.

By the claim above, we have two cases: $(-K_N.\varphi(l), \det \mathcal{F}.\varphi(l)) = (r+1, n+1)$ and $(-K_N.\varphi(l), \det \mathcal{F}.\varphi(l)) = (r, n)$. Let l' denote $\varphi(l)$ and let C_1' denote $\pi(C_1)$. Put $s = A.C_1'$ and $t = B.l'$. We have $\varphi^* B = xL + y\pi^* A$ for some $x, y \in \mathbf{Z}$. Restricting this formula on l, we get $0 < t = x$, and restricting this formula on C_1, we obtain $0 = x + ys$. Hence $y < 0$. Since $\pi^* A \in \mathbf{Z} \cdot L \oplus \mathbf{Z} \cdot \varphi^* B$, y is a unit in $\mathbf{Z}$. Hence $y = -1$ and $s = x = t$. Thus $\varphi^* B = sL - \pi^* A$. Put $\mathbf{P}^1_\eta = l$. Note that $\mathbf{P}^1_\eta = l \to l'$ is the normalization. Let X denote

$P \times_N \mathbf{P}^1_\eta$, and let π_X denote the composite of $X \to P$ and π.

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{P}^1_\eta \\ \downarrow & & \downarrow \\ P & \xrightarrow{\varphi} & N \end{array}$$

Suppose that $(-K_N.l', \det \mathcal{F}.l') = (r+1, n+1)$. Then

$$X = \mathbf{P}(\mathcal{F} \otimes \mathcal{O}_l) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)).$$

Let $p : X \to \mathbf{P}^n_\xi$ be the morphism determined by $|H(\mathcal{O}_{\mathbf{P}^1}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2))|$. Note that $L_X = H_\xi + H_\eta$, where $H_\xi = H(\mathcal{O}_{\mathbf{P}^1}^{\oplus(n-1)} \oplus \mathcal{O}(1)) = \mathcal{O}_{\mathbf{P}^n_\xi}(1) \otimes \mathcal{O}_X$ and $H_\eta = \mathcal{O}_{\mathbf{P}^1_\eta}(1) \otimes \mathcal{O}_X$. Hence $\pi_X^* A = sL_X - (\varphi^* B)_X = sH_\xi + sH_\eta - tH_\eta = sH_\xi$. Thus we obtain a unique finite morphism $h : \mathbf{P}^n_\xi \to M$ forming a commutative diagram

$$\begin{array}{ccc} \mathbf{P}^n_\xi & \xleftarrow{p} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \pi_X} \\ M & = & M. \end{array}$$

This implies that $M \cong \mathbf{P}^n$ by Lazarsfeld's theorem [L, §4]. This contradicts the assumption that $l(R) = n$. Hence this case does not occur.

Suppose that $(-K_N.l', \det \mathcal{F}.l') = (r, n)$. Then

$$X = \mathbf{P}(\mathcal{F} \otimes \mathcal{O}_l) = \mathbf{P}^{n-1}_\xi \times \mathbf{P}^1_\eta.$$

Let $p : X \to \mathbf{P}^{n-1}_\xi$ be the projection. We have $L_X = H_\xi + H_\eta$, where $H_\xi = \mathcal{O}_{\mathbf{P}^n_\xi}(1) \otimes \mathcal{O}_X$ and $H_\eta = \mathcal{O}_{\mathbf{P}^1_\eta}(1) \otimes \mathcal{O}_X$. Hence $\pi_X^* A = sL_X - (\varphi^* B)_X = sH_\xi + sH_\eta - tH_\eta = sH_\xi$. Thus there exists a unique finite morphism $h : \mathbf{P}^{n-1}_\xi \to M$ forming a commutative diagram

$$\begin{array}{ccc} \mathbf{P}^{n-1}_\xi & \xleftarrow{p} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \pi_X} \\ M & = & M. \end{array}$$

Put $D_M = \pi(X)$. D_M is a prime divisor on M. For every point $z \in l'$, $\pi(\varphi^{-1}(z)) = D_M$. This implies that for every line l_1 in a fiber of π we have $\pi(\varphi^{-1}(z)) = \pi(\varphi^{-1}(z'))$ for all points $z, z' \in \varphi(l_1)$. Since every two points in the fiber $\pi^{-1}(\pi(l))$ can be joined by a line, we know that $\pi(\varphi^{-1}(z)) = D_M$ for every point $z \in \varphi(\pi^{-1}(\pi(l)))$. Moreover for every point $x \in D_M$ and $x' \in h^{-1}(x)$, $x' \times \mathbf{P}^1_\eta$ is embedded as a line in $\pi^{-1}(x)$ because $L_X = H_\xi + H_\eta$, and $\varphi(x' \times \mathbf{P}^1_\eta) = l'$. Therefore it follows from the above argument that $\pi(\varphi^{-1}(z)) = D_M$ for every point $z \in \varphi(\pi^{-1}(x))$. Hence $\pi(\varphi^{-1}(z)) = D_M$ for every point $z \in \varphi(\pi^{-1}(D_M))$. Putting $D_P = \pi^*(D_M)$, we get $\pi(\varphi^{-1}(\varphi(D_P))) = D_M$. Thus $\varphi^{-1}(\varphi(D_P)) = \pi^{-1}(D_M) = D_P$. Therefore $D_P.C_1 = 0$. On the other hand, since $D_M = \alpha A$ for some positive integer α, we have $D_P.C_1 = \alpha\pi^* A.C_1 = \alpha A.C_1' = \alpha s > 0$. This is a contradiction. Therefore there is no $(n-1)$-dimensional fiber in φ and $d = n$.

Now take z as a general point of N. Then $\tilde{W} = W = A_z = \varphi^{-1}(z)$. It follows from $(K_P + nL).C_1 = 0$ that $K_{\varphi^{-1}(z)} + nL|_{\varphi^{-1}(z)} = 0$. Applying Kobayashi and Ochiai's theorem [KO], we infer that $\varphi^{-1}(z) \cong \mathbf{Q}^n$. Hence we obtain $M \cong \mathbf{P}^n$ or $\mathbf{Q}^n$ by [CS] or [PS]. Now we are in the assumption that $l(R) = n$, so that M is in fact isomorphic to $\mathbf{Q}^n$. Furthermore since $\det \mathcal{E}.D = -r/nK_M.D = r$ for any line D in $\mathbf{Q}$ we have $\mathcal{E}|_D \cong \mathcal{O}_D(1)^{\oplus r}$ for any line $D \subset \mathbf{Q}$. Hence $\mathcal{E} \cong \mathcal{O}(1)^{\oplus r}$. □

Finally we give a proof of Theorem 0.5.

Proof of Theorem 0.5. Let τ denote the nef value $\tau(M, \det \mathcal{E})$ of $(M, \det \mathcal{E})$. Let R be an extremal ray of $\overline{\mathrm{NE}}(M)$ such that $(K_M + \tau \det \mathcal{E}).R = 0$ and $\psi : M \to C$ the contraction morphism of R. Let D be an extremal rational curve belonging to R. Since $(n-1)/r < \tau$, $(K_M + (n-1)/r \det \mathcal{E}).R < 0$. Hence we have $n - 1 \leq (n-1)/r \det \mathcal{E}.D < -K_M.D$, and therefore $n \leq -K_M.D$. On the other hand, $(K_M + n/r \det \mathcal{E}).R > 0$ since $\tau < n/r$. If we have $-K_M.D = n$, this implies that $\det \mathcal{E}.D > r$. Hence $\det \mathcal{E}.D \geq r + 1$. Therefore we have $-K_M.D > (n-1)/r \det \mathcal{E}.D \geq n - 1 + (n-1)/r \geq n$ since $r \leq n - 1$. This is a contradiction. Thus we have $-K_M.D = n + 1$, so that the length $l(R)$ of R is $n + 1$. Applying Ionescu's theorem [I, Theorem (0.4)], we know that $\dim C = 0$. Therefore M is a Fano manifold of Picard number one. It follows from Theorem 1.1 that $M \cong \mathbf{P}^n$. For every line D in $\mathbf{P}^n$, we have $\det \mathcal{E}.D < r(n+1)/(n-1) = r + (2r/(n-1)) \leq r + 2$ and $\det \mathcal{E}.D > r(n+1)/n = r + (r/n)$. Hence $\det \mathcal{E}.D = r + 1$ and $1 < 2r/(n-1)$. Therefore $\mathcal{E} \cong \mathcal{O}(1)^{\oplus(r-1)} \oplus \mathcal{O}(2)$ and $r > (n-1)/2$. □

Remark 2.1. *Without using Theorem 1.1, we can show Theorem 0.5.*

3 Outline of Proof of Theorems 0.6

Outline of Proof of Theorem 0.6. Let P be the projective space bundle $\mathbf{P}(\mathcal{E})$ over M, $\pi : P \to M$ the projection, and L the tautological line bundle $H(\mathcal{E})$. Let R be an extremal ray of $\overline{\mathrm{NE}}(M)$ such that $(K_M + ((n-1)/r) \det \mathcal{E}).R = 0$ and let $\psi : M \to S$ be the contraction morphism of R. Since $r \leq n - 1$, we have $(K_M + \det \mathcal{E}).R \leq 0$ so that $-\pi^*(K_M + \det \mathcal{E})$ is $\psi \circ \pi$-nef. Thus $-K_P$ is $\psi \circ \pi$-ample because $-K_P = rL - \pi^*(K_M + \det \mathcal{E})$. Let R_π be the extremal ray corresponding to $\pi : P \to M$. Then $\overline{\mathrm{NE}}(M/S) = R_\pi + R_1$, where R_1 is an extremal ray of $\overline{\mathrm{NE}}(P/S)$ different from R_π. Let $\varphi : P \to N$ be the contraction morphism of R_1, which is naturally an S-morphism. Let $\pi' : N \to S$ be the structural morphism. We have the following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & N \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\psi} & S. \end{array}$$

Let $z \in N$ be a point such that $\dim \varphi^{-1}(z) > 0$ and put $d = \dim \varphi^{-1}(z)$. Let A_z be a d-dimensional irreducible component of $\varphi^{-1}(z)$. Since $\pi|_{A_z} : A_z \to M$ is finite, we have $d \leq n$. Hence we have $l(R_1) \leq n + 1$ by Wiśniewski's theorem [W, Theorem (1.1)]. Let $C_1 \subset P$ be a rational curve which belongs to R_1 and which attains the length $l(R_1)$ of R_1. Since

$\psi(\pi(C_1))$ is a point, $\pi(C_1)$ belongs to R, and therefore $(K_M+((n-1)/r)\det\mathcal{E}).\pi(C_1)=0$. Hence we have

$$\begin{aligned} n+1\geq -K_P.C_1 &= rL.C_1-\pi^*(K_M+\det\mathcal{E}).C_1 \\ &= rL.C_1+(((n-1)/r)-1)\det\mathcal{E}.\pi_*(C_1) \\ &\geq n-1. \end{aligned}$$

If $L.C_1\geq 2$, then $\det\mathcal{E}.\pi_*(C_1)\geq r+1$. Hence

$$\begin{aligned} n+1 &\geq rL.C_1+(((n-1)/r)-1)\det\mathcal{E}.\pi_*(C_1) \\ &\geq 2r+(n-1)(1+(1/r))-r-1=r-1+n-1+(n-1)/r. \end{aligned}$$

However this contradicts the assumption that $2\leq r\leq n-2$. Therefore we have $L.C_1=1$.

Since $L.C_1=1$, we know that $C_1\to\pi(C_1)$ is birational. Let $f:W\to A_z$ be the normalization, $\tilde{W}\to W$ a desingularization, and $g:\tilde{W}\to W\to A_z$ the composite of these two morphisms.

The case where $-K_P.C_1=n+1$ is ruled out by the same argument in the proof of Theorem 0.4. If $-K_P.C_1=n$, then we know that φ is birational and that $(M,\mathcal{E})\cong(\mathbf{P}^n,\mathcal{O}(1)^{\oplus(r-1)}\oplus\mathcal{O}(2))$ where $r=(n-1)/2$ and n is odd by the similar argument in the proof of Theorem 0.4. This is the case 0) of the theorem.

Assume that $-K_P.C_1=n-1$ in the following. Then $(((n-1)/r)-1)\det\mathcal{E}.\pi(C_1)=n-1-r$ by the inequality above. Since $r\leq n-2$, it follows that $\det\mathcal{E}.\pi(C_1)=r$. Hence $-K_M.\pi(C_1)=n-1$ and $l(R)=n-1$. Suppose that ψ is birational. Then φ is also birational by the analogous argument in [ABW, Lemma 1.8]. Since $-K_P.C_1=n-1$, it follows from the analogous statement in [ABW, Lemma 1.13] that S is smooth. Let E be the exceptional locus of ψ. Since $l(R)=n-1$, E is an irreducible divisor which is contracted to a point by ψ. Thus ψ is the blowing-up of S at a point $\psi(E)$ by [ES, Theorem 1.1]. Hence we have the case 4) of the theorem by the standard argument.

Now suppose that ψ is of fiber type. Then $\dim S\leq 2$ because $l(R)=n-1$. If $\dim S=2$, then we have the case 3) of the theorem by the same argument as in [ABW]. Assume that $\dim S=1$ and let F be a general fiber of ψ. Then $K_F+((n-1)/r)\det\mathcal{E}_F=0$. Since $r\leq n-2$, it follows from Theorem 0.4 that $(F,\mathcal{E}_F)$ is isomorphic to $(Q,\mathcal{O}_Q(1)^{\oplus r})$ or $(\mathbf{P}(\mathcal{F}),H(\mathcal{F})\otimes\psi'^*\mathcal{G})$, where $\mathcal{F}$ is a vector bundle of rank $n-1$ on a smooth proper curve C, $\psi':\mathbf{P}(\mathcal{F})\to C$ is the projection, and $\mathcal{G}$ is a vector bundle of rank r on C. If $F=\mathbf{P}(\mathcal{F})$, then we have $h^1(\mathcal{O}_C)=h^1(\mathcal{O}_F)=0$ since F is Fano. Hence $C=\mathbf{P}^1$ and $\mathcal{F}$ and $\mathcal{G}$ can be written as direct sums of line bundles. Now we can derive a contradiction by the assumption that $2\leq r\leq n-2$ and the fact that $K_F+((n-1)/r)\det\mathcal{E}_F=0$. Thus we have $(F,\mathcal{E}_F)\cong(Q,\mathcal{O}_Q(1)^{\oplus r})$. Hence we obtain the case 2) of the theorem by the standard argument.

Finally let us consider the case $\dim S=0$. Note that M is a Fano manifold of Picard number one and that $K_M+((n-1)/r)\det\mathcal{E}=0$. If φ has an n-dimensional fiber, then it follows from the argument in [PSW, §4] that every fiber of φ is n-dimensional and that M is a Del Pezzo manifold. Let $\mathcal{O}_M(1)$ be the ample line bundle such that $K_M+(n-1)\mathcal{O}_M(1)=0$. Since $-K_M.\pi(C_1)=n-1$, we have $\mathcal{O}_M(1).\pi(C_1)=1$. Hence $H(\mathcal{E}(-1)).C_1=0$ and $H(\mathcal{E}(-1))$ is nef. Therefore $H(\mathcal{E}(-1))$ is a supporting function for φ and semiample. Thus $\mathcal{E}(-1)\cong\mathcal{O}^{\oplus r}$ by [PSW, Cor.1.2], and hence $\mathcal{E}\cong\mathcal{O}_M(1)^{\oplus r}$. Let us assume that φ has no n-dimensional fibers in the following. Moreover we can show that φ has no $(n-1)$-dimensional fibers by the similar argument as in [PSW, §5].

Hence it follows from $-K_P.C_1 = n-1$ that φ is of fiber type and that every fiber of φ is $(n-2)$-dimensional. Then $(\varphi^{-1}(z), L|_{\varphi^{-1}(z)}) \cong (\mathbf{P}^{n-2}, \mathcal{O}(1))$ for a general point $z \in N$. Thus N is smooth of dimension $r+1$ and φ makes (P, L) a scroll over N by [F1, (2.12)]. Let $\mathcal{F}$ be $\varphi_* L$. $\mathcal{F}$ is an ample vector bundle of rank $n-1$ on N. Note that C_1 is a line in $W = \mathbf{P}^{n-2}$. Since $\det \mathcal{E}_W.C_1 = r$, we have $\det \mathcal{E}_W = \mathcal{O}_W(r)$. Hence $\mathcal{E}_W = \mathcal{O}_W(1)^{\oplus r}$. Since $-rL + \pi^*(K_M + \det \mathcal{E}) = -(n-1)L + \varphi^*(K_N + \det \mathcal{F})$, we have $n-r-1 = \varphi^*(K_N + \det \mathcal{F}).l$, where l denotes a line in a fiber of π. Hence we obtain $(K_N + \det \mathcal{F}).l'$, where l' denotes $\varphi(l)$. Thus $-K_N.l' = \det \mathcal{F}.l' + r - (n-1) \geq r$.

Assume that $-K_N.l' \geq r+3$. Then l' can be deformed to a sum $\sum_{i=1}^{\delta} l_i$ of at least two rational curves l_i's (some of which may be equal) $(i = 1, \ldots, \delta, \delta \geq 2)$ such that $-K_N.l_i \leq r+2$ by [M1, Theorem 4]. Thus $n-r-1 = \sum_{i=1}^{\delta}(K_N + \det \mathcal{F}).l_i \geq \delta(-r-2+n-1)$. Hence $(\delta-1)(n-r-1) \leq 2\delta$. Since $r \leq n-2$ by the assumption, we have $1 \leq n-r-1 \leq 2+(2/(\delta-1)) \leq 4$. We can rule out the case $n-r-1=1$ by the same reason as before. If $n-1-r = 2$ or 3, then $(K_N + \det \mathcal{F}).l_i = 1$ for some i. Hence $r+2 \geq -K_N.l_i = \det \mathcal{F}.l_i - 1 \geq n-2$. If $n-1-r=2$, then $r+2 = n-1$. If $-K_N.l_i = r+2$, then we know that $(N, \mathcal{F}) \cong (\mathbf{P}^{r+1}, \mathcal{O}(1)^{\oplus(n-2)} \oplus \mathcal{O}(2))$ by Kobayashi-Ochiai's theorem [KO] as before. However this contradicts the fact that π is of fiber type. If $-K_N.l_i = r+1$, then again by Kobayashi-Ochiai's theorem [KO] we infer that $(N, \mathcal{F}) \cong (Q^{r+1}, \mathcal{O}(1)^{\oplus(n-1)})$. However this implies that $\mathrm{Im}(\pi) = \mathbf{P}^{n-2}$, which is also a contradiction. If $n-1-r=3$, then $-K_N.l_i = r+2$. Hence we obtain $(N, \mathcal{F}) \cong (\mathbf{P}^{r+1}, \mathcal{O}(1)^{\oplus(n-1)})$, which contradicts the fact that $\mathrm{Im}(\pi) = \mathbf{P}^{n-2}$. If $n-1-r=4$, then $\delta = 2$. If $(K_N + \det \mathcal{F}).l_i = 1$ for some i, then $n-3 = r+2 \geq -K_N.l_i = \det \mathcal{F}.l_i - 1 \geq n-2$. This is a contradiction. Hence we may assume that $(K_N + \det \mathcal{F}).l_i = 2$ for $i = 1$ and 2. This implies that $n-3 = r+2 \geq -K_N.l_i = \det \mathcal{F}.l_i - 2 \geq n-3$. Thus $-K_N.l_i = r+2$ and $\det \mathcal{F}.l_i = n-1$. Hence there exits a rational curve $\tilde{l}_i$ on P such that $L.\tilde{l}_i = 1$ and $\varphi(\tilde{l}_i) = l_i$. If $\pi(\tilde{l}_i)$ is a point, then we may assume that $\tilde{l}_i = l$ and this contradicts the assumption that $-K_N.l' \geq r+3$. Thus $\pi(\tilde{l}_i)$ is a rational curve. On the other hand, $-\pi^*(K_M + \det \mathcal{E}).\tilde{l}_i = n-1-r-2 = 2$. This gives that $(((n-1)/r)-1)\det \mathcal{E}.\pi(\tilde{l}_i) = 2$. Therefore we get $r \leq \det \mathcal{E}.\pi(\tilde{l}_i) = r/2$, which is a contradiction. Hence we have $-K_N.l' \leq r+2$.

By the consideration above, we have three cases: $(-K_N.l', \det \mathcal{F}.l') = (r+2, n+1)$, $(r+1, n)$ or $(r, n-1)$. Let A be the ample generator of $\mathrm{Pic}\, M$ and B the ample generator of $\mathrm{Pic}\, N$. Let C_1' denote $\pi(C_1)$. Put $s = A.C_1'$ and $t = B.l'$. Then we obtain $s = t$ and $\varphi^* B = sL - \pi^* A$ by the same argument as before. We can rule out the case where $\det \mathcal{F}.l' = n+1$ by the argument before and the case where $\det \mathcal{F}.l' = n$ by the argument as in [PSW].

Let us consider the case $(-K_N.l', \det \mathcal{F}.l') = (r, n-1)$ in the following. This part is the heart of this proof of the theorem. Let F denote any fiber of π. We have $\mathcal{F}|_F \cong \mathcal{O}_F(1)^{\oplus(n-1)}$. Note that $F \to \varphi(F)$ and $W \to \pi(W)$ are birational. For any point $z \in N$, we have $\mathcal{E}|_{\varphi^{-1}(z)} \cong \mathcal{O}_{\mathbf{P}^{n-2}}(1)^{\oplus r}$. Hence we have a birational morphism $\mathbf{P}^{n-2} \times \mathbf{P}^{r-1} \to \pi^{-1}(\pi(\varphi^{-1}(z)))$. Since $\pi^{-1}(\pi(\varphi^1(z))) \supset \varphi^{-1}(z) \cong \mathbf{P}^{n-2}$, it induces a birational morphism $\mathbf{P}^{r-1} \to \varphi(\pi^{-1}(\pi(\varphi^1(z))))$. Fix a point $z_0 \in N$ and take an irreducible reduced curve C on N such that C is not contained in $\varphi(\pi^{-1}(\pi(\varphi^1(z_0))))$. For any $z_1 \in C \setminus \varphi(\pi^{-1}(\pi(\varphi^1(z_0))))$, we have $\pi(\varphi^{-1}(z_1)) \cap \pi(\varphi^{-1}(z_0)) = \emptyset$. Since $\dim \varphi^{-1}(C) =$

$1+n-2=n-1$, we know that $\dim \pi(\varphi^{-1}(C)) = n-1$. Put $D_M = \pi(\varphi^{-1}(C))$. D_M is a prime divisor on M. Put $D_P = \pi^*(D_M)$. D_P is a prime divisor on P. It follows from $D_M = \cup_{z\in C}\pi(\varphi^{-1}(z))$ that $D_P = \cup_{z\in C}\pi^{-1}(\pi(\varphi^{-1}(z)))$. Hence $\varphi(D_P) = \cup_{z\in C}\varphi(\pi^{-1}(\pi(\varphi^1(z))))$. Thus $D_P \to \varphi(D_P)$ has $(n-2)$-dimensional fibers and $\dim \varphi(D_P) = n+r-2-n-2 = r$. Putting $D_N = \varphi(D_P)$, we know that D_N is a prime divisor on N and $D_P = \varphi^*(D_N)$. This implies that $D_P = \pi^*(D_M) = \pi^*(D_N)$, which is impossible. Therefore if $\dim S = 0$ then M is a Del Pezzo manifold and $\mathcal{E} \cong \mathcal{O}_M(1)^{\oplus r}$. This is the case 1) of the theorem. □

References

[ABW] Andreatta, M, Ballico, E.and Wiśniewski, J. A.: Vector bundles and adjunction. Int. J. Math. **3**, 331–340 (1992)

[CM] Cho, K. and Miyaoka, Y.: A characterization of projective spaces in terms of the minimum degrees of rational curves. Preprint, March 1997.

[CS] Cho, K. and Sato, E.: Smooth projective varieties dominated by smooth quadric hypersurfaces in any characteristic. Math. Z. **217**, 553–565 (1994)

[ES] Ein, L. and Shepherd-Barron, N.: Some Special Cremona transformation. Amer. J. Math. **111**, 783–800 (1989)

[F1] Fujita, T.: On Polarized Manifolds Whose Adjoint Bundles Are Not Semipositive. in Algebraic Geometry, Sendai, 1985. (Adv. Stud. Pure Math. **10**, 167–178) Kinokuniya 1987

[F2] Fujita, T.: Remarks on quasi-polarized varieties. Nagoya Math. J. **115**, 105–123 (1989)

[F3] Fujita, T.: Classification Theory of Polarized Varieties. (London Math. Soc. Lecture Note Ser., 155) Cambridge New York Port Chester Melbourne Sydney: Cambridge University Press 1990

[F4] Fujita, T.: On adjoint bundles of ample vector bundles. (Lecture Notes in Math. **1507**, 105–112) Berlin Heidelberg New York: Springer 1992

[Fu] Fulton, W.: Intersection Theory. (Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd 2) Berlin Heidelberg New York: Springer 1984

[H] Hartshorne, R.: Algebraic Geometry. (Graduate Texts in Math., 52) Berlin Heidelberg New York: Springer 1983

[I] Ionescu, P.: Generalized adjunction and applications. Math. Proc. Camb. Phil. Soc. **99**, 457–472 (1986)

[KMM] Kawamata, Y., Matsuda, K. and Matsuki, K.: Introduction to the Mimimal Model Problem. in Algebraic Geometry Sendai 1985. (Adv. Stud. Pure Math. **10**, 283–360) Kinokuniya 1987

[KO] Kobayashi, S. and Ochiai, T.: Characterizations of complex projective spaces and hyperquadrics. J. Math. Kyoto Univ. **13**, 31–47 (1973)

[Ko] Kollár, J.: Rational Curves on Algebraic Varieties. (Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd 32) Berlin Heidelberg New York: Springer 1996

[L] Lazarsfeld, R.: Some applications of the theory of positive vector bundles. (Lecture Notes in Math. **1092**, 29–61) Berlin Heidelberg New York: Springer 1984

[Ma] Maeda, H.: Ramification divisors for branched coverings of $\mathbf{P}^n$. Math. Ann. **288**, 195–199 (1990)

[M1] Mori, S.: Projective manifolds with ample tangent bundles. Ann. of Math. (2) **110**, 593–606 (1979)

[M2] Mori, S.: Threefolds whose canonical bundles are not numerically effective. Ann. of Math. (2) **116**, 133–176 (1982)

[OSS] Okonek, C., Schneider, M. and Spindler, H.: Vector Bundles on Complex Projective Spaces. (Progr. Math., 3) Boston Basel Stuttgart: Birkhäuser 1987.

[PS] Paranjape K. H. and Srinivas, V.: Self maps of homogeneous spaces. Invent. math. **98**, 425–444 (1989)

[P1] Peternell, T.: A characterization of $\mathbb{P}_n$ by vector bundles. Math. Z. **205**, 487–490 (1990)

[P2] Peternell, T.: Ample vector bundles on Fano manifolds. Int. J. Math. **2**, 311–322 (1991)

[PSW] Peternell, T., Szurek, M. and Wiśniewski, J. A.: Fano manifolds and vector bundles. Math. Ann. **294**, 151–165 (1992)

[W] Wiśniewski, J. A.: On contractions of extremal rays of Fano manifolds. J. reine angew. Math. **417**, 141–157 (1991)

[YZ] Ye, Y. and Zhang, Q.: On ample vector bundles whose adjunction bundles are not numerically effective. Duke Math. **60** No. 3, 671–687 (1990)