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Effective base point freeness on normal surfaces

東工大理工学研究科 川内 毅 (Takeshi Kawachi)

1. Introduction

Let M be a divisor on a normal variety Y. Our main aim is to get criteria which provide the base point freeness of the adjoint linear system $|K_Y + \lceil M \rceil|$ where $\lceil M \rceil$ is the round-up of M. For smooth manifolds, there are many good results in higher dimension. On the other hand, since singularity has much information, we would conclude the same result by a weaker condition. It is true in the two dimensional case, we introduce that worse singularity causes better base point freeness.

2. The invariant

Let Y be a projective normal two dimensional variety over $\mathbb{C}$ (we will call "normal surface" for short), and y be a fixed point on Y. Let $f\colon X \to Y$ be the blowing up at y if y is a smooth point, or the minimal resolution of y if y is singular.

Definition 1. (MRLT) Let Y, y and f be as above. Let B be an effective $\mathbb{Q}$-divisor on Y. (Y, B) is called *minimal resolutional log terminal* (MRLT) at y if the following conditions are satisfied:

(1) the round-down $\lfloor B \rfloor = 0$,

(2) if we write $K_X + f^{-1}B = f^*(K_Y + B) - \Delta_B$ and $\Delta_B = \sum e_i E_i$ then all $e_i < 1$,

where $f^{-1}B$ means the strict transformation of B by f. □

Definition 2. Let Z be the fundamental cycle of y. We define $\delta_{B,y} = -(Z - \Delta_B)^2$. □

We set $\Delta = \Delta_0$, which is the case of $B = 0$; and also $\delta_y = \Delta_{0,y}$. Since B is effective, we have $\Delta_B > \Delta$ and then $0 \leq \delta_{B,y} \leq \delta_y$ (cf. [F]). We have the following bound of δ_y.

Proposition 1. [KM, Theorem 1]

(1) *$\delta_y = 4$ if y is a smooth point, and $\delta_y = 2$ if y is a rational double point.*

(2) *$0 < \delta_y < 2$ if Y is Kawamata log terminal at y.*

Note that if (Y, B) is MSLT at y then Y is Kawamata log terminal at y. Hence $\delta_{B,y}$ is also bounded if (Y, B) is MRLT. Now we will take the above invariant a little bit smaller.

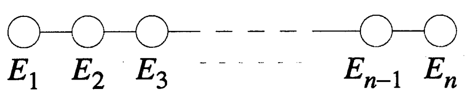
Definition 3.

$$\delta_{\min} = \min\{-(Z - \Delta_B + x)^2 \mid x \text{ is an effective } f\text{-exceptional divisor.}\}$$

$$\delta = \begin{cases} \delta_{\min}, & (Y, B) \text{ is an MRLT at } y \\ 0, & \text{otherwise} \end{cases}$$

$$\delta' = \begin{cases} 1 - \max\{e_1, e_n\}, & y \text{ is of type } A_n, \\ \text{any positive number}, & y \text{ is of type } D_n, \\ 0, & \text{otherwise.} \end{cases}$$ □

Note that if y is of type A_n, the indices are taken in the standard way.



3. The main result

Theorem 2. *Let M be a nef and big $\mathbb{Q}$-Weil divisor on Y, and $B = \lceil M \rceil - M$. Assume that $K_Y + \lceil M \rceil$ is Cartier. If $M^2 > \delta$ and $M \cdot C \geq \delta'$ for any curve C on Y passing through y, then y is not a base point of $|K_Y + \lceil M \rceil|$.*

Note that if y is of type D_n then the assumption $M \cdot C \geq \delta'$ is equivalent to assume $M \cdot C > 0$ by the definition of δ'.

Proof. If y is not an MRLT, the proof is well known. (cf. [KM, (2.1)]). So we assume that y is an MRLT point.

Since the assertion is local, we may assume $Y - \{y\}$ is smooth.

First we take a good effective $\mathbb{Q}$-divisor D such that $\mathbb{Q}$-linearly equivalent to M.

Lemma 3. *There exists an effective $\mathbb{Q}$-divisor D on Y such that $D \equiv M$ (numerically equivalent) and $f^*D > Z - \Delta_B + x$ where x attains the minimum $\delta_{\min}$.*

Proof. Since $M^2 > \delta_{\min}$, we have $(f^*M - (Z - \Delta_B + x))^2 > 0$ and $f^*M \cdot (f^*M - (Z - \Delta_B + x)) > 0$. Hence $f^*M - (Z - \Delta_B + x)$ is big, we can get an effective $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to $f^*M - (Z - \Delta_B + x)$. $\square$

Let D be an $\mathbb{Q}$-divisor satisfying the above lemma. We set $D = \sum d_i C_i$, $B = \sum b_i C_i$, $D_i = f^{-1}C_i$, $f^*D = \sum d_i D_i + \sum d'_j E_j$, $f^*B = \sum b_i D_i + \sum b'_j E_j$. We choose the rational number c as the following.

$$c = \min \left\{ \frac{1-b_i}{d_i}, \frac{1-e_j}{d'_j} \mid d_i > 0, D_i \cap f^{-1}(y) \neq \emptyset \text{ and } f(E_j) = \{y\} \right\}.$$

Since (Y, B) is MRLT and the choice of D, we have $0 < c < 1$.

Let $R = f^*M - cf^*D$. Since $0 < c < 1$ and $D \equiv M$ is nef and big, R is also nef and big. By a simple calculation, we have

$$\lceil R \rceil = f^*(K_Y + \lceil M \rceil) - K_X - \lfloor cf^*D + f^*B + \Delta \rfloor = R + \{cf^*D + f^*B + \Delta\},$$

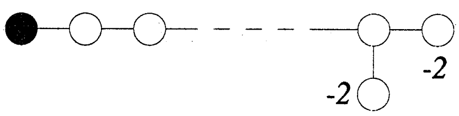
where $\{\cdot\}$ means the fractional part. Hence we have

$$K_X + \lceil R \rceil = f^*(K_Y + M) - \sum \lfloor cd_i + b_i \rfloor D_i + \sum \lfloor cd'_j + e_j \rfloor E_j.$$

We write $\sum \lfloor cd_i + b_i \rfloor D_i = A + N$ where all components of A meet with $f^{-1}(y)$ and N is disjoint from $f^{-1}(y)$. Let $E = \sum \lfloor cd'_j + e_j \rfloor E_j$. By the choice of c, both A and E are reduced or only one of them is zero. Let $A = D_1 + \cdots + D_t$.

Lemma 4. *If $A \neq 0$ then (Y, f_*A) is log canonical at y and the dual graph is one of the followings.*

(1) 

(2) 

(3) 

In the above lemma, we denote prime components of E and f_*A by ○ and ● respectively. Note that only the case (1) is log terminal.

Proof. Because of $f^*(K_Y + f_*A) - K_X - A \leq E$, (Y, f_*A) is log canonical at y. These are classified as in [A] and [K], they are only above 3 cases. □

We divide the proof of the main theorem in two cases according to E.

Case 1: $E \neq 0$.

If $t > 0$ then y is of type A_n or D_n. Note that if y is of type E_n then A must be 0.

Since R is nef and big, each D_i is integral in R and $R \cdot D_i \geq \delta' > 0$, we have the following vanishing due to Kawamata-Viehweg.

$$H^1(X, K_X + \lceil R \rceil + A) = H^1(X, f^*(K_Y + \lceil M \rceil) - N - E) = 0.$$

Hence the morphism

$$H^0(X, f^*(K_Y + \lceil M \rceil) - N) \to H^0(E, (f^*(K_Y + \lceil M \rceil) - N)|_E)$$

is surjective.

Case 2: $E = 0$.

In this case, (Y, f_*A) is log terminal of type A_n at y and $t = 1$. So we let $A = D_1$. Hence the morphism

$$H^0(X, f^*(K_Y + \lceil M \rceil) - N) \to H^0(D_1, (f^*(K_Y + \lceil M \rceil) - N)|_{D_1})$$

is surjective. Since $(f^*(K_Y + \lceil M \rceil) - N)|_{D_1} = K_{D_1} + \lceil R \rceil |_{D_1}$, if $\lceil R \rceil \cdot D_1 > 1$ then there exists a section in $H^0(D_1, K_{D_1} + \lceil R \rceil |_{D_1})$ which does not vanish at $D_1 \cap f^{-1}(y)$ by [H]. Hence it is enough to show $\lceil R \rceil \cdot D_1 > 1$.

Note that $\lceil R \rceil \cdot D_1 \geq R \cdot D_1 + \sum(cd'_j + e_j)E_j \cdot D_1$ and $y \in \operatorname{Supp} f_*D_1$, we have $R \cdot D_1 \geq (1-c)\delta'$. By changing the indices we may assume $e_1 \leq e_n$. Hence $\delta' = 1 - e_n$.

If D_1 meets E_n then the inequalities $f^*D > Z - \Delta_B$ and

$$\lceil R \rceil \cdot D_1 \geq (1-c)(1-e_n) + cd'_n + e_n = 1 + c(d'_n + e_n - 1)$$

imply $\lceil R \rceil \cdot D_1 > 1$.

So we assume that D_1 meets E_1.

Let $A = A(w_1, \ldots, w_n) = (-E_i \cdot E_j)_{ij}$ be the intersection matrix of the exceptional divisors of type A_n. Let $a(w_1, \ldots, w_n) = \det A(w_1, \ldots, w_n)$ be the determinant. We set

$a() = 1$ for convenience. Let L_i be an irreducible curve on Y such that $f^{-1}L_i \cdot E_i = 1$ and $f^{-1}L_i \cdot E_j = 0$ for all $j \neq i$. We set $f^*L_i = f^{-1}L_i + \sum c_{ij}E_j$.

By simple calculation of matrices, we have the following proposition.

Proposition 5. *Let* $\Delta = \sum a_j E_j$.

$$1 - a_i = \frac{a(w_1, \ldots, w_{i-1}) + a(w_{i+1}, \ldots, w_n)}{a(w_1, \ldots, w_n)},$$

$$c_{ij} = \frac{a(w_1, \ldots, w_{i-1})a(w_{j+1}, \ldots, w_n)}{a(w_1, \ldots, w_n)}, \text{ if } i \leq j, \quad c_{ij} = c_{ji}.$$

Let $f^*C_1 = D_1 + \sum c_j E_j$. Let $y_{D,j} = d'_j - d_1 c_j$, the coefficients of E_j arising from D_i's except D_1. We also let $y_{B,j} = b'_j - b_1 c_j$ and $y_j = c y_{D,j} + y_{B,j}$. Since the minimality of c, we have $cd_1 + b_1 = 1$. Hence we have $cd'_1 + b'_1 = c_1 + y_1$. Therefore we have

$$\lceil R \rceil \cdot D_1 \geq (1-c)\delta' + cd'_1 + e_1 = (1-c)(1-e_n) + a_1 + c_1 + y_1.$$

By Proposition 5, we have $a_1 + c_1 = 1/\alpha$, where $\alpha = \det A(w_1, \ldots, w_n)$. Since $E = 0$, we also have $y_1 \leq 1/\alpha$.

Claim 6.

$$(1-c)(1-e_n) > \frac{a(w_1, \ldots, w_{n-1})}{\alpha} \quad \textit{and} \quad y_n \leq a(w_1, \ldots, w_{n-1}) y_1.$$

By this claim, we have $\lceil R \rceil \cdot D_1 > 1 + (a(w_1, \ldots, w_{n-1}) - 1)(1/\alpha - y_1)$. Since $a(w_1, \ldots, w_{n-1}) \geq 1$ and $y_1 < 1/\alpha$, we have $\lceil R \rceil \cdot D_1 > 1$.

Proof of Claim 6. By the choice of D, we have $d'_n > 1 - a_n - b'_n$. Hence

$$(d'_n - 1 + a_n + b'_n)\frac{c}{1-a_n} > 0 = \frac{cd_1 + b_1 - 1}{1 + a(w_1, \ldots, w_{n-1})},$$

since $cd_1 + b_1 = 1$. We set $\alpha' = a(w_1, \ldots, w_{n-1})$ for convenience. Then we have

$$\left((d'_n - 1 + a_n + b'_n)\frac{1}{1-a_n} - \frac{d_1}{1+\alpha'} \right) c > \frac{b_1 - 1}{1+\alpha'}.$$

Since $(1-a_n)\alpha = 1 + \alpha'$ and $d'_n = d_1/\alpha + y_{D,n}$, the left-hand-side equals to

$$\left(\frac{d'_n}{1-a_n} - 1 + \frac{b'_n}{1-a_n} - \frac{d_1}{1+\alpha'}\right) c = \left(\frac{y_{D,n}}{1-a_n} + \frac{b'_n}{1-a_n} - 1\right) c.$$

On the other hand, the right-hand-side equals to

$$\frac{b_1 - 1}{1+\alpha'} = \frac{b_1 + \alpha y_{B,n}}{1+\alpha'} - \frac{1+\alpha y_{B,n}}{1+\alpha'} = \frac{b'_n}{1-a_n} - 1 + \frac{\alpha' - \alpha y_{B,n}}{1+\alpha'}.$$

Thus we have

$$(1-c)\left(1 - \frac{b'_n}{1-a_n}\right) > \frac{\alpha'/\alpha - y_{B,n} - c y_{D,n}}{1-a_n}.$$

The second assertion follows from Proposition 5 and the inequalities $c_{11} > c_{12} > \cdots > c_{1n}$ and $c_{n1} < c_{n2} < \cdots < c_{nn}$. □

References

[A] V. Alexeev, *Classification of log-canonical surface singularities*, Flips and abundance for algebraic threefolds, Astérisque **211** (1992), 47–58.

[F] T. Fujita, *An appendix to Kawachi-Maşek's paper on global generation of adjoint bundles on normal surfaces*, J. Alg. Geom. **7** (1998), 251-252.

[H] R. Hartshone, *Generalized divisor on Gorenstein curves and a theorem of Noether*, J. Math. Kyoto Univ. **26** (1986), 375–386.

[K] Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its applications to degeneration of surfaces*, Ann. Math. **127** (1988), 93–163.

[KM] T. Kawachi and V. Maşek, *Reider-type theorems on normal surfaces*, J. Alg. Geom. **7** (1998), 239–249.