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REMARKS ON NONSMOOTH DYNAMIC VECTOR OPTIMIZATION PROBLEMS

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1. Introduction. This paper deals with vector optimization problems. By convention, throughout this paper we will use the following notations. For $y=(y_1,\cdots,y_n)$, $z=(z_1,\cdots,z_n)\in R^n$, we say that

(i) $y\leq z$, if and only if $y_i\leq z_i$ for any $i\in\{1,\cdots,n\}$,
(ii) $y<z$ if and only if $y_i\leq z_i$ for any $i\in\{1,\cdots,n\}$ with $y\neq z$,
(ii) $y\ll z$ if and only if $y_i<z_i$ for any $i\in\{1,\cdots,n\}$.

Recently, many papers have been devoted to optimality conditions for the vector-valued programming and optimal control problems under some smooth or convex assumptions (see [2], [6], [7], [9], [10]). In [11], we derived the Kuhn-Tucker type proper-efficiency conditions for vector optimal control problems in general case. In this paper we use analogous method to discuss weak-efficiency and efficiency conditions for the following problem,

$$
\begin{aligned}
(P):\qquad & \textit{minimize}: && \mathcal{F}(x,u):=(\mathcal{F}_1(x,u),\cdots,\mathcal{F}_k(x,u))\\
& \textit{subject to}: && \dot{x}(t)=\Phi(t,x(t),u(t))\quad a.e.,\\
& && x(0)\in D,\quad u(t)\in U(t)\quad a.e.,\\
& && \mathcal{G}(x,u):=(\mathcal{G}_1(x,u),\cdots,\mathcal{G}_l(x,u))\leq 0
\end{aligned}
$$

where

$$
\mathcal{F}_i(x,u):=\int_0^1 F_i(t,x(t),u(t))dt+f_i(x(1)) \text{ for } i\in I:=\{1,\cdots,k\}
$$
$$
\mathcal{G}_j(x,u):=\int_0^1 G_j(t,x(t),u(t))dt+g_j(x(1)) \text{ for } i\in J:=\{1,\cdots,l\};
$$

$x(\cdot)\in AC([0,1],R^m)$ and $u(\cdot)\in M([0,1],R^n)$; F_i, $G_j:[0,1]\times R^m\times R^n\to R$, f_i, $g_j:R^m\to R$ for $i\in I$, $j\in J$ and $\Phi:[0,1]\times R^m\times R^n\to R^m$ are given functions; D is a subset of R^m and $U(\cdot):[0,1]\to 2^{R^n}$ is a set-valued function. Here, $AC([0,1],R^m)$ is the space of absolutely continuous functions on [0,1] with value in R^m, $M([0,1],R^n)$ is the space of Lebesgue measurable functions on [0,1] with value in R^n.

For this optimal control problem (P), we say that (x,u) is an admissible process iff $F_i(\cdot,x(\cdot),u(\cdot))$ and $G_j(\cdot,x(\cdot),u(\cdot))$ are integrable for every $i\in I$ and $j\in J$, (x,u) satisfies state equation $\dot{x}(t)=\Phi(t,x(t),u(t))$ a.e. with $x(0)\in D$, $u(t)\in U(t)$ a.e. and $\mathcal{G}(x,u)\leq 0$. The first component of a process (x,u) is called a state and the second is called a control. We denote by Ω the set of all admissible processes of (P). The optimal solutions for (P) are defined in the following meaning.

Definition 1: $(x_*,u_*)\in\Omega$ is said to be
(i) a weakly-efficient solution for (P) if there exists no $(x,u)\in\Omega$ such that

$$
\mathcal{F}(x,u)\ll\mathcal{F}(x_*,u_*);
$$

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(ii) an efficient solution for (P) if there exists no $(x,u) \in \Omega$ such that

$$\mathcal{F}(x,u) < \mathcal{F}(x_*,u_*).$$

Definition 2: $(x_*,u_*) \in \Omega$ is called a local weakly-efficient solution of type (I) (resp. (II)) for (P) if and only if there is no $(x,u) \in \Omega$ with $\|x - x_*\|_{L^\infty} \leq \epsilon$ for some $\epsilon > 0$ (resp. with $x(t) \in x_*(t) + \epsilon B_m$ *and* $u(t) \in u_*(t) + \epsilon B_n$ for some $\epsilon > 0$, where B^m and B^n are unit closed balls of R^m and R^n, respectively) such that $\mathcal{F}(x,u) \ll \mathcal{F}(x_*,u_*)$.

The main method to obtain optimality conditions for multiobjective optimization problems is based on a replacement of the multiobjective problems by single-objective (scalar) optimization problems. The following results give the relationship between (P) and scalar optimization problems.

Lemma 1: *$(x_*,u_*) \in \Omega$ is a weakly-efficient (local weakly-efficient) solution of (P) if and only if (x_*,u_*) is an optimal (local optimal) solution of the following scalar optimization problem,*

$$\begin{aligned} min: &\quad \max_{i\in I} \left(\mathcal{F}_i(x,u) - \mathcal{F}_i(x_*,u_*)\right) \\ s.\,t.: &\quad (x,u) \in \Omega. \end{aligned}$$

Proof. By the definitions, it is easy to see that (x_*,u_*) is a weakly efficient of (P) if and only if there is no $(x,u) \in \Omega$ satisfying

$$\max_{i\in I} \left(\mathcal{F}_i(x,u) - \mathcal{F}_i(x_*,u_*)\right) < 0.$$

Thus, this lemma hold. □

Lemma 2: *([6, Lemma 3.1]) $(x_*,u_*) \in \Omega$ is an efficient solution of (P) if and only if (x_*,u_*) is an optimal solution of the following scalar optimal control problem (P_i) for each $i \in I$.*

$$(P_i): \qquad \begin{aligned} minimize: &\quad \mathcal{F}_i(x,u) \\ subject\ to: &\quad (x,u) \in \Omega \\ &\quad \mathcal{F}_j(x,u) - \mathcal{F}(x_*,u_*) \leq 0 \quad j \in I/\{i\}. \end{aligned}$$

Lemma 3: *Suppose that Ω is convex set and $\mathcal{F}_i(x,u)$, $i = 1,\cdots,k$ are convex functions. Then, $(x_*,u_*) \in \Omega$ is a weakly-efficient solution of (P) if and only if (x_*,u_*) is an optimal solution of (P_i) stated in Lemma 2 for some $i \in I$.*

Proof. Assume that (x_*,u_*) is a weakly-efficient solution of (P). If for every (P_i), (x_*,u_*) is not an optimal solution, i.e. for any $i \in I$ there exists $(x_i,u_i) \in \Omega$ with

$$\begin{aligned} \mathcal{F}_i(x_i,u_i) &< \mathcal{F}_i(x_*,u_*) \\ \mathcal{F}_j(x_i,u_i) - \mathcal{F}_j(x_*,u_*) &\leq 0 \quad \text{for } j \in I/\{i\}. \end{aligned}$$

Putting $(x_0,u_0) := \frac{1}{k}\sum_{i\in I}(x_i,u_i)$, we see that $(x_0,u_0) \in \Omega$. Notice that $\mathcal{F}_i(x,u)$ is convex, we have

$$\mathcal{F}_i(x_0,u_0) \leq \sum_{j\in I} \frac{1}{k}\mathcal{F}_i(x_j,u_j) < \mathcal{F}_i(x_*,u_*).$$

Thus, $\mathcal{F}(x_0, u_0) \ll \mathcal{F}(x_*, u_*)$, which contradicts that (x_*, u_*) is a weakly-efficient solution of (P).

Conversely, let (x_*, u_*) be an optimal solution of (P_i) for some $i \in I$. If (x_*, u_*) is not a weakly-efficient solution of (P), then there is $(x, u) \in \Omega$ satisfying

$$\mathcal{F}_i(x,u) < \mathcal{F}_i(x_*, u_*) \text{ and } \mathcal{F}_j(x,u) - \mathcal{F}_j(x_*, u_*) < 0 \text{ for } j \in I/\{i\},$$

which contradicts that (x_*, u_*) is an optimal solution of (P_i). □

2. Optimality conditions. For simplicity, throughout this section we omit the variable t when it does not cause confusion, and abbreviate the arguments $(t, x_*(t), u_*(t))$ to $[t]$, for instance, we write $G_i[t] = G_i(t, x_*(t), u_*(t))$. In Theorem 1 and 2 below, the notations ∂ denote the Clarke generalized gradients and N_D, $N_{U(t)}$ indicate the Clarke normal cones, while in Theorem 3 and 4, these notations stand for the subdifferentials and the normal cones in the sense of convex analysis, respectively.

The following assumptions are required. The pair (x_*, u_*) in (A2) and (A3) will be assumed to be a local weakly efficient solution of type (I) for (P).

(A1): D is closed, $U(\cdot)$ is a nonempty compact set-valued map and the graph GrU is $\mathcal{L} \times \mathcal{B}$ measurable.

(A2): $f_i(\cdot)$, $g_j(\cdot)$ $(i \in I,\ j \in J)$ are Lipschitz continuous in a neighborhood of $x_*(1) \in R^m$.

(A3): For every admissible control $u(\cdot)$, there are real-valued measurable function $\epsilon(t) > 0$ and $h_i(t) \geq 0$, $i = 0, \cdots, k+l$, such that

$$\begin{aligned} |F_i(t,x,u(t)) - F_i(t,x',u(t))| &\leq h_i(t)\,|x - x'| \text{ for } i \in I \\ |G_j(t,x,u(t)) - G_j(t,x',u(t))| &\leq h_{k+j}(t)\,|x - x'| \text{ for } j \in J \\ |\Phi(t,x,u(t)) - \Phi(t,x',u(t))| &\leq h_0(t)\,|x - x'| \end{aligned}$$

whenever $|x - x_*(t)| \leq \epsilon(t), |x' - x_*(t)| \leq \epsilon(t)$, $t \in [0,1]$; for $u(\cdot) = u_*(\cdot)$ these functions can be chosen in such a way that $\epsilon(t) = \epsilon > 0$ and $h_i(t)$ $(i = 0, \cdots, k+l)$ are integrable.

(A4): For any $u(\cdot) \in \mathcal{U} := \{u(\cdot) \in M([0,1], R^n) : u(t) \in U(t)\ a.e.\}$, $F_i(t,x,u(t))$ for $i \in I$, $G_j(t,x,u(t))$ for $j \in J$ and $\Phi(t,x,u(t))$ are measurable.

Theorem 1. *Let assumptions (A1)-(A4) be satisfied. Suppose that $(x_*; u_*)$ is a local weakly efficient solution of type (I) for (P). Then, there exist $\lambda = (\lambda_1, \cdots, \lambda_{k+l}) > 0$ and an absolutely continuous function $p(\cdot) : [0,1] \to R^n$, such that*

$$(1) \qquad -\dot{p}(t) \in \partial_x H(t, x_*(t), p(t), u_*(t), \lambda) \qquad a.e.$$

$$(2) \qquad p(0) \in N_D(x_*(0)),\ -p(1) \in \sum_{i \in I} \lambda_i \partial f_i(x_*(1)) + \sum_{j \in J} \lambda_{k+j} \partial g_j(x_*(1))$$

$$(3) \qquad H(t, x_*(t), p(t), u_*(t), \lambda) = \max_{v \in U(t)} H(t, x_*(t), p(t), v, \lambda) \qquad a.e.$$

$$(4) \qquad \lambda_{k+j} \left(\int_0^1 G_j[t]dt + g_j(x_*(1)) \right) = 0 \qquad for\ j \in J$$

where $H(t,x,p,u,\lambda) := \langle p, \Phi(t,x,u)\rangle - \sum_{i\in I} \lambda_i F_i(t,x,u) - \sum_{j\in J} \lambda_{k+j} G_j(t,x,u)$

Proof. We consider the following problem,

$$(P') \qquad min: \ \Gamma_0(y) := \max_{i\in I} \{y_i(1) + f_i(x(1)) - \mathcal{F}_i(x_*, u_*)\}$$

$$s.\ t.: \ L_0(y,u) := x(t) - x(0) - \int_0^t \Phi(t, x(t), u(t))dt = 0$$

$$L_i(y,u) := y_i(t) - \int_0^t F_i(t, x(t), u(t))dt = 0 \qquad i \in I$$

$$L_{k+j}(y,u) := y_{k+j}(t) - \int_0^t G_j(t, x(t), u(t))dt = 0 \qquad j \in J$$

$$\Gamma_j(y) := y_{k+j}(1) + g_j(x(1)) \le 0 \qquad j \in J$$

$$y(\cdot) \in \mathcal{S},\ u(\cdot) \in \mathcal{U},$$

where $y(\cdot) := (x(\cdot), y_1(\cdot), \cdots, y_{k+l}(\cdot)) \in C([0,1], R^{m+k+l})$ is the state and $u(\cdot) \in M([0,1], R^n)$ is the control, $\mathcal{S} := \{x \in C([0,1], R^m) : x(0) \in D\} \times C([0,1], R^{2k})$.

Let $y_{i_*}(t) := \int_0^t F_i[t]dt$ for $i \in I$ and $y_{(k+j)_*}(t) := \int_0^t G_j[t]dt$ for $j \in J$. Thus, by Lemma 1, we see that $y_* := (x_*, y_{i_*}, \cdots, y_{(k+l)_*})$ corresponding u_* minimizes $\Gamma_0(y)$ over all admissible processes (y,u) for (P') with x being sufficiently close to x_* in the norm of L^∞.

By [4, Theorem 2], we see that there exist Lagrange multipliers $\delta := (\delta_0, \cdots, \delta_l) \ge 0$, $x^* \in C^*([0,1], R^m)$, and $y_i^* \in C^*([0,1], R)$ $i = 1, \cdots, k+l$ not all zero such that

$$(5) \qquad 0 \in \partial_y \mathcal{L}(y_*, y^*, u_*, \kappa) + N_{\mathcal{S}}(y_*)$$

$$(6) \qquad \mathcal{L}(y_*, y^*, u_*, \kappa) = \min_{u\in\mathcal{U}} \mathcal{L}(y_*, y^*, u, \kappa)$$

$$(7) \qquad \delta_j \Gamma_j(y_*) = 0 \qquad j \in J$$

where $\mathcal{L}(y, y^*, u, \kappa) := \sum_{i=0}^l \delta_i \Gamma_i(y) + \langle x^*, L_0(y,u)\rangle + \sum_{i=1}^{k+l} \langle y_i^*, L_i(y,u)\rangle$.

According to the formulas of the Clarke gradients (see [3]), we see that

(i) For any $\xi \in \partial\Gamma_0(y_*)$, there are $\bar\lambda_i \ge 0$, $\nu_i \in \partial f_i(x_*(1))$ for $i \in I$ with $\sum_{i\in I} \bar\lambda_i = 1$ such that for any $y \in C([0,1], R^{n+2k})$

$$\langle \xi, y\rangle = \sum_{i\in I} \bar\lambda_i y_i(1) + \sum_{i\in I} \bar\lambda_i \langle \nu_i, x(1)\rangle.$$

for every $\xi \in \sum_{i=1}^l \delta_i \Gamma_i(y_*)$, there exist $\nu_{k+j} \in \partial g_j(x_*(1))$ for $j \in J$ such that for any $y \in C([0,1], R^{n+2k})$

$$\langle \xi, y\rangle = \sum_{j\in J} \delta_j y_i(1) + \sum_{j\in J} \delta_j \langle \nu_{k+j}, x(1)\rangle.$$

Analyzing as in [4], we have the following.

(ii) The above multipliers x^*, y_1^*, $\cdots$, y_{2k}^* can be expressed by pairs of the nonnegative Radon measure and Radon-integrable functions (μ_i, ξ_i), $i = 0, \cdots, 2k$. For every $\xi \in \partial_y \left(\langle x^*, L_0(x_*, u_*)\rangle + \sum_{i=1}^{k+l} \langle y_i^*, L_i(x_*, u_*)\rangle\right)$, there is a Lebesgue measurable function $\eta(\cdot)$ with

$$(8) \qquad \begin{aligned} \eta(t) \in \ & \partial_x \Bigg(\left\langle \int_t^1 \xi_0 d\mu_0, \Phi[t] \right\rangle + \sum_{i\in I} \left\langle \int_t^1 \xi_i d\mu_i, F_i(t, x_*(t), u_*(t)) \right\rangle \\ & + \sum_{j\in J} \left\langle \int_t^1 \xi_{k+i} d\mu_{k+i}, G_i(t, x_*(t), u_*(t)) \right\rangle \Bigg) \ a.e., \end{aligned}$$

such that for any $y \in C([0,1], R^{n+2k})$,

$$\langle \xi, y\rangle = \int_0^1 \langle x(t) - x(0), \xi_0\rangle \, d\mu_0 + \sum_{i=1}^{k+l} \int_0^1 \langle y_i, \xi_i\rangle \, d\mu_i - \int_0^1 \langle \eta, x\rangle \, dt.$$

(iii) For each $\xi \in N_S(y_*)$, there is $\alpha \in N_D(x_*(0))$, such that

$$\langle \xi, y\rangle := \langle \alpha, x(0)\rangle \qquad \text{for any } y \in C([0,1], R^{n+k}).$$

Combining (i), (ii) and (iii), from (5) we see that there are $\bar{\lambda}_i$, $i = 1, \cdots, l$; ν_i, $i = 1, \cdots, k+l$; (μ_i, ξ_i), $i = 0, \cdots, k+l$, η and α stated above such that

$$\begin{aligned} 0 = &\sum_{i \in I} \delta_0 \bar{\lambda}_i y_i(1) + \sum_{j \in J} \delta_j y_{k+j}(1) + \sum_{i \in I} \delta_0 \bar{\lambda}_i \langle \nu_i, x(1)\rangle + \sum_{j \in J} \delta_j \langle \nu_{k+j}, x(1)\rangle + \\ &\sum_{i=1}^{k+l} \int_0^1 \langle y_i, \xi_i\rangle \, d\mu_i + \int_0^1 \langle x(t) - x(0), \xi_0\rangle \, d\mu_0 - \int_0^1 \langle \eta, x\rangle \, dt + \langle \alpha, x(0)\rangle \end{aligned}$$

for any $x \in C([0,1], R^n)$ and $y_i \in C([0,1], R)$, $i = 1, \cdots, k+l$.

Setting $\lambda_i = \delta_0 \bar{\lambda}_i$ for $i \in I$, $\lambda_{k+j} := \delta_j$ for $j \in J$ and $p(t) := \int_t^1 \xi_0 d\mu_0$, from the above equation, we see that

$$\begin{aligned} &\lambda_i y_i(1) + \int_0^1 \left\langle \int_t^1 \xi_i d\mu_i, \dot{y}_i \right\rangle dt = 0 \qquad (\forall\ y_i \in AC \ with \ y_i(0) = 0,\ i \in I \cup J), \\ &\langle \alpha, x(0)\rangle + \sum_{i=1}^{k+l} \lambda_i \langle \nu_i, x(1)\rangle + \int_0^1 \left\langle p(t) - \int_t^1 \eta d\tau, \dot{x} \right\rangle dt = 0 \qquad (\forall\ x \in AC). \end{aligned}$$

These yield that (refer to the proof of [4, Theorem 3])

$$\begin{aligned} &\int_t^1 \xi_i d\mu_i = -\lambda_i, \quad i = 1, \cdots, k+l \\ &\dot{p}(t) = -\eta(t) \text{ a.e.}, \ p(0) = \alpha, \ p(1) = -\sum_{i=1}^{k+l} \lambda_i \nu_i. \end{aligned} \tag{9}$$

Therefore, (9), (8) and (7) imply (1), (2) and (4)

Here, if $\delta = 0$, then $(\lambda_1, \cdots, \lambda_{k+l}) = (y_1^*, \cdots, y_{k+l}^*) = 0$. From (1) and (2), we can get $p(\cdot) = 0$. Thus, $y^* = 0$ which contradicts that δ and y^* are not all zero. Hence, we have $(\lambda_1, \cdots, \lambda_{k+l}) > 0$.

On other hand, By (6) and (9), we see that

$$\int_0^1 H(t, x_*, p, u_*, \lambda) dt = \max_{u \in \mathcal{U}} \int_0^1 H(t, x_*, p, u, \lambda) dt.$$

Discussing as in the proof of [4, Theorem 3], we can obtain (3). □

According to the results of [8], we see that the above necessary conditions (1)-(4) (Maximum Principle-type) may fail to be sufficient conditions for weak-efficient solutions of (P) even in the "convex" case given below. Next, we give another type necessary weakly-efficiency conditions for (P), which is an extension of [8]. In the "convex" case, the latter necessary conditions are necessary-sufficient for weakly-efficiency under Slater constraint qualifications. Moreover , these conditions are also necessary-sufficient for efficient solutions of (P) under further assumptions.

We impose the following assumption, in which the process (x_*, u_*) will be assumed to be a weakly-efficient solution of type (II) for (P).

(A5): $F_i(\cdot, x, u)$, $G_i(\cdot, x, u)$, $i = 1, \cdots, k$, $\Phi(\cdot, x, u)$ are Lebesgue measurable, and there exist $\epsilon > 0$ and $h_i(t) \in L^1([0,1], R)$, $i = 0, \cdots, k+l$, such that

$$|F_i(t,x,u) - F_i(t,x',u')| \le h_i(t)\,(|x-x'| + |u-u'|) \quad \text{for } i \in I$$
$$|G_j(t,x,u) - G_j(t,x',u')| \le h_{k+j}(t)\,(|x-x'| + |u-u'|) \quad \text{for } j \in J$$
$$|\Phi(t,x,u(t)) - \Phi(t,x',u')| \le h_0(t)\,(|x-x'| + |u-u'|)$$

whenever $x, x' \in x_*(t) + \epsilon B_n$, $u, u' \in u_*(t) + \epsilon B_m$ *a.e.*.

Theorem 2: *Assume that (A1), (A2) and (A5) be satisfied. Let (x_*, u_*) be a local weakly efficient solution of type (II) for (P). Then there exist $\lambda = (\lambda_1 \cdots, \lambda_{k+l}) > 0$, an absolutely continuous function $p(\cdot) : [0,1] \to R^n$ and an integrable function $\zeta(\cdot) : [0,1] \to R^m$ such that*

$$(10) \qquad (-\dot{p}(t), \zeta(t)) \in \partial_{(x,u)} H(t, x_*(t), p(t), u_*(t), \lambda) \qquad a.e.$$

$$(11) \qquad p(0) \in N_D(x_*(0)), \quad -p(1) \in \sum_{i\in I} \lambda_i \partial f_i(x_*(1)) + \sum_{j \in J} \lambda_{k+j} \partial g_j(x_*(1))$$

$$(12) \qquad \zeta(t) \in N_{U(t)}(u_*(t)) \qquad a.e.$$

$$(13) \qquad \lambda_{k+j} \left(\int_0^1 G_j[t]dt + g_j(x_*(1)) \right) = 0 \qquad for\ j \in J$$

where $H(t,x,p,u,\lambda)$ is defined in Theorem 1.

Proof. It is obvious that the scalar optimization problem in Lemma 1 can be rewritten as follows

$$
\begin{aligned}
(P^\dagger): \quad & minimize: && \Gamma(y(1)) := \max_{i \in I, j \in J} \{y_i(1) + f_i(x(1)) - \mathcal{F}_i(x_*, u_*), \\
& && \qquad y_{k+j}(1) + g_j(x(1))\} \\
& subject\ to: && \dot{x}(t) = \Phi(t, x(t), u(t)) \quad a.e. \\
& && \dot{y}_i(t) = F_i(t, x(t), u(t)) \quad a.e. \quad i \in I \\
& && \dot{y}_{k+i}(t) = G_i(t, x(t), u(t)) \quad a.e. \quad i \in I \\
& && x(0) \in C,\ y_i(0) = 0 \quad i = 1, \cdots, 2k, \\
& && u(t) \in U(t) \quad a.e.
\end{aligned}
$$

where $y := (x, y_1, \cdots, y_{2k}) \in AC([0,1], R^{m+2k})$ is the state and $u \in M([0,1], R^n)$ is the control.

Define y_* as in proof of Theorem 1. By Lemma 1, we see that (y_*, u_*) is a minimizer over all admissible process for $(P^\dagger)$ with $x(t) \in x_*(t) + \epsilon B_n$, $u(t) \in u_*(t) + \epsilon B_m$ *a.e.* for some $\epsilon > 0$. Thus, by [8, Proposition 6.1], there exist an absolutely continuous function $\bar{p} = (p, p_1, \cdots, p_{k+l})$ and an integrable function ζ such that (12) and the following hold

$$(14) \qquad (-\dot{\bar{p}}(t), \dot{y}(t), \zeta(t)) \in \partial_{(y,\bar{p},u)} \bar{H}(t, y_*(t), \bar{p}(t), u_*(t)) \qquad a.e.$$

$$(15) \qquad \bar{p}(0) \in N_{C \times \underbrace{\{0\} \times \cdots \times \{0\}}_{2k}}(y_*(0))$$

(16) $$-\bar{p}(1) \in \partial\Gamma(y_*(1))$$

where $\bar{H}(t,y,\bar{p},u) := \langle p, \Phi(t,x,u)\rangle + \sum_{i\in I} \langle p_i, F_i(t,x,u)\rangle + \sum_{i\in I} \langle p_{k+i}, G_i(t,x,u)\rangle$.

First, let us discuss inclusion (16). Notice that for every $i \in I$ and $j \in J$,

$$\Gamma_i(y(1)) := y_i(1) + f_i(x(1)) - \mathcal{F}_i(x_*, u_*),$$
$$\Gamma_j(y(1)) := y_{k+j}(1) + g_j(x(1))$$

only contains the arguments x and y_i, and $\Gamma_i(y_*(1)) = \Gamma(y_*(1)) = 0$. So by the formulas of the Clarke gradients, there are $\gamma_i \in \partial_x f_i(x_*(1))$ for $i \in I$, $\gamma_{k+j} \in \partial_x g_j(x_*(1))$ for $j \in J$ and $(\lambda_1, \cdots, \lambda_{k+l}) > 0$ such that

(17) $$-p(1) = \sum_{i\in I} \lambda_i \gamma_i, \quad -p_i(1) = \lambda_i, \; i = 1, \cdots, k+l.$$

where we can set $\lambda_j = 0$ for $j \in \{j \in J : \mathcal{G}_i(x_*, u_*) < 0\}$.

Thus, (11) and (13) follow from (15) and (17).

On the other hand, since $\bar{H}$ does not contain the arguments y_i, $i = 1, \cdots, k+l$, (14) implies that $\dot{p}_i(\cdot) = 0$, $i = 1, \cdots, k+l$. Thus, $p_i(\cdot) = -\lambda_i$, $i = 1, \cdots, k+l$ and

$$(-\dot{p}(t), \dot{x}(t), \zeta(t)) \in \partial_{(x,\bar{p},u)} \left(\langle p(t), \Phi[t]\rangle - \sum_{i\in I} \lambda_i F_i[t] - \sum_{i\in I} \lambda_{k+i} G_i[t] \right) \qquad a.e.$$

From this inclusion, by the definition of the Clarke generalized gradients, we can easily deduce (10).

Next, we proceed to the optimality conditions for the following problem.

$$(P^*): \quad \begin{array}{ll} min: & \mathcal{F}(x,u) \\ s.\,t.: & \dot{x}(t) = A(t)x(t) + B(t)u(t)) + b(t) \qquad a.e. \\ & x(0) \in D, \; u(t) \in U(t) \quad a.e. \\ & \mathcal{G}(x,u) \le 0 \end{array}$$

where $x(\cdot) \in AC([0,1], R^m)$ and $u(\cdot) \in L^1([0,1], R^n)$, $\mathcal{F}$ and $\mathcal{G}$ are given above, $A(\cdot) : [0,1] \to R^{n\times n}$, $B(\cdot) : [0,1] \to R^{n\times m}$ are integrable, $b(\cdot) : [0,1] \to R^n$ is measurable.

We impose the following hypotheses:

(H1): For every $i \in I$, $F_i(\cdot, x(\cdot), u(\cdot))$ and $G_i(\cdot, x(\cdot), u(\cdot))$ are integrable for any $(x,u) \in AC \times L^1$.

(H2): $F_i(t,\cdot,\cdot)$ for $i \in I$ and $G_i(t,\cdot,\cdot)$ for $j \in J$ are convex lower semicontinuous, and there are $v_i(t) \in L^\infty([0,1], R^{m+n})$ and $w_i(t) \in L^1([0,1], R)$, $i = 1, \cdots, k+l$ such that for any $x \in R^m$, $u \in R^n$, $F_i(t,x,u) \ge \langle v_i(t), (x,u)\rangle + w_i(t)$ for $i \in I$ and $G_j(t,x,u) \ge \langle v_j(t), (x,u)\rangle + w_j(t)$ for $j \in J$ a.e..

(H3): The functions $f_i(\cdot)$ for $i \in I$ and $g_i(\cdot)$ for $j \in J$ are proper convex and lower semicontinuous.

(H4): The set C is convex, $U(t)$ is convex a.e., and there is $\rho(t) \in L^1$ such that $|u| \le \rho(t)$ for any $u \in U(t)$ a.e..

(H5): There exists an admissible process (x_i, u_i) for (P^*), such that $\mathcal{G}_j(x_i, u_i) - \mathcal{G}_j(x_*, u_*) < 0$ for any $j \in \{j \in J : \mathcal{G}_j(x_*, u_*) = 0\}$.

Here, (x_*, u_*) will be assumed to be an admissible process for (P^*).

Theorem 3: *Assume that (H1)-(H5) and (A1) be satisfied. An admissible process* (x_*, u_*) *is a weakly-efficient solution for* (P^*) *if and only if there exist* $\lambda = (\lambda_1 \cdots, \lambda_{k+l}) \geq 0$ with $(\lambda_1 \cdots, \lambda_k) > 0$, $p(\cdot) \in AC([0,1], R^m)$, *and* $\zeta(\cdot) \in L^\infty([0,1], R^n)$ *such that*

$$(18)\quad (\dot{p}(t) + p(t)A(t), p(t)B(t) - \zeta(t)) \in \partial_{(x,u)} \left(\sum_{i \in I} \lambda_i F_i[t] + \sum_{j \in J} \lambda_{k+j} G_j[t] \right) \quad a.e.$$

$$(19)\quad p(0) \in N_C(x_*(1)), \ -p(1) \in \sum_{i \in I} \lambda_i \partial f_i(x_*(1) + \sum_{j \in J} \lambda_{k+j} \partial g_i(x_*(1))$$

$$(20)\quad \zeta(t) \in N_{U(t)}(u_*(t)) \qquad a.e,$$

$$(21)\quad \lambda_{k+j} \left(\int_0^1 G_j[t]dt + g_j(x_*(1)) \right) = 0 \qquad for\ j \in J.$$

Proof. [Necessity] By Lemma 3, we know that there exists $i \in I$ such that (x_*, u_*) is an optimal solution for the following scalar optimal control problem,

$$\begin{aligned}
minimize: \ & \mathcal{F}_i(x,u) \\
subject\ to: \ & \dot{x}(t) - A(t)x(t) - B(t)u(t) - b(t) = 0 \qquad a.e. \\
& \mathcal{G}_j(x,u) \leq 0 \qquad j \in J \\
& \mathcal{F}_j(x,u) \leq 0 \qquad j \in I/\{i\} \\
& x \in \{x \in AC([0,1], R^m) : \ x(0) \in D\} \\
& u \in \mathcal{C} := \{u \in L^1([0,1], R^n) : \ u(t) \in U(t) \ a.e.\}.
\end{aligned}$$

This means that $(x_*, u_*, x_*(0), x_*(1))$ is a minimizer for the following scalar optimization problem.

$$\begin{aligned}
minimize: \ & \Lambda_i(z,u,\alpha,\beta) := \int_0^1 F_i(t,z,u)dt + f_i(\beta) \\
subject\ to: \ & \Gamma_1(z,u,\alpha,\beta) := z(t) - \alpha - \int_0^t (Az + Bu + b)d\tau = 0 \qquad a.e. \\
& \Gamma_2(z,u,\alpha,\beta) := \beta - \alpha - \int_0^1 (Az + Bu + b)d\tau = 0 \\
& \Lambda_j(z,u,\alpha,\beta) := \int_0^1 F_j(t,z,u)dt + f_j(\beta) - \mathcal{F}_j(x_*,u_*) \leq 0 \text{ for } j \in I/\{i\} \\
& \Lambda_j(z,u,\alpha,\beta) := \int_0^1 G_j(t,z,u)dt + g_j(\beta) \leq 0 \text{ for } j \in J \\
& (z,u,\alpha,\beta) \in \mathcal{M} := L^1([0,1], R^m) \times \mathcal{C} \times D \times R^m,
\end{aligned}$$

where $(z,u,\alpha,\beta) \in L^1([0,1], R^m) \times L^1([0,1], R^n) \times R^m \times R^m$

Put $\theta := (z,u,\alpha,\beta)$ and $\theta_* := (x_*, u_*, x_*(0), x_*(1))$. It is obvious that $\Lambda_i(\theta)$ is convex, $\Gamma_1(\theta)$ and $\Gamma_2(\theta)$ are affine mappings. By [5, Theorem 5 p74], there exist $\lambda := (\lambda_1, \cdots, \lambda_{k+l}) \geq 0$, $q(\cdot) \in (L^1)^*$ and $\sigma \in R^m$ not all zero, such that

$$(22)\quad \begin{aligned}
& \sum_{j=1}^{k+l} \lambda_j \Lambda_j(\theta_*) + \int_0^1 \langle q, \Gamma_1(\theta_*) \rangle \, dt + \langle \sigma, \Gamma_2(\theta_*) \rangle \\
&= \min_{\theta \in \mathcal{M}} \left(\sum_{j=1}^{k+l} \lambda_j \Lambda_j(\theta) + \int_0^1 \langle q, \Gamma_1(\theta) \rangle \, dt + \langle \sigma, \Gamma_2(\theta) \rangle \right),
\end{aligned}$$

$$\lambda_{k+j}\Lambda_j(\theta_*) = \lambda_{k+j}\left(\int_0^1 G_j[t]dt + g_j(x_*(1))\right) = 0 \quad \text{for } j \in J$$

Let $I_{\mathcal{M}}(\theta)$ denote the indicator function of $\mathcal{M}$. Notice that the functions $I_{\mathcal{M}}$, Λ_j $(j \in I)$, $\int_0^1 \langle p, \Gamma_1 \rangle \, dt$, $\langle \sigma, \Gamma_2 \rangle$ are proper convex and lower semicontinuous, from (22) we see that

$$(23) \qquad 0 \in \sum_{j=1}^{k+l} \lambda_j \partial \Lambda_j(\theta_*) + \partial \int_0^1 \langle q, \Gamma_1(\theta_*) \rangle \, dt + \partial \langle \sigma, \Gamma_2(\theta_*) \rangle + N_{\mathcal{M}}(\theta_*)$$

(refer to Section 1 of Chapter 1 in [1]).

Now, we analyze (23). By the formulas of subdifferential (see [1], [5]), we have the following conclusions.

For every $\xi \in \sum_{j=1}^{k+l} \lambda_j \partial \Lambda_j(\theta_*)$, there are $(\mu_j, \eta_j) \in L^\infty$ with $(\mu_j(t), \eta_j(t)) \in \partial_{(x,u)} F_i[t]$ and $\nu_j \in \partial f_j(x_*(1))$ for $j \in I$, $(\mu_{k+j}, \eta_{k+j}) \in L^\infty$ with $(\mu_{k+j}(t), \eta_{k+j}(t)) \in \partial_{(x,u)} G_i[t]$ and $\nu_{k+j} \in \partial g_j(x_*(1))$ for $j \in J$ such that for any $\theta \in L^1 \times L^1 \times R^m \times R^m$

$$\langle \xi, \theta \rangle = \sum_{j=1}^{k+l} \lambda_j \left(\int_0^1 (\langle \mu_j, x \rangle + \langle \eta_j, u \rangle) \, dt + \langle \nu_j, \beta \rangle \right).$$

Corresponding to any $\xi \in N_{\mathcal{M}}(\theta_*)$, there are $\gamma \in N_D(x_*(0))$, and $\zeta(\cdot) \in N_C(u_*(\cdot))$ such that for any $\theta \in L^1 \times L^1 \times R^m \times R^m$, one has

$$\langle \xi, \theta \rangle = \langle \gamma, \alpha \rangle + \int_0^1 \langle \zeta, u \rangle \, dt.$$

Notice that $\int_0^1 \langle q, \Gamma_1(\theta) \rangle \, dt$ is affine on θ, thus $\partial \int_0^1 \langle q, \Gamma_1(\theta_*) \rangle \, dt = \{\xi\}$ with

$$\langle \xi, \theta \rangle = \int_0^1 \left\langle q, z - \alpha - \int_0^t (Az - Bu) d\tau \right\rangle dt$$

for any $\theta \in L^1 \times L^1 \times R^m \times R^m$.

Similarly, $\partial \langle \sigma, \Gamma_2(\theta_*) \rangle = \{\xi\}$ with

$$\langle \xi, \theta \rangle = \left\langle \sigma, \beta - \alpha - \int_0^1 (Az - Bu) dt \right\rangle$$

for any $\theta \in L^1 \times L^1 \times R^m \times R^m$.

Then, (23) implies that there are (μ_j, η_j), ν_j, $j = 1, \cdots, k+l$, γ and ζ stated above such that

$$(24) \quad \begin{aligned} &\sum_{j=1}^{k+l} \lambda_j \int_0^1 (\langle \mu_j, z \rangle + \langle \eta_j, u \rangle) \, dt + \sum_{j=1}^{k+l} \lambda_j \langle \nu_j, \beta \rangle + \int_0^1 \left\langle q, z - \int_0^t (Az + Bu) d\tau \right\rangle dt \\ &- \left\langle \int_0^1 q dt, \alpha \right\rangle + \left\langle \sigma, \beta - \alpha - \int_0^1 (Az + Bu) dt \right\rangle + \langle \gamma, \alpha \rangle + \int_0^1 \langle \zeta, u \rangle \, dt = 0 \end{aligned}$$

for any $(z, u, \alpha, \beta) \in L^1 \times L^1 \times R^m \times R^m$.

Put $p(t) := \int_t^1 q(\tau) d\tau + \sigma$. From (24) we see that

$$\begin{aligned} &\int_0^1 \left\langle \sum_{i=1}^{k+l} \lambda_i \mu_i, z \right\rangle dt - \int_0^1 \langle \dot{p} + pA, z \rangle \, dt + \int_0^1 \left\langle \sum_{i=1}^{k+l} \lambda_i \eta_i, u \right\rangle dt - \int_0^1 \langle pB - \zeta, u \rangle \, dt \\ &+ \left\langle \sum_{i=1}^{k+l} \lambda_i \nu_i, \beta \right\rangle + \langle \sigma, \beta \rangle - \left\langle \int_0^1 q dt, \alpha \right\rangle - \langle \sigma, \alpha \rangle + \langle \gamma, \alpha \rangle = 0 \end{aligned}$$

for any $(z,u,\alpha,\beta)\in L^1\times L^1\times R^m\times R^n$, which implies that

$$\begin{aligned}\dot p+pA&=\sum_{i=1}^{k+l}\lambda_i\mu_i,\ pB-\zeta=\sum_{i=1}^{k+l}\lambda_i\eta_i,\\ p(1)=\sigma&=-\sum_{i=1}^{k+l}\lambda_j\nu_j,\ p(0)=\int_0^1 q(\tau)d\tau+\sigma=\gamma.\end{aligned}\tag{25}$$

From (25), we obtain (18) and (19).

By $\zeta(\cdot)\in N_{\mathcal{C}}(u_*(\cdot))$, we have $\zeta(t)(u(t)-u_*(t))\le 0$ for any $u(\cdot)\in\mathcal{U}$. Thus, from the theory of measurable selection (20) follows.

Finally, if $\lambda=0$, then (28) and (29) imply that $\sigma=0$ and $p(\cdot)=0$, thus λ, q and σ all are zero. Hence, $\lambda>0$. If $(\lambda_1,\cdots,\lambda_k)=0$, then $(\lambda_k,\cdots,\lambda_{k+l})>0$. By the Slater constraint qualifications (H5) and the conditions (18)-(21), we have that

$$\begin{aligned}0>&\sum_{j\in J}\lambda_{k+j}\left(\mathcal{G}_j(x_i,u_i)-\mathcal{G}_j(x_*,u_*)\right)\\ =&\sum_{j\in I/\{i\}}\lambda_j\left(\int_0^1(G_j(t,x_i,u_i)-G_j[t])\,dt+g_j(x_i(1))-g_j(x_*(1))\right)\\ \ge&\int_0^1(\langle\dot p+pA,x_i-x_*\rangle+\langle pB-\zeta,u_i-u_*\rangle)\,dt-p(1)\,(x_i(1)-x_*(1))\\ =&-p(0)\,(x_i(0)-x_*(0))-\int_0^1\langle\zeta,u_i-u_*\rangle\,dt\\ \ge&\ 0,\end{aligned}$$

a contradiction. Hence, $(\lambda_1,\cdots,\lambda_k)>0$.

[Sufficiency] Assume that there exist $(\lambda_1\cdots,\lambda_k)>0$, $p(\cdot)\in AC$, and $\zeta(\cdot)\in L^\infty$ satisfying (18)-(21). Notice that $\sum_{i\in I}\lambda_i>0$, so we can set $\sum_{i\in I}\lambda_i=1$. Let (x,u) be an arbitrary admissible process for (P^*). Using (18)-(21) again, we see that

$$\begin{aligned}&max\{\mathcal{F}_i(x,u)-\mathcal{F}_i(x,u):\ i\in I\}\\ \ge&\sum_{i\in I}\lambda_i\left(\int_0^1 F_i(t,x,u)dt+f_i(x(1))-\int_0^1 F_i[t]dt-f_i(x_*(1))\right)\\ &\sum_{j\in J}\lambda_{k+j}\left(\int_0^1 G_i(t,x,u)dt+g_i(x(1))-\int_0^1 G_i[t]dt-g_i(x_*(1))\right)\\ &+\int_0^1\langle p,\dot x-Ax-Bu-b\rangle\,dt-\int_0^1\langle p,\dot x_*-Ax_*-Bu_*-b\rangle\,dt\\ =&\int_0^1\left(\sum_{i\in I}\lambda_iF_i(t,x,u)+\sum_{j\in J}\lambda_{k+j}G_i(t,x,u)dt-\sum_{i\in I}\lambda_iF_i[t]-\sum_{j\in J}\lambda_{k+j}G_j[t]\right)dt\\ &+\sum_{i\in I}\lambda_if_i(x(1))+\sum_{j\in J}\lambda_{k+j}g_j(x(1))-\sum_{i\in I}\lambda_if_i(x_*(1))-\sum_{j\in J}\lambda_{k+j}g_j(x_*(1))\\ &-\int_0^1(\langle\dot p+pA,x-x_*\rangle+\langle pB-\zeta,u-u_*\rangle)\,dt-\int_0^1\langle\zeta,u-u_*\rangle\,dt\\ &+\langle p(1),x(1)-x_*(1)\rangle-\langle p(0),x(0)-x_*(0)\rangle\\ \ge&\ 0.\end{aligned}$$

By Lemma 1, (x_*,u_*) is a weakly-efficient solution for (P). □

Using Theorem 3 and Lemma 3, we can easily show that the conditions (18)-(21) in Theorem 3 are also necessary-sufficient for efficient solutions of (P^*) under the following Slater constraint qualifications (H6).

(H6): For every $i \in I$, there is an admissible process (x_i, u_i) for (P^*), such that $\mathcal{F}_j(x_i, u_i) - \mathcal{F}_i(x_*, u_*) < 0$ for any $j \in I/\{i\}$ and $\mathcal{G}_j(x_i, u_i) - \mathcal{G}_j(x_*, u_*) < 0$ for any $j \in \{j \in J : \mathcal{G}_j(x_*, u_*) = 0\}$

Theorem 4: *Assume that (H1)-(H6) and (A1) are satisfied. An admissible process* (x_*, u_*) *is an efficient solution for* (P^*) *if and only if there exist* $(\lambda_1 \cdots, \lambda_{k+l}) \geq 0$ *with* $(\lambda_1 \cdots, \lambda_k) \gg 0$, $p(\cdot) \in AC([0,1], R^m)$, *and* $\zeta(\cdot) \in L^\infty([0,1], R^n)$ *such that (18)-(21) hold.*

Remark. It is easy to see that the sufficiency in Theorem 3 and Theorem 4 also hold under the following simpler assumptions: F_i for $i \in I$ and G_j for $j \in I$ are convex in (x, u) and measurable in t, f_i for $i \in I$ and g_j for $j \in I$ are convex functions, C is convex set and $U(t)$ is convex a.e..

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