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A Multiple Choice Secretary Problem With a Random Number of Objects

愛知大学経営学部 玉置光司 (Mitsushi Tamaki)

1. Introduction

We consider here a multiple choice secretary problem, that can be stated as follows. A set of N rankable objects appear one at a time in random order with all N! permutations equally likely. As each object appears, we decide either to select or reject it based on the relative ranks of the objects. For the m choice problem, we are allowed to choose at most m objects and win if either of the chosen objects is the best overall. Obviously only relatively best object, sometimes referred to as a *candidate*, can be chosen. The objective is to find a strategy that will maximize the probability of win. For the m choice problem, we consider a class of strategies which, whenever there remain k choices yet to be made, selects a candidate if it appears after or on time s_k, $1 \le k \le m$. We call this strategy a *multi-valued threshold rule with decision sequence* $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ or simply a multi-valued threshold rule if s_k is non-increasing in k, i.e., $s_1 \ge s_2 \ge \ldots \ge s_m$.

Gilbert and Mosteller(1966) solved the above problem when the value of N is known exactly in advance and showed that its optimal strategy is a multi-valued threshold rule. In this paper, we allow N to be a bounded random variable having probability distribution $p_i = P(N = i)$, $i=1,2,\ldots,n$ with $p_n>0$ and derive a simple sufficient condition on $\{p_i\}_{i=1}^n$ for a multi-valued threshold rule to be optimal.

Presman and Sonin(1972) are the first to consider a problem with a random number of objects, though their research interest is restricted to the study of the one choice problem. Presman and Sonin give a sufficient condition for a threshold rule to be optimal. Define the sequence $\{d_i\}_{i=1}^n$ as

$$d_i = p_i - \sum_{j=i+1}^{n} \frac{p_j}{j}.$$

Then this condition can be stated as "d_i changes sign from negative to positive only once (as i increases)". As will be seen, this condition remains a sufficient condition for a multi-valued threshold rule to be optimal. See also Irle(1980), Petruccelli(1983), Lehtinen(1993), Mori(1985) and Tamaki(1979) for the secretary problem with a random number of objects.

2. Optimal Strategy

The following theorem summarizes the main result of this note.

Theorem 2.1

Let

$$G_i^{(1)} = \sum_{j=i}^{n} \frac{p_j}{j} - \sum_{j=i+1}^{n} \frac{1}{j-1} \sum_{k=j}^{n} \frac{p_k}{k}, \qquad 1 \le i \le n. \tag{2.1}$$

Then the optimal strategy is a multi-valued threshold rule with decision sequence $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ if $G_i^{(1)}$ satisfies the following two conditions :

(a) $G_i^{(1)}$ is non-decreasing in i where $G_i^{(1)} \le 0$.
(b) If $G_i^{(1)} \ge 0$, then $G_{i+1}^{(1)} \ge 0$ for $1 \le i < n$.

Moreover, s_k is determined by

$$s_k = \min\left\{ i : G_i^{(k)} \ge 0 \right\}, \tag{2.2}$$

where $G_i^{(k)}$, $1 \le i \le n$, $1 \le k \le m$, is defined recursively as

$$G_i^{(k)} = G_i^{(1)} + \sum_{j=\max(i+1,\, s_{k-1})}^{n} \frac{1}{j-1} G_j^{(k-1)}, \qquad k \ge 2 \tag{2.3}$$

starting from $G_i^{(1)}$.

Proof. See Appendix.

We immediately have the following corollary from this theorem.

Corollary 2.2

If $G_i^{(1)}$ is a unimodal function of i, then the optimal strategy is a multi-valued threshold rule.

We have from (2.1)

$$G_i^{(1)} - G_{i+1}^{(1)} = \frac{1}{i}\left(p_i - \sum_{j=i+1}^{n} \frac{p_j}{j} \right) = \frac{1}{i} d_i .$$

Hence, the unimodality of $G_i^{(1)}$ with $G_n^{(1)} > 0$ assures that if $d_i \ge 0$ then $d_{i+1} \ge 0$. Thus Presman and Sonin condition remains a sufficient condition for a multi-valued threshold rule to be optimal.

3 Asymptotic results

It is of interest to investigate the asymptotic behaviors of s_k, $1 \le k \le m$, as n tends to infinity. To do this, we here employ an intuitive approach of approximating the infinite sum by the corresponding integral. We examine in detail four distributions for which the corresponding $G_i^{(1)}$ is unimodal.

3.1. Arithmetic distribution 1 : $p_i = 2i/n(n+1)$, $\quad 1 \le i \le n$.

Let $F_i^{(k)} = n\, G_i^{(k)}$. For k=1, we easily see that

$$F_i^{(1)} = \frac{2}{n+1}\left[(n-i+1) - \sum_{j=i+1}^{n} \sum_{t=i}^{j-1} \frac{1}{t} \right]$$

is a Riemann approximation, if one lets $i/n \to x$ as $n \to \infty$, to the integral

$$F^{(1)}(x) = 2\left[(1-x) - \int_x^1 dy \int_x^y \frac{dz}{z} \right] = 2[2(1-x) + \log x].$$

Thus, from (2.2), $s_1^* = \lim_{n\to\infty} \frac{s_1}{n} = 0.2032$ is a unique root $x \in (0, 1)$ of the equation $F^{(1)}(x) = 0$, that is,

$$2(1-x) + \log x = 0.$$

Define in general $s_k^* = \lim_{n\to\infty} \frac{s_k}{n}$. Then, in a similar way, we can obtain s_k^* for $k \ge 2$ successively as a unique root $x \in (0, s_{k-1}^*)$ of the equation

$$F^{(k)}(x) = 0 \tag{3.1}$$

if $F^{(k)}(x)$, $0<x<1$, is defined recursively as

$$F^{(k)}(x) = F^{(1)}(x) + \int_{\max(x,\, s_{k-1}^*)}^{1} \frac{1}{y} F^{(k-1)}(y)\, dy, \qquad k \ge 2 \tag{3.2}$$

starting with $F^{(1)}(x)$ (note that $F_i^{(k)}$ is a Riemann approximation to $F^{(k)}(x)$ if one lets $i/n \to x$ as $n \to \infty$).

From (3.1) and (3.2), s_k^* is a root of the equation

$$F^{(1)}(x) = -\int_{s_{k-1}^*}^{1} \frac{1}{y} F^{(k-1)}(y)\, dy.$$

In other words, if we denote by g(a), a>0, the unique root $x \in (0, 1)$ of the equation

$$2(1-x) + \log x = -\frac{a}{2}, \tag{3.3}$$

then

$$s_k^* = g(A_{k-1}), \tag{3.4}$$

where

$$A_{k-1} = \int_{s_{k-1}^*}^{1} \frac{1}{x} F^{(k-1)}(x)\, dx. \tag{3.5}$$

To calculate A_{k-1}, we can invoke the following recursive formula(see Lemmas 2.1 and 2.2 of Tamaki et al.(1998))

$$A_k = \sum_{i=1}^{k} \left[\frac{a_{k,\,k-i}}{(k-i)!} - \frac{(\log s_i^*)^{k-i+1}}{(k-i+1)!} A_{i-1} \right],$$

where

$$a_{k,\,k-i} = \int_{s_i^*}^{1} \frac{(\log x)^{k-i}}{x} F^{(1)}(x)\, dx.$$

If we let

$$I_n(s) = \int_{s}^{1} (\log x)^n dx,$$

we have

$$\frac{a_{k,k-i}}{(k-i)!} = -2\left[\frac{2(\log s_i^*)^{k-i+1}}{(k-i+1)!} + \frac{(\log s_i^*)^{k-i+2}}{(k-i)!(k-i+2)} + 2\frac{I_{k-i}(s_i^*)}{(k-i)!} \right].$$

Applying this to (3.5) yields

$$A_k = 2\sum_{i=1}^{k}\left[\frac{(\log s_i^*)^{k-i+2}}{(k-i+2)!} + 2\frac{I_{k-i+1}(s_i^*)}{(k-i+1)!}\right],$$

where we have used from (3.3) and (3.4)

$$A_{i-1} = -\left[4(1 - s_i^*) + 2\log s_i^*\right].$$

Consequently we can calculate s_k^* recursively through

$$s_k^* = g\left(2\sum_{i=1}^{k-1}\left\{\frac{(\log s_i^*)^{k-i+1}}{(k-i+1)!} + 2\frac{I_{k-i}(s_i^*)}{(k-i)!}\right\}\right).$$

For numerical calculation of $I_n(s)$, we can use the identity

$$I_n(s) = n!\left[(-1)^n - s\sum_{k=0}^{n}(-1)^{n-k}\frac{(\log s)^k}{k!}\right],$$

which can be obtained by repeated use of integration by parts.
Table 1 presents some numerical values of s_k^*.

Table 1

k	s_k^*
1	0.2032
2	0.1259
3	0.0810
4	0.0529
5	0.0350

3. 2. Arithmetic distribution 2 : $p_i = 2(n+1-i)/n(n+1)$, $1 \le i \le n$.

Since the derivation is quite similar to that for arithmetic distribution 1, we omit the detail. Let g(a), a>0, be the unique root $x \in (0, 1)$ of the equation

$$\log x = -(2 + \sqrt{4x + a}).$$

Then s_k^* can be calculated revcrsively through

$$s_k^* = g\left((-2)\sum_{i=1}^{k-1}\left\{2\frac{I_{k-i}(s_i^*)}{(k-i)!} + 2\frac{(\log s_i^*)^{k-i+1}}{(k-i+1)!} + \frac{(k-i+1)(\log s_i^*)^{k-i+2}}{(k-i+2)!}\right\}\right).$$

3.3. Uniform distribution : $p_i = 1/n$, $1 \le i \le n$.

s_k^* can be calculated revcrsively by

$$s_k^* = \exp\left[-\left\{1 + \sqrt{1 - 2\sum_{i=1}^{k-1}\frac{[(k-i+2)+(k-i+1)\log s_i^*]}{(k-i+2)!}(\log s_i^*)^{k-i+1}}\right\}\right].$$

See Tamaki et.al(1998) in detail.

3.5. Geometric distribution : $p_i = p(1-p)^{i-1}$, $1 \le i$.

For this distribution, N is not a bounded random variable, but it is not difficult to show that the optimal strategy becomes a multi-valued threshould rule in a similar manner as developed in Presman and Sonin(1972). Let $\lambda = E(N) = 1/p$ and $\gamma_i = \lim_{\lambda \to \infty} s_i/\lambda$. Then γ_1, γ_2 and γ_3 are calculated from the following system of equations :

$$\int_1^\infty \frac{e^{-\gamma_1 s}}{s}(1 - \log s)ds = 0$$

$$\int_1^\infty \frac{e^{-\gamma_2 s}}{s}(1 - \log s)ds + \frac{1}{2}\int_1^\infty \frac{e^{-\gamma_1 s}}{s}\log s\,(2 - \log s)ds = 0$$

$$\int_1^\infty \frac{e^{-\gamma_3 s}}{s}(1 - \log s)ds + \int_1^\infty \frac{e^{-\gamma_2 s}}{s}\log s\left(1 - \frac{1}{2!}\log s\right)ds$$
$$+ \int_1^\infty \frac{e^{-\gamma_1 s}}{s}\log s\left\{\log\left(\frac{\gamma_1}{\gamma_2}\right) + \frac{1}{2!}\left(1 - \log\left(\frac{\gamma_1}{\gamma_2}\right)\right)\log s - \frac{1}{3!}\log^2 s\right\}ds = 0$$

Remarks

1. Though we only derive the limiting values of the decision numbers, we can also derive the limiting values of the probability of win.
2. Recently the author has come to know Mori(1985), which also considers the multiple choice problems. Mori obtained the recursive formula for the decision numbers, but it is not appropriate to the real calculation. I think this paper could be a counterpart of Mori.

Appendix

Proof of Theorem 2.1

We consider our problem as a Markovian decision process. Since serious decision of either selection or rejection takes place only when a candidate appears, we describe the state of the process as (i,k), $1 \le i \le n$, $1 \le k \le m$ if the i th object is a candidate and there remain k more choices to be made. For the above process to be a Markov chain, we must further introduce additional absorbing state (n+1,k) denoting the situation where the process comes to an end with k choices left, $1 \le k \le m$.

Let $W_i^{(k)}$ be the probability of win under an optimal strategy starting from state (i,k), $1 \le i \le n$, $1 \le k \le m$, and also let $U_i^{(k)}\left(V_i^{(k)}\right)$ be the probability of win when we select(reject) the i th object and then continues search in an optimal manner. Then the principle of optimality yields, for $1 \le k \le m$,

$$W_i^{(k)} = \max\left\{U_i^{(k)},\ V_i^{(k)}\right\}, \quad 1 \le i \le n, \tag{A.1}$$

where

$$U_i^{(k)} = \sum_{j=i}^{n} \frac{i}{j}\frac{p_j}{\pi_i} + \sum_{j=i+1}^{n} \frac{i}{j(j-1)}\frac{\pi_j}{\pi_i} W_j^{(k-1)} \tag{A.2}$$

and

$$V_i^{(k)} = \sum_{j=i+1}^{n} \frac{i}{j(j-1)}\frac{\pi_j}{\pi_i} W_j^{(k)} \quad . \tag{A.3}$$

Define $G_i^{(k)}$, $k \ge 1$, as

$$G_i^{(k)} = G_i^{(1)} + \sum_{j=i+1}^{n} \frac{\pi_j}{j(j-1)}\left\{W_j^{(k-1)} - V_j^{(k-1)}\right\}, \qquad k \ge 2 \tag{A.4}$$

starting with $G_i^{(1)}$. We will naturally find in the course of the proof that this definition in fact agrees with that given in (2.3). Suppose that we are in state (i,k). If we select a current candidate we receive $U_i^{(k)}$. If instead, we continue and select the next candidate if any, we expect to receive $\tilde{V}_i^{(k)}$ defined as

$$\begin{aligned} \tilde{V}_i^{(k)} &= \sum_{j=i+1}^{n} \frac{i}{j(j-1)}\frac{\pi_j}{\pi_i} U_j^{(k)} \\ &= \sum_{j=i+1}^{n} \frac{i}{j(j-1)}\frac{\pi_j}{\pi_i}\left\{\frac{j}{\pi_j}\sum_{t=j}^{n}\frac{p_t}{t} + V_j^{(k-1)}\right\}. \end{aligned} \tag{A.5}$$

The one-stage look-ahead rule immediately calls for selection if $U_i^{(k)} \geq \tilde{V}_i^{(k)}$. Then the strategy specified by (2.1) - (2.3) is in fact the one-stage look-ahead rule, because $G_i^{(k)}$ can be written, from (A.2) and (A.5), as

$$G_i^{(k)} = \left(\frac{\pi_i}{i}\right)\left(U_i^{(k)} - \tilde{V}_i^{(k)}\right) . \tag{A.6}$$

Since the horizon is finite, the one-stage look-ahead rule is optimal if the problem is monotone. To prove that the problem is monotone and that s_k is non-increasing in k, it suffices to show that, for each k, $G_i^{(k)}$ has the following properties :

$(P1)_k$ If $G_i^{(k)} \geq 0$, then $G_{i+1}^{(k)} \geq 0$ for $1 \leq i < n$

$(P2)_k$ $G_i^{(k+1)} \geq G_i^{(k)}$ for $1 \leq i \leq n$.

$(P1)_k$ implies that the k choice problem is monotone and $(P2)_k$, combined with the definition (2.2), guarantees $s_{k+1} \leq s_k$. We show $(P1)_k$ and $(P2)_k$ simultaneously by induction on k. $(P1)_1$ holds from the condition (a). $(P2)_1$ is immediate since, from (A.4),

$$G_i^{(2)} = G_i^{(1)} + \sum_{j=i+1}^{n} \frac{\pi_j}{j(j-1)}\left\{W_j^{(1)} - V_j^{(1)}\right\} \geq G_i^{(1)}.$$

Assume now that $(P1)_j$ and $(P2)_j$ hold for j=1,2,...,k. Then, considering that the one-stage look-ahead rule of the k choice problem is optimal from the induction hypothesis $(P1)_k$, we have that, for $s_k = \min\left\{i : G_i^{(k)} \geq 0\right\}$ as defined in (2.2),

$$W_j^{(k)} = \left\{\begin{array}{ll} V_j^{(k)}, & j<s_k \\ U_j^{(k)}, & j \geq s_k \end{array}\right\}$$

and

$$V_j^{(k)} = \tilde{V}_j^{(k)}, \quad j \geq s_k - 1.$$

These are combined to yield, from (A.6),

$$W_j^{(k)} - V_j^{(k)} = \left\{\begin{array}{ll} 0, & j<s_k \\ \frac{j}{\pi_j} G_j^{(k)}, & j \geq s_k \end{array}\right\},$$

and then applying this to (A.6)(with k replaced by k+1) gives

$$G_i^{(k+1)} = G_i^{(1)} + \sum_{j=\max(i+1,\, s_k)}^{n} \frac{1}{j-1} G_j^{(k)}. \tag{A.7}$$

It is easy to see $G_i^{(k+1)} \geq 0$ for $i \geq s_k$ from the induction hypotheses $(P1)_k$ and $(P2)_k$. Thus, to prove $(P1)_{k+1}$, it suffices to show that $G_i^{(k+1)}$ is non-decreasing in i for $i \leq s_k$. This is immediate from (A.7) and the conditions (a) and (b) since $G_i^{(1)}$ is non-dereasing in i for $i \leq s_k \leq s_1$ through the induction hypotheses $(P2)_j$ for j=1,2,...,k-1.

Now we turn to $(P2)_{k+1}$. Since the one-stage look-ahead rule of the k+1 choice problem is optimal from $(P1)_{k+1}$, we come to have, for $s_{k+1} = \min\{i : G_i^{(k+1)} \geq 0\}$,

$$G_i^{(k+2)} = G_i^{(1)} + \sum_{j=\max(i+1,\, s_{k+1})}^{n} \frac{1}{j-1} G_j^{(k+1)}, \tag{A.8}$$

in a similar way as (A.7) was derived.

Therefore from (A.7) and (A.8)

$$G_i^{(k+2)} - G_i^{(k+1)}$$

$$= \sum_{j=\max(i+1,\, s_{k+1})}^{n} \frac{1}{j-1} G_j^{(k+1)} - \sum_{j=\max(i+1,\, s_k)}^{n} \frac{1}{j-1} G_j^{(k)}$$

$$\geq \sum_{j=\max(i+1,\, s_k)}^{n} \frac{1}{j-1} \left\{ G_j^{(k+1)} - G_j^{(k)} \right\} \qquad \text{(use } s_{k+1} \leq s_k \text{ from the induction hypothesis } (P2)_k\text{)}$$

$$\geq 0, \qquad \text{(again from the induction hypothesis } (P2)_k\text{)}$$

which proves $(P2)_{k+1}$ and hence completes the induction.

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