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REASONABLE OUTCOMES IN COOPERATIVE TU-GAMES

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Abstract

The reasonable set of a cooperative game was defined by Milnor (1952). In this article several subsets of the reasonable set are defined by extending the definition of the reasonable set. Some inclusion-relation between solution-concepts in cooperative games and these subsets are examined.

1 Introduction and Preliminaries.

The reasonable set of a cooperative game was first introduced by Milnor (1952). Adding a lower bound to the reasonable set, Gerard-Valet and Zamir (1987) defined a set of reasonable outcomes and justified it by axiomatization. These sets as well as the imputation set are considered as pre-solution-concepts. I.e., it is asserted not that outcomes within the sets are necessarily plausible, but only that those outside the sets are implausible.

Since the reasonable set is large, it would be preferable if we could define a subset of the reasonable set and if it enjoys any reasonability in the sense that those outcomes outside the set are implausible. A decision-maker would get a more precise guideline, if he has a smaller set as a pre-solution. Milnor (1952) defined other two pre-solutions, known as "L" and "D", which put lower and upper bounds on the payoff to any coalition. Milnor proved that "L" and the efficient part of "D" are non-empty for certain classes of games. Kikuta and Shapley(1986) gave an example showing "L" and the efficient part of "D" are empty sets.

In this paper, we try to define a subset of the reasonable set, extending the definition of the reasonable set, and give properties of it.

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An n-person cooperative game with side payments (abbreviated as a *game*) is an ordered pair (N, v), where $N \equiv \{1, 2, ..., n\}$ is the set of *players* and v, called the *characteristic function*, is a real-valued function on the power set of N, satisfying $v(\emptyset) = 0$. For simplicity we express a game (N, v) as v. A subset of N is called a *coalition*. For a coalition S, (S, v) is a game in which S is the set of players and v is restricted on 2^S. For any set Z, $|Z|$ denotes the cardinality of Z. For a coalition S, $\mathbf{R}^S$ is the $|S|$-dimensional product space $\mathbf{R}^{|S|}$ with coordinates indexed by players in S. The i-th component of $x \in \mathbf{R}^S$ is denoted by x_i. For $S \subseteq T \subseteq N$ and $x \in \mathbf{R}^T$, $x|S$ means the projection of x to $\mathbf{R}^S$. For $x, y \in \mathbf{R}^S$, $x \leq y$ (or $y \geq x$) means $x_i \leq y_i$ for all $i \in S$. For $S \subseteq N$ and $x \in \mathbf{R}^N$, we define $x(S) = \sum_{i \in S} x_i$ (if $S \neq \emptyset$) and $= 0$ (if $S = \emptyset$). A *pre-imputation* for a game v is a vector $x \in \mathbf{R}^N$ that satisfies

(1) $$x(N) = v(N).$$

$X^*(v) \equiv X^*(N, v)$ is the set of all pre-imputations for v. For a coalition S, $X^*(S, v)$ is the set of pre-imputations for (S, v).

2 Reasonable Set.

For a game v, let $r_i(S, v) \equiv Max\{v(T) - v(T\backslash\{i\}) : i \in T \subseteq S\}$ for all $i \in S$ and all $\emptyset \neq S \subseteq N$. $v(T) - v(T\backslash\{i\})$ is the marginal contribution of Player i to the coalition T. $r_i(S, v)$ is the maximal marginal contribution of Player i in the game (S, v). Suppose $x \in X^*(v)$. We say x is *reasonable* (See Milnor (1952)) if for each $i \in N$,

(2) $$x_i \leq r_i(N, v).$$

(2) means if Player i is to get the amount x_i, there should be at least one coalition to which he contributes at least x_i. The reasonable set, written as $R(v) \equiv R(N, v)$, is the set of all reasonable pre-imputations. Milnor (1952) showed that the Shapley value(Shapley (1953)) is in $R(v)$, and that von Neumann-Morgenstern solution is in $R(v)$ also. The reasonable set considers the maximization over all coalitions which the player belongs. We extend (2) to subcoalitions so that $x_i + \varepsilon_i^S \leq r_i(S, v)$ for $x \in X^*(v)$ and $\sum_{i \in S}(x_i + \varepsilon_i^S) = v(S)$. More precisely, first we extend (2) as follows. For

$x \in X^*(v)$, for some *weight* $w = \{w^S\}_{S \subseteq N}$ with $\sum_{i \in S} w_i^S = 1$ for all $S \subseteq N$, and for all $S \subseteq N$,

$$(3) \qquad x_i + w_i^S[v(S) - x(S)] \leq r_i(S, v).$$

$v(S) - x(S)$ is called the excess of S at x (See Maschler (1992)). When $x(S) = v(S)$ (i.e., $x|S \in X^*(S, v)$), then $x|S$ is a distribution of $v(S)$ and (3) becomes to $x_i \leq r_i(S, v)$. When $S \neq N$ and $x(S) \neq v(S)$, let $x_i^S \equiv x_i + w_i^S[v(S) - x(S)]$. The excess $v(S) - x(S)$ must be added to x_i, according to a weight w, so that $x^S(S) = v(S)$. Let W be the set of all weights, i.e.,$W \equiv \{w = \{w^S\}_{S \subseteq N} | \sum_{i \in S} w_i^S = 1 \text{ for all } S \subseteq N\}$. For a fixed weight $w = \{w^S\}_{S \subseteq N}$, denote by $R(v; w)$ the set of all pre-imputations satisfying (3).

$$(4) \qquad R(v; w) = \{x \in X^*(v) | (3) \text{ for all } i \in S \text{ and all } S \subseteq N\}^1 .$$

Theorem 1 $\bigcup_{w \in W} R(v; w) = R(v)$.

Proof: (3) implies (2) by letting $S = N$ and by (1). So $R(v; w) \subseteq R(v)$ for all $w \in W$. Next suppose $x \in R(v)$. For any $S \subset N$ such that $S \neq N$, a system of inequalities and an equality with respect to w^S:

$$(5) \qquad w_i^S[v(S) - x(S)] \leq r_i(S, v) - x_i \text{ for all } i \in S, \text{ and } w^S(S) = 1.$$

is feasible since $\sum_{i \in S} w_i^S[v(S) - x(S)] = v(S) - x(S) \leq \sum_{i \in S} r_i(S, v) - x(S) = \sum_{i \in S}[r_i(S, v) - x_i]$. That is, there is w^S which satisfies (5) for each $S \subset N$ such that $S \neq N$. For $S = N$, we have (2) for all $i \in N$ since $x \in R(v)$. Consequently, $x \in \bigcup_{w \in W} R(v; w)$. ∎

By Theorem 1, $R(v)$ is interpreted alternatively. I.e., $x \in X^*(v)$ is reasonable if and only if for some $w = \{w^S\}_{S \subseteq N}$, and for any $S \subset N$, $x^S \equiv \{x_i^S\}$ is in the reasonable set of (S, v).

[1] By Theorem 1 below, there is $w \in W$ such that $R(v; w)$ is not empty.

3 Subsets of the Reasonable Set.

In this section we reduce $R(v)$, by considering a subset of W as a set of *admissible* weights. Players $i, j \in N$ is called *symmetric* if $v(S \cup \{i\}) = v(S \cup \{j\})$ whenever $i, j \in S$ (See p.606 of Maschler (1992)). Player i is called *dummy* if $v(S \cup \{i\}) = v(S) + v(\{i\})$ whenever $i \notin S$. A weight $w = \{w_i^S\}$ is called *symmetric w.r.t.* v if, whenever $i, j \in N$ is symmetric,

$$(6) \qquad \begin{cases} w_i^S = w_j^S \ \ for\ all\ S : i, j \in S \\ w_i^{S \cup \{i\}} = w_j^{S \cup \{j\}} \ \ for\ all\ S : i, j \notin S. \end{cases}$$

A weight $w = \{w_i^S\}$ is called *dummy w.r.t.* v if $w_i^S = v(\{i\})$ for all $S : i \in S$ when i is dummy. $W^s(v)$, $W^d(v)$, and, $W^{sd}(v)$ are the sets of weights which are symmetric, dummy w.r.t. v, and symmetric and dummy w.r.t. v respectively. Let $R^*(v) \equiv \bigcup_{w \in W^*} R(v; w)$ where $*$ is s, d or sd. Let ϕ be a function defined on the set of all games with $\phi(N, v) \subseteq X^*(v)$ for (N, v). We call ϕ a solution function. In particular, if ϕ satisfies $\phi(N, v) \subseteq R(N, v)$ for (N, v) then we say ϕ is reasonable. We consider two properties of ϕ.

Symmetry: ϕ is called symmetric if symmetric players in (N, v) receive equal payments in each point in $\phi(N, v)$.

Dummy : ϕ is called dummy if each dummy player $i \in N$ in (N, v) receives $v(\{i\})$ in each point in $\phi(N, v)$.

Theorem 2 *Suppose ϕ is a reasonable solution function.*
(i) If ϕ is symmetric, then $\phi(N, v) \subseteq R^s(N, v)$ for all (N, v).
(ii) If ϕ is dummy, then $\phi(N, v) \subseteq R^d(N, v)$ for all (N, v).
(iii) If ϕ is symmetric and dummy, then $\phi(N, v) \subseteq R^{sd}(N, v)$ for all (N, v).

Proof: (i) Suppose $i, j \in N$ are symmetric and $x \in \phi(N, v)$. Then $x_i = x_j$. By Theorem 1 there is $w \in W$ such that $x \in R(v; w)$. Suppose $i, j \in S$. By (3),

$$(7) \quad x_i + w_i^S[v(S) - x(S)] \leq r_i(S, v) \text{ and } x_j + w_j^S[v(S) - x(S)] \leq r_j(S, v).$$

Here by the symmetry of i, j we have $v(T \cup \{i\}) - v(T) = v(T \cup \{j\}) - v(T)$ for all $T \subseteq S : i, j \notin T$. Furthermore, $v(T) - v(T \backslash \{i\}) = v(T) - v(T \backslash \{j\})$ for all $T \subseteq S : i, j \in T$. This implies $r_i(S, v) = r_j(S, v)$. So let $w\prime_i^S = w\prime_j^S = (w_i^S + w_j^S)/2$. Then (7) holds even if w_i^S and w_j^S are replaced by $w\prime_i^S$ and $w\prime_j^S$ respectively. Next suppose $i, j \notin S$. For all $k \in S$,

$$x_k + w_k^{S \cup \{i\}}[v(S \cup \{i\}) - x(S \cup \{i\})] \leq r_k(S \cup \{i\}, v).$$

$$x_k + w_k^{S \cup \{j\}}[v(S \cup \{j\}) - x(S \cup \{j\})] \leq r_k(S \cup \{j\}, v).$$

Here we note that $r_k(S \cup \{i\}, v) = r_k(S \cup \{j\}, v)$ since $v(T \cup \{i\}) - v(T \cup \{i\} \backslash \{k\}) = v(T \cup \{j\}) - v(T \cup \{j\} \backslash \{k\})$ by the symmetry. Further, $v(S \cup \{i\}) = v(S \cup \{j\})$, and $x(S \cup \{i\}) = x(S \cup \{j\})$. So let $w\prime_k^{S \cup \{i\}} = w\prime_k^{S \cup \{j\}} = (w_k^{S \cup \{i\}} + w_k^{S \cup \{j\}})/2$ for all $k \in S$. For the other components of $w\prime$, let $w\prime_k^U = w_k^U$ for all $k \in U$. Then $w\prime$ satisfies (6) for i and j. If there are more than 2 symmetric players, the same argument applies.
(ii) Suppose $i \in N$ is dummy in v and $x \in \phi(N, v)$. By Theorem 1 there is $w \in W$ such that $x \in R(v; w)$. By assumption, $x_i = v(\{i\})$. By definition, $r_i(S, v) = v(\{i\})$ for all $S : i \in S$. (3) becomes

$$w_i^S[v(S) - x(S)] \leq 0 \text{ and } x_j + w_j^S[v(S) - x(S)] \leq r_j(S, v) \text{ for all } j \in S, j \neq i.$$

Since i is dummy, $r_j(S, v) = r_j(S \backslash \{i\}, v)$ for all $j \in S \backslash \{i\}$. So let $w\prime_i^S = 0$, and $w\prime_j^S = w_j^{S \backslash \{i\}}$ for all $j \in S \backslash \{i\}$.

(iii) Suppose $x \in \phi(N, v)$. Suppose $i, j \in N$ are symmetric and $k \in N$ is dummy, and suppose $x \in \phi(N, v)$. Then $x_i = x_j$ and $x_k = v(\{k\})$. If $k = i$ or $k = j$, then $x_i = x_j = v(\{k\})$ and the argument reduces to that of (i). Suppose $k \neq i$ and $k \neq j$. By (i) and (ii) of this theorem, there are $w \in W^s(v)$ and $w' \in W^d(v)$ such that $x \in R(v; w)$ and $x \in R(v; w')$ respectively. Define $w'' \in W^{sd}(v)$ as follows. If $i, j \in S$ and $k \notin S$ then let $w''^S = w^S$. If $i, j \notin S$ and $k \in S$ then let $w''^S = w'^S$. If $i, j \notin S$ and $k \notin S$ then let $w''^S = w^S$ or $= w'^S$. Suppose $i, j \in S$ and $k \in S$.

$$x_i + w'^S_i[v(S) - x(S)] \leq r_i(S, v) \text{ and } x_j + w'^S_j[v(S) - x(S)] \leq r_j(S, v).$$

Noting that $x_i = x_j$, $r_i(S, v) = r_j(S, v)$, let

$$w''^S_i = w''^S_j = \frac{w'^S_i + w'^S_j}{2}, w''^S_k = w'^S_k = 0, w''^S_\ell = w'^S_\ell \text{for all } \ell \in S\backslash\{i,j,k\}.$$

Suppose $i, j \notin S$.

$$x_\ell + w'^{S\cup\{i\}}_\ell[v(S \cup \{i\}) - x(S \cup \{i\})] \leq r_\ell(S \cup \{i\}), v)$$

and

$$x_\ell + w'^{S\cup\{j\}}_\ell[v(S \cup \{j\}) - x(S \cup \{j\})] \leq r_\ell(S \cup \{j\}), v).$$

Let

$$w''^{S\cup\{i\}}_\ell = w''^{S\cup\{j\}}_\ell = \frac{w'^{S\cup\{i\}}_\ell + w'^{S\cup\{j\}}_\ell}{2} \text{ for all } \ell \in S$$

and

$$w''^{S\cup\{i\}}_i = w''^{S\cup\{j\}}_j = \frac{w'^{S\cup\{i\}}_i + w'^{S\cup\{j\}}_j}{2}.$$

For other components of w'', let $w''^U_\ell = w^U_\ell$ for all $k \in U$ and all $U \subseteq N$. ∎

Several solution functions are symmetric and dummy. We have the next corollary. The definitions of solution-concepts are omitted here. See Shapley(1953) for the Shapley value, and Maschler(1992) and Schmeidler (1969) for others.

Corollary 3 *The Shapley value, the nucleolus, the prenucleolus, the kernel, and the prekernel of v are in $R^{sd}(v)$.*

4 Core and a Subset of the Reasonable Set.

The *core* of a game v is defined by $C(N, v) \equiv \{x \in X^*(v) | x(S) \geq v(S)$ for all $S \subseteq N\}$. A game v is called *i-zero-monotonic*[2] if $v(S) - v(S\backslash\{i\}) \geq v(\{i\})$ for all S such that $i \in S$. A game v is called *zero-monotonic* if v is *i-zero-monotonic* for all $i \in N$. A weight $w = \{w^S_i\}$ is called *i-non-negative* if $w^S_i \geq 0$ for all S such that $i \in S$. For $i \in N$, W^{i+} is the set of all i- non-negative weights. Let $M(v)$ be the set of $i \in N$ such that v is i-zero-monotonic. For $i \in M(v)$, we let $R^{i+}(v) \equiv \bigcup_{w \in W^{i+}} R(v; w)$. Then we let $R^+(v) \equiv \bigcap_{i \in M(v)} R^{i+}(v)$

[2] Sprumont(1990) defines the convexity of Player i and also an i-veto game.

Theorem 4 *If a game v is zero-monotonic and the core of v is non-empty, then the core is in $R^+(v)$.*

Proof: First we need a lemma.
Lemma. Suppose a game v is i-zero-monotonic. For $x \in X^*(v)$, suppose $x(S) \geq v(S)$ for all S such that $i \in S$. Then $x \in R^{i+}(v)$.

Suppose x is in the core. By the definition of the core and by Lemma, we have $x \in R^{i+}(v)$ for all $i \in N$. Hence $x \in R^+(v)$.

Next prove the lemma. Assume $v(S) - x(S) < 0$ for S such that $i \in S$. (3) becomes $w_j^S \geq [r_j(S,v) - x_j]/[v(S) - x(S)]$ for all $j \in S$. Hence a set $\{w^S |$ (3) holds for all $j \in S, w^S(S) = 1\}$ is compact. So let $\alpha \equiv \max\{w_i^S | w \in W$ and (3) holds for all $j \in S\}$. Assume $\alpha < 0$. Suppose for w'^S, $w_i'^S = \alpha$. If $w_j'^S[v(S) - x(S)] < r_j(S,v) - x_j$ for some $j \in S$, then define w''^S by $w_i''^S = w_i'^S + t$, $w_j''^S = w_j'^S - t$ for $t > 0$. Then w''^S will satisfy (3). This contradicts the definition of α. Hence we must have $w_j'^S[v(S) - x(S)] = r_j(S,v) - x_j$ for all $j \in S \backslash \{i\}$. Adding these together, and noting that $1 - \alpha = \sum_{S\backslash\{i\}} w_j'^S$,

$$1 - \alpha = \frac{\sum_{S\backslash\{i\}} r_j(S,v) - x(S\backslash\{i\})}{v(S) - x(S)} > 1,$$

which implies $\sum_{S\backslash\{i\}} r_j(S,v) + x_i < v(S)$. But the left hand side is greater than or equal to $v(S)$, since $x_i \geq v(\{i\})$. This is a contradiction. Hence $\alpha \geq 0$. Assume $v(S) - x(S) = 0$. In this case we see $\alpha \geq 0$ trivially. Consequently, we can find $w \in W^{i+}$. This completes the proof of the lemma.
∎

A game v is called *totally balanced* if each subgame (S, v) has the non-empty core. It is well-known that if a game is totally balanced then the core is non-empty and the game is zero-monotonic. So we have the next corollary.

Corollary 5 *If a game v is totally balanced then the core of v is in $R^+(v)$.*

5 Examples.

Gerard-Valet and Zamir (1987) characterized and justified the set of reasonable outcomes, which is defined by $R'(v) \equiv \prod_{i \in N} R_i'(v)$, where $R_i'(v) \equiv [m_i(v), r_i(N,v)]$, and $m_i(v) \equiv \min_{S:i\in S}\{v(S) - v(S\backslash\{i\})\}$. Both of $R'(v)$

and $R^s(v)$ are included in $R(v)$. In general, there is no inclusion relation between these two, as is seen by the next example. $R^s(v)$ is not a convex set in the next example.

Example 1. Let $n = 3$ and $v(123) = 100$, $v(23) = 80$, $v(12) = v(13) = 30$, and $v(1) = v(2) = v(3) = 0$.

$$R(v) = \{(x_1, x_2, x_3) \in X^*(v) | x_1 \leq 30, x_2 \leq 80, x_3 \leq 80\}.$$

$R'(v)$ is the non-negative part of $R(v)$. $R^s(v)$ is the region included in $R(v)$, surrounded by $x_2 + \frac{x_1}{2} \leq 90$, $x_3 + \frac{x_1}{2} \leq 90$, $1 = \frac{30-x_2}{x_3-70} + \frac{30-x_1}{x_2-70}$, and $1 = \frac{30-x_1}{x_3-70} + \frac{30-x_3}{x_2-70}$. Note that $w_2^{23} = w_3^{23} = \frac{1}{2}$ and $w_2^{12} = w_3^{13}$ since Players 1 and 2 are symmetric. $R^s(v)$ includes a point such that $x_1 < 0$, which is not contained in $R'(v)$, while $R^s(v)$ does not contain $(0, 80, 20)$, which is contained in $R'(v)$. $R^s(v)$ is not a convex set in this game v. The core is the region in $R'(v)$ such that $x_1 \leq 20$, $x_2 \leq 70$, and $x_3 \leq 70$. The core is included in $R^s(v)$. $R^+(v)$ is a quadrangle, the corner-points of which are $(30, -10, 80)$,$(0, 20, 80)$,$(0, 80, 20)$, and $(30, 80, -10)$.

Figure 1

In the next example, the core includes $R^s(v)$.

Example 2. Let $n = 3$ and $v(123) = 4$, $v(12) = v(13) = v(23) = 1$, and $v(1) = v(2) = v(3) = 0$. $R^s(v)$ is the region in $X^*(v)$ such that $x_i + \frac{x_j}{2} \leq \frac{5}{2}$ for $i \neq j$. The core is the region in $X^*(v)$ such that $0 \leq x_i \leq 3$ for $i = 1, 2, 3$. So $R^s(v)$ is included in the core. $R^+(v)$ coincides with the core.

Figure 2

$x \in X^*(v)$ is called *pairwise reasonable* (See pp.606-7 of Maschler (1992)) if for every pair of Players $i, j \in N$,

$$x_i - x_j \leq \max_{S \subseteq N \backslash \{i,j\}} [v(S \cup \{i\}) - v(S \cup \{j\})].$$

The next example says that there is no inclusion relation between $R(v)$ and the set of pairwise reasonable preimputations.

Example 3. Let $n = 3$ and $v(123) = 6$, $v(12) = v(13) = 0$, $v(23) = 10$, and $v(1) = v(2) = v(3) = 0$. $x \in X^*(v)$ is pairwise reasonable if and only if $x_2 = x_3$ and $x_1 \leq x_2 \leq x_1 + 10$, while $x \in R^s(v)$ if and only if $x_2 = x_3$ and $-14 \leq x_1 \leq 0$. $R^+(v)$ coincides with $R(v)$.

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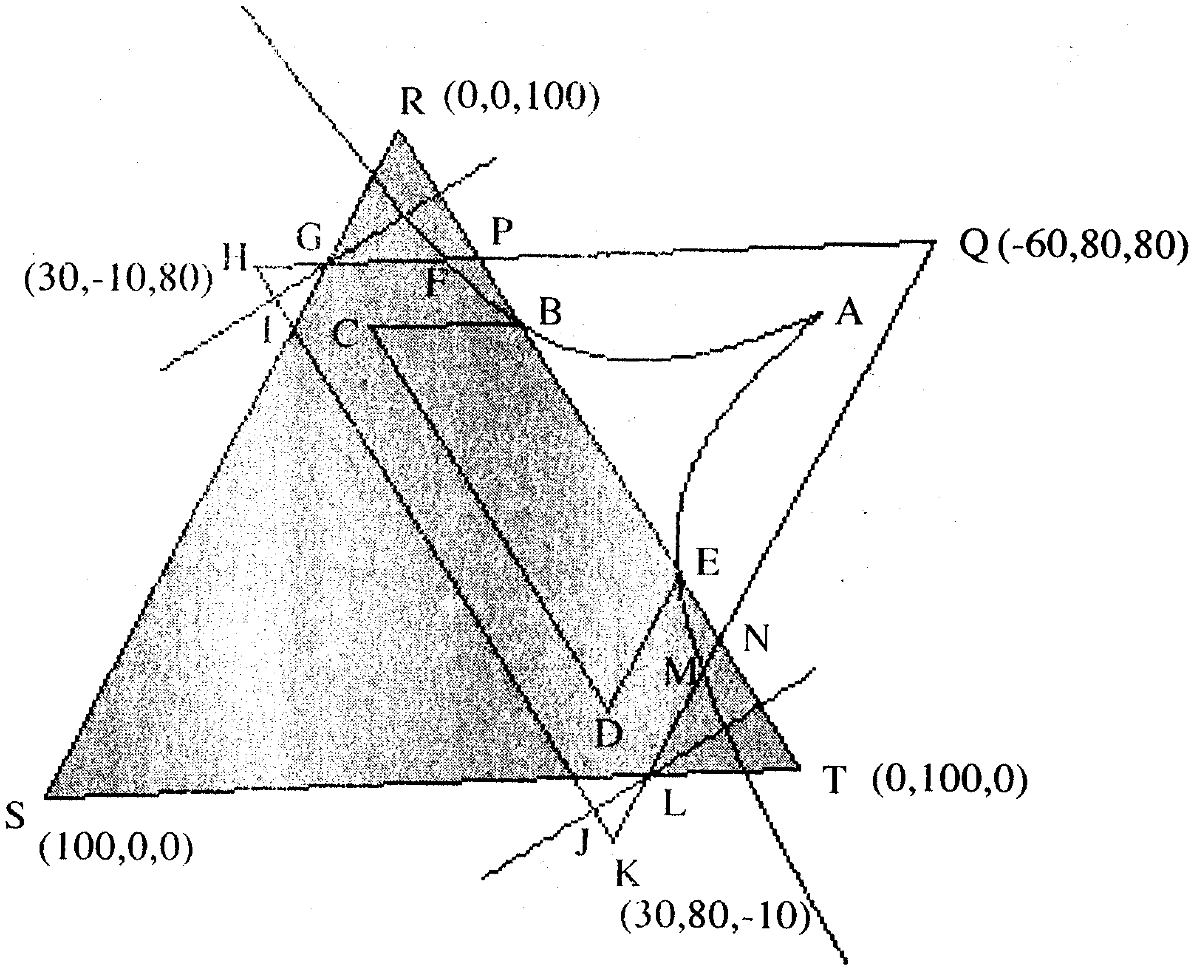
Figure 1



$R^S(v)$: AFGIJLMA

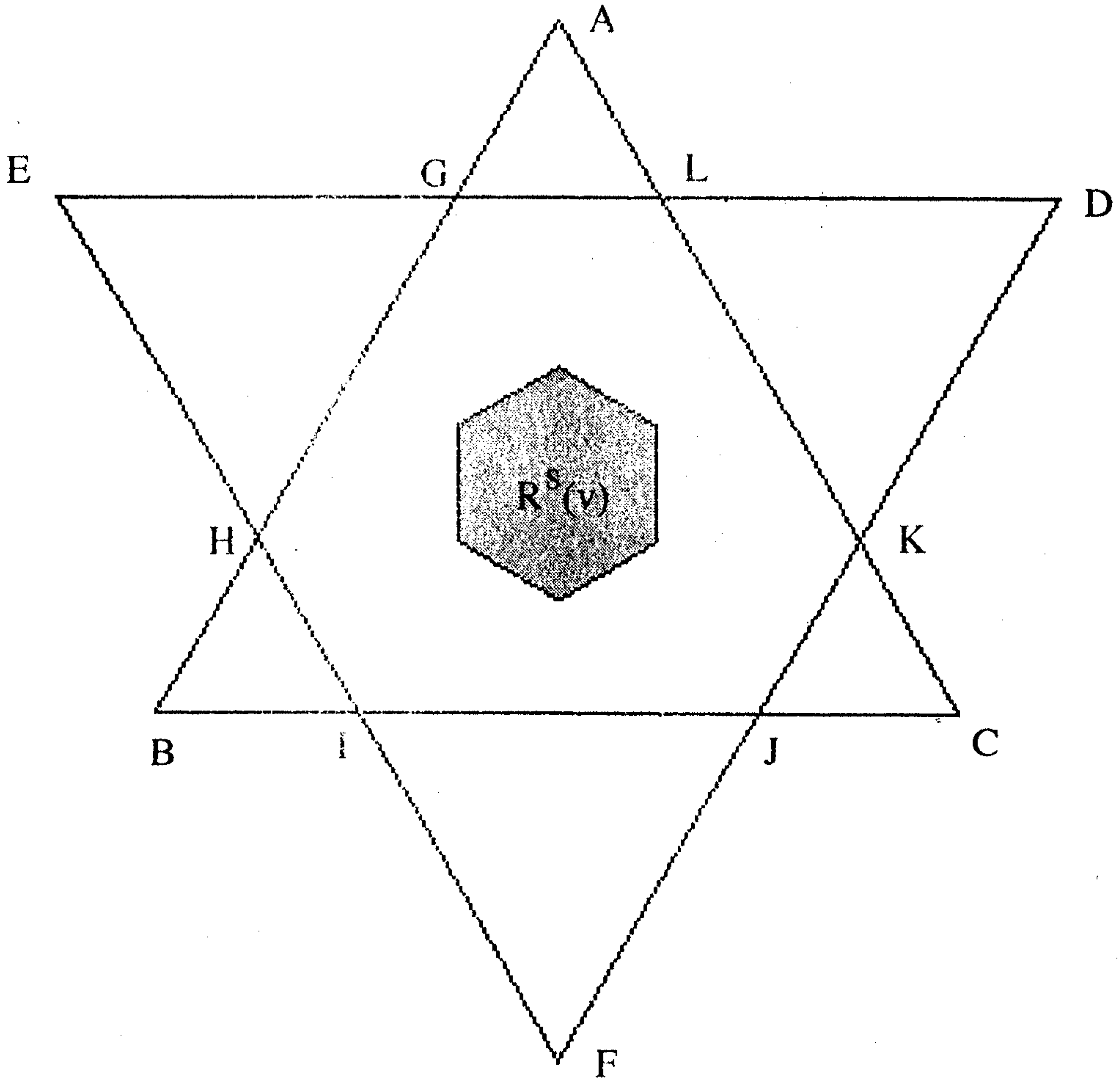
$R^+(v)$: PHKNP

core : BCDEB

R(v) : QHKQ

Imputation Set : RSTR

Figure 2



R(v) · DEFD

Imputation Set : ABCA

Core=$R^+(v)$: GHIJKLG