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Complementary inequalities of the Furuta inequality

大阪教育大学 藤井正俊 (Masatoshi Fujii*)
前橋工科大学 亀井栄三郎 (Eizaburo Kamei**)
大阪府立東豊中高校 松本明美 (Akemi Matumoto***)

ABSTRACT. As a continuation of our preceding note, we discuss inequalities on the complementary domain of the Furuta inequality. For positive operators $A \geq B > 0$, it is shown that

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \geq A^\delta \geq B^\delta \geq B^t \natural_{\frac{\delta-t}{p-t}} A^p,$$

for $0 \leq \delta \leq t < p \leq 1$. This inequality is opposite to the inequality in [12].

1. Introduction. Throughout this note, a capital letter means a bounded linear operator on a Hilbert space H. An opertor A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible. The Furuta inequality [5] established by Furuta himself in 1987 (cf.[6]) was given by the following form.

Furuta inequality:([5],cf.[6]) *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(A^r A^p A^r)^{\frac{1}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$$

and

$$(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$$

holds for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

The best possibility of the conditions for p, q and r for the Furuta inequality is proved in [15]. In this inequality, if we take $r = 0$, then the following Löwner-Heinz inequality is obtained.

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Löwner-Heinz inequality: *If* $A \geq B \geq 0$, *then*

$$A^{\alpha} \geq B^{\alpha} \ for \ \alpha \in [0,1].$$

We can review the Furuta inequality by using the operator mean theory established by Kubo-Ando[14]. Especially we use the *α-power mean*, $\sharp_{\alpha}$ which corresponds to the Löwner-Heinz inequality and is given by

$$A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}, \quad for \ \alpha \in [0,1].$$

Using it, we can reformulate the Furuta inequality as follows (cf.[1],[10],[13]):

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \leq A \ \ and \ \ B \leq B^t \sharp_{\frac{1-t}{p-t}} A^p \quad for \ p \geq 1 \ and \ t \leq 0.$$

In our arguments of the Furuta inequality in [10], we obtained a chain of the following inequalities:

Satellite theorem of the Furuta inequality: *If* $A \geq B \geq 0$, *then for* $p \geq 1$ *and* $t \leq 0$,

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \leq B \leq A \leq B^t \sharp_{\frac{1-t}{p-t}} A^p.$$

In our preceding notes [2],[3],[4] and [11], we discussed about the domain on which a similar formula to the Furuta inequality holds. In [12], we have given a unified form of the inequalities shown in [3], [4] and [11]; if $A \geq B > 0$ and $0 \leq t < p \leq 1$, then for each δ with $t \leq \delta \leq \min\{1, 2p\}$,

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq A^{\delta},$$

and

$$B^t \natural_{\frac{\delta-t}{p-t}} A^p \geq B^{\delta}.$$

In particular, for each δ with $p \leq \delta \leq \min\{1, 2p\}$, we have another chain of inequalities:

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^{\delta} \leq A^{\delta} \leq B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

Recently, in [9], Furuta, Yamazaki and Yanagida have researched precisely the Furuta type inequalities on the complementary domain, $0 \leq t \leq 1$ and $0 \leq p \leq 1$, and investigate the relations.

In this note, we consider Furuta's type operator inequality in the case of $0 \le \delta \le t < p \le 1$. Then we have the following inequality contrary to the above: If $A \ge B > 0$ and $\delta \le t \le \frac{1+\delta}{2}$, then

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \ge A^\delta \ge B^\delta \ge B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

In particular, if $0 \le t \le \frac{1}{2}$ and $0 \le t < p \le 1$, then

$$A^t \natural_{\frac{-t}{p-t}} B^p \ge 1 \ge B^t \natural_{\frac{-t}{p-t}} A^p.$$

2. Complementary inequalities.

The following lemmas shown in [11] are rewritten for the sake of convenience.

Lemma 1. *If $A \ge B > 0$, then the following inequalities hold;*

(i)
$$A^{-t} \sharp_s B^{-p} \ge A^{-(p-t)s-t}$$

for $0 \le p \le 1, 0 \le s \le 1$ and $t \in \mathbf{R}$,

(ii)
$$A^{-t} \natural_s B^{-p} \ge B^{-(p-t)s-t}$$

for $p \in \mathbf{R}, 1 \le s \le 2$ and $0 \le t \le 1$.

The following is proved from Lemma 1.

Lemma 2. *If $A \ge B > 0$, then the following inequalities hold;*
(i) if $2n \le s \le 2n+1$ and $0 \le p \le 1$, then

$$\begin{aligned} A^{-t} \sharp_s B^{-p} &= (B^{-p}A^t)^n (A^{-t} \sharp_{s-2n} B^{-p})(A^t B^{-p})^n \\ &\ge (B^{-p}A^t)^n A^{-(p-t)(s-2n)-t}(A^t B^{-p})^n, \end{aligned}$$

(ii) if $2n+1 \le s \le 2(n+1)$ and $0 \le t \le 1$, then

$$\begin{aligned} A^{-t} \natural_s B^{-p} &= (B^{-p}A^t)^n (A^{-t} \natural_{s-2n} B^{-p})(A^t B^{-p})^n \\ &\ge (B^{-p}A^t)^n B^{-(p-t)(s-2n)-t}(A^t B^{-p})^n. \end{aligned}$$

The next lemma is necessary to apply the Löwner Heinz inequality in the below.

Lemma 3. *Let $0 \le \delta \le t < p \le 1$ and $(\frac{\delta}{2} \le)t \le \frac{1+\delta}{2}$. Then either*
(1) $2n \le \frac{t-\delta}{p-t} \le 2n+1$, that is, $\frac{t-\delta}{2n+1} \le p-t \le \frac{t-\delta}{2n}$
or
(2) $2n+1 \le \frac{t-\delta}{p-t} \le 2(n+1)$, that is, $\frac{t-\delta}{2(n+1)} \le p-t \le \frac{t-\delta}{2n+1}$
ensures

(a) $0 \le 2(n-l)(p-t)+\delta \le 1$
and
(b) $-1 \le 2(n-l)p-2(n-l+1)t+\delta \le 0,$
where $l = 0, 1, \ldots, n-1$.

First of all, we discuss on the case of $\delta = 0$. Technically we will be along with our preceding argument in [2,3,4,11].

Theorem 1. *Let* $A \ge B > 0$, $0 \le t < p \le 1$ *and* $0 \le t \le \frac{1}{2}$. *Then the following inequality holds;*

$$A^t \natural_{\frac{-t}{p-t}} B^p \ge 1 \ge B^t \natural_{\frac{-t}{p-t}} A^p.$$

Consequently, if $0 \le \delta \le t \le \frac{1}{2}$, *then*

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \ge A^\delta \ge B^\delta \ge B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

Proof. Under the assumption $A \ge B > 0$, $A^t \natural_{\frac{-t}{p-t}} B^p \ge 1$ is equivalent to $1 \ge B^t \natural_{\frac{-t}{p-t}} A^p$. We first consider the cases $\frac{t}{p-t} \in [0,1]$ and $\frac{t}{p-t} \in [1,2]$:

$$\begin{aligned} A^t \natural_{\frac{-t}{p-t}} B^p &= A^t(A^{-t} \sharp_{\frac{t}{p-t}} B^{-p})A^t \\ &\ge A^t(A^{-t} \sharp_{\frac{t}{p-t}} A^{-p})A^t = 1. \end{aligned}$$

If $1 \le \frac{t}{p-t} \le 2$, then

$$\begin{aligned} A^t \natural_{\frac{-t}{p-t}} B^p &= A^t(A^{-t} \natural_{\frac{t}{p-t}} B^{-p})A^t \\ &\ge A^t(B^{-t} \natural_{\frac{t}{p-t}} B^{-p})A^t \ge A^tB^{-2t}A^t \ge 1. \end{aligned}$$

In general, if $2n \le \frac{t}{p-t} \le 2n+1$, then

$$\begin{aligned} A^t \natural_{\frac{-t}{p-t}} B^p &= A^t(A^{-t} \natural_{\frac{t}{p-t}} B^{-p})A^t \\ &= A^t(B^{-p}A^t)^n(A^{-t} \sharp_{\frac{t}{p-t}-2n} B^{-p})(A^tB^{-p})^nA^t \\ &\ge A^t(B^{-p}A^t)^nA^{2np-2(n+1)t}(A^tB^{-p})^nA^t \quad \text{by Lemma 2 (i)} \\ &= A^t(B^{-p}A^t)^{n-1}B^{-p}A^{2n(p-t)}B^{-p}(A^tB^{-p})^{n-1}A^t \\ &\ge A^t(B^{-p}A^t)^{n-1}B^{2(n-1)p-2nt}(A^tB^{-p})^{n-1}A^t \quad \text{by Lemma 3 (a)} \\ &\ge A^t(B^{-p}A^t)^{n-1}A^{2(n-1)p-2nt}(A^tB^{-p})^{n-1}A^t \quad \text{by Lemma 3 (b)} \\ &\ge \ldots \\ &\ge A^t(B^{-p}A^t)^{n-l}A^{2(n-l)p-2(n-l+1)t}(A^tB^{-p})^{n-l}A^t \quad \text{by Lemma 3 (b)} \\ &= A^t(B^{-p}A^t)^{n-l-1}B^{-p}A^{2(n-l)(p-t)}B^{-p}(A^tB^{-p})^{n-l-1}A^t \\ &\ge A^t(B^{-p}A^t)^{n-l-1}B^{2(n-l-1)p-2(n-l)t}(A^tB^{-p})^{n-l-1}A^t \quad \text{by Lemma 3 (a)} \end{aligned}$$

$$
\begin{aligned}
&\geq \ \ldots \\
&\geq \ A^t(B^{-p}A^t)A^{2p-4t}(A^tB^{-p})A^t \\
&= \ A^tB^{-p}A^{2p-2t}B^{-p}A^t \geq A^tB^{-2t}A^t \geq 1.
\end{aligned}
$$

Moreover, if $2n+1 \leq \frac{t}{p-t} \leq 2(n+1)$, then

$$
\begin{aligned}
A^t \natural_{\frac{-t}{p-t}} B^p &= A^t(A^{-t} \natural_{\frac{t}{p-t}} B^{-p})A^t \\
&= A^t(B^{-p}A^t)^n(A^{-t} \natural_{\frac{t}{p-t}-2n} B^{-p})(A^tB^{-p})^nA^t \\
&\geq A^t(B^{-p}A^t)^nB^{2np-2(n+1)t}(A^tB^{-p})^nA^t \quad \textit{by Lemma 2 (ii)} \\
&\geq A^t(B^{-p}A^t)^nA^{2np-2(n+1)t}(A^tB^{-p})^nA^t \quad \textit{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-1}B^{-p}A^{2n(p-t)}B^{-p}(A^tB^{-p})^{n-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-1}B^{2(n-1)p-2nt}(A^tB^{-p})^{n-1}A^t \quad \textit{by Lemma 3 (a)} \\
&\geq \ldots \\
&\geq A^t(B^{-p}A^t)^{n-l}B^{2(n-l)p-2(n-l+1)t}(A^tB^{-p})^{n-l}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l}A^{2(n-l)p-2(n-l+1)t}(A^tB^{-p})^{n-l}A^t \quad \textit{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-l-1}B^{-p}A^{2(n-l)(p-t)}B^{-p}(A^tB^{-p})^{n-l-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l-1}B^{2(n-l-1)p-2(n-l)t}(A^tB^{-p})^{n-l-1}A^t \\
&\geq \ldots \\
&\geq A^t(B^{-p}A^t)A^{2p-4t}(A^tB^{-p})A^t = A^tB^{-p}A^{2p-2t}B^{-p}A^t \\
&\geq A^tB^{-2t}A^t \geq 1.
\end{aligned}
$$

As a consequence, we obtain the second inequality as follows:

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p = A^t \sharp_{\frac{\delta-t}{-t}} (A^t \natural_{\frac{-t}{p-t}} B^p) \geq A^t \sharp_{\frac{\delta-t}{-t}} I = A^\delta.$$

Theorem 2. *Let $A \geq B > 0$, $0 \leq \delta \leq t < p \leq 1$ and $t \leq \frac{1+\delta}{2}$. Then the following inequality holds;*

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \geq A^\delta \geq B^\delta \geq B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

Proof. If $0 \leq \frac{t-\delta}{p-t} \leq 1$, then

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p = A^t(A^{-t} \sharp_{\frac{t-\delta}{p-t}} B^{-p})A^t \geq A^t(A^{-t} \sharp_{\frac{t-\delta}{p-t}} A^{-p})A^t = A^\delta,$$

and if $1 \leq \frac{t-\delta}{p-t} \leq 2$, then

$$A^t \natural_{\frac{-t}{p-t}} B^p = A^t(A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p})A^t \geq A^tB^{\delta-2t}A^t \geq A^\delta.$$

In general, if $2n \leq \frac{t-\delta}{p-t} \leq 2n+1$, then

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \ = \ A^t(A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p})A^t$$

$$
\begin{aligned}
&= A^t(B^{-p}A^t)^n(A^{-t} \natural_{\frac{t-\delta}{p-t}-2n} B^{-p})(A^tB^{-p})^nA^t \\
&\geq A^t(B^{-p}A^t)^n A^{2np-2(n+1)t+\delta}(A^tB^{-p})^nA^t \quad \textit{by Lemma 2 (i)} \\
&= A^t(B^{-p}A^t)^{n-1}B^{-p}A^{2n(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-1}B^{2(n-1)p-2nt+\delta}(A^tB^{-p})^{n-1}A^t \quad \textit{by Lemma 3 (a)} \\
&\geq A^t(B^{-p}A^t)^{n-1}A^{2(n-1)p-2nt+\delta}(A^tB^{-p})^{n-1}A^t \quad \textit{by Lemma 3 (b)} \\
&\geq \cdots \\
&\geq A^t(B^{-p}A^t)^{n-l}A^{2(n-l)p-2(n-l+1)t+\delta}(A^tB^{-p})^{n-l}A^t \quad \textit{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-l-1}B^{-p}A^{2(n-l)(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-l-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l-1}B^{2(n-l-1)p-2(n-l)t+\delta}(A^tB^{-p})^{n-l-1}A^t \quad \textit{by Lemma 3 (a)} \\
&\geq \cdots \\
&\geq A^t(B^{-p}A^t)A^{2p-4t+\delta}(A^tB^{-p})A^t \\
&= A^tB^{-p}A^{2p-2t+\delta}B^{-p}A^t \geq A^tB^{-2t+\delta}A^t \geq A^\delta.
\end{aligned}
$$

On the other hand, if $2n+1 \leq \frac{t-\delta}{p-t} \leq 2(n+1)$, then

$$
\begin{aligned}
A^t \natural_{\frac{\delta-t}{p-t}} B^p &= A^t(A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p})A^t \\
&= A^t(B^{-p}A^t)^n(A^{-t} \natural_{\frac{t-\delta}{p-t}-2n} B^{-p})(A^tB^{-p})^nA^t \\
&\geq A^t(B^{-p}A^t)^n B^{2np-2(n+1)t+\delta}(A^tB^{-p})^nA^t \quad \textit{by Lemma 2 (ii)} \\
&\geq A^t(B^{-p}A^t)^n A^{2np-2(n+1)t+\delta}(A^tB^{-p})^nA^t \quad \textit{by Lemma 3 (b)} \\
&= A^t(B^{-p}A^t)^{n-1}B^{-p}A^{2n(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-1}B^{2(n-1)p-2nt+\delta}(A^tB^{-p})^{n-1}A^t \quad \textit{by Lemma 3 (a)} \\
&\geq \cdots \\
&\geq A^t(B^{-p}A^t)^{n-l}B^{2(n-l)p-2(n-l+1)t+\delta}(A^tB^{-p})^{n-l}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l}A^{2(n-l)p-2(n-l+1)t+\delta}(A^tB^{-p})^{n-l}A^t \\
&= A^t(B^{-p}A^t)^{n-l-1}B^{-p}A^{2(n-l)(p-t)+\delta}B^{-p}(A^tB^{-p})^{n-l-1}A^t \\
&\geq A^t(B^{-p}A^t)^{n-l-1}B^{2(n-l-1)p-2(n-l)t+\delta}(A^tB^{-p})^{n-l-1}A^t \\
&\geq \cdots \\
&\geq A^t(B^{-p}A^t)A^{2p-4t+\delta}(A^tB^{-p})A^t \\
&= A^tB^{-p}A^{2p-2t+\delta}B^{-p}A^t \geq A^tB^{\delta-2t}A^t \geq A^\delta.
\end{aligned}
$$

Remark. The assumption $t \leq \frac{1+\delta}{2}$ is needed to ensure the final inequality $A^tB^{\delta-2t}A^t \geq A^\delta$ in the proofs.

3. Brief proof of Theorem 2. Professor Furuta pointed out that the following known results [11] shorten proofs of Theorems above.

Theorem A. *If* $A \geq B > 0$, *then*
(i) in the case of $\frac{1}{2} \leq p \leq 1$ *and* $0 \leq t < p$,

$$A^t \natural_{\frac{1-t}{p-t}} B^p \le B \le A$$

and (ii) in the case of $0 \le t < p \le \frac{1}{2}$

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \le B^{2p} \le A^{2p}.$$

Brief proof of Theorem 2. The assumption says $0 \le \frac{t-\delta}{1-t} \le 1$. If $\frac{1}{2} \le p \le 1$, then the above (i) implies as follows;

$$\begin{aligned} A^t \natural_{\frac{\delta-t}{p-t}} B^p &= A^t(A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p})A^{-t} \\ &= A^t(A^{-t} \sharp_{\frac{t-\delta}{1-t}} (A^{-t} \natural_{\frac{1-t}{p-t}} B^{-p})A^{-t} \\ &\ge A^t(A^{-t} \sharp_{\frac{t-\delta}{1-t}} A^{-1})A^t = A^\delta. \end{aligned}$$

Suppose $0 \le p \le \frac{1}{2}$. Since $0 \le \frac{t-\delta}{2p-t} \le 1$, a similar calculation leads us the conclusion by the use of the result (ii) of Theorem A.

$$\begin{aligned} A^t \natural_{\frac{\delta-t}{p-t}} B^p &= A^t(A^{-t} \natural_{\frac{t-\delta}{p-t}} B^{-p})A^{-t} \\ &= A^t(A^{-t} \sharp_{\frac{t-\delta}{2p-t}} (A^{-t} \natural_{\frac{2p-t}{p-t}} B^{-p})A^{-t} \\ &\ge A^t(A^{-t} \sharp_{\frac{t-\delta}{2p-t}} A^{-2p})A^t = A^\delta. \end{aligned}$$

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*) Department of Mathematics, Osaka kyoiku University,
Kashiwara, Osaka, 582-8582, Japan
e-mail: mfujii@cc.osaka-kyoiku,ac.jp

**) Maebashi Institute of Technology,
Kamisadori, Maebashi, Gunma, 371-0816, Japan
e-mail: kamei@maebashi-it.ac.jp

***) Higashi-Toyonaka Senior High School,
Shinsenriminami, Toyonaka, Osaka, 565-0084, Japan