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AN APPLICATION OF ALUTHGE TRANSFORM TO PUTNAM INEQUALITY FOR LOG-HYPONORMAL OPERATORS

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ABSTRACT. In this note, we give a short proof to the Putnam inequality for log-hyponormal operators due to Tanahashi: If T is invertible and log-hyponormal, i.e., $\log T^*T \geq \log TT^*$, then

$$\| \log T^*T - \log TT^* \| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{-1}\, dr d\theta,$$

where $\sigma(T)$ is the spectrum of T. It is based on his original idea that the log-hyponormality is regarded as 0-hyponormality.

1. Introduction. After the Furuta inequality was originated by Furuta [12], see also [5,13,16,18], we initiated to study it under the chaotic order $A \gg B$, i.e., $\log A \geq \log B$, for positive invertible operators A and B [11]. We finally characterized $A \gg B$ by a Furuta-type inequality [6] and [7,8]: For $A, B > 0$, $A \gg B$ if and only if

$$(A^r B^p A^r)^{\frac{2r}{p+2r}} \leq A^{2r} \tag{1}$$

holds for all $p, r \geq 0$.

Furthermore we interpolated between the Furuta inequality and Theorem A as follows [9,10]:

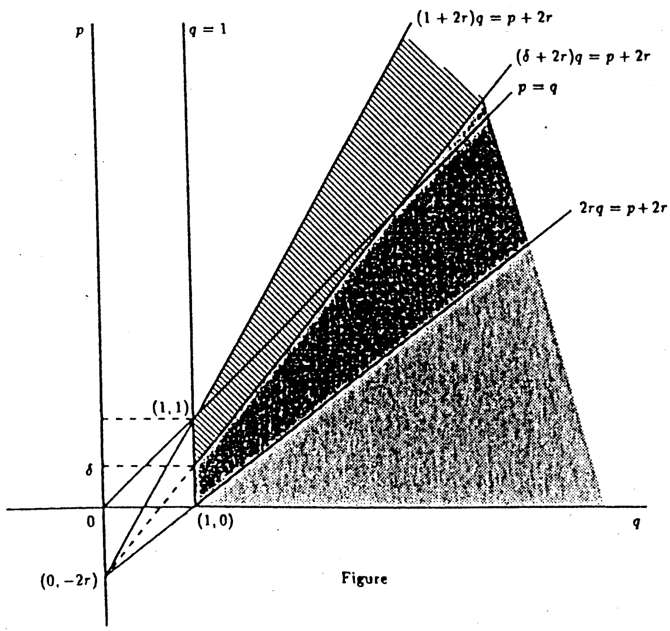
Theorem A. *For a fixed $\delta > 0$, $A^\delta \geq B^\delta$ for $A, B \geq 0$ if and only if for each $r \geq 0$*

$$A^{\frac{p+2r}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}} \tag{2}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(\delta + 2r)q \geq p + 2r. \tag{3}$$

We note that (1) is equivalent to the case $\delta = 0$ in Theorem A and the Furuta inequality is just the case $\delta = 1$. The domain given by (3) is explained by the figure below:



Figure

From the viewpoint of this, Tanahashi [19] defined the log-hyponormality for invertible operators by $|T| \gg |T^*|$, where $|X|$ is the squere root of X^*X, and constructed the Putnam inequality for log-hyponormal operators:

Theorem T. *If T is an invertible log-hyponormal operator, i.e., $\log T^*T - \log TT^* \geq 0$, then*

$$(4) \qquad \|\log T^*T - \log TT^*\| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{-1}\, drd\theta,$$

where $\sigma(T)$ is the spectrum of T.

It was conjectured from the Putnam inequality for p-hyponormal operators by Cho and Itoh [3]:

If T is a p-hyponormal operator, i.e., $(T^*T)^p - (TT^*)^p \geq 0$, then

$$(5) \qquad \|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \iint_{\sigma(T)} r^{2p-1}\, drd\theta.$$

As a matter of fact, he understood (5) as follows:

$$(6) \qquad \|\frac{(T^*T)^p - (TT^*)^p}{p}\| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{2p-1}\, drd\theta.$$

By taking $p \to \infty$, he constructed Theorem T and proved it by the idea developed in (5).

The purpose of this note is to continue his consideration directly. That is, we here propose a straight and simple proof of Theorem T which might be along with his intention;Tanahashi might regards the log-hyponormality as the 0-hyponormality. Our tool in this note is the Aluthge transform which is grown up the p-hyponormality, see [1,2,4,14,15,20].

2. Preliminary.

For the sake of convenience, we cite the following characterization of chaotic order which implies (1) by the help of the Furuta inequality [7], see also [8] and [9].

Theorem B. *For $A, B > 0$, $A \gg B$, i.e., $\log\ A \geq \log\ B$, if and only if for any $\delta \in (0,1]$ there exists an $\alpha = \alpha_\delta > 0$ such that*

$$(e^\delta A)^\alpha > B^\alpha.$$

The essential part of Theorem B is as follows: If A and B are selfadjoint and $A > B$, then there exists an $\alpha \in (0,1]$ such that

$$(*) \qquad e^{\alpha A} > e^{\alpha B}.$$

It has the following simple proof: The assumption $A > B$ means that $A - B \geq \epsilon > 0$ for some ϵ. We here take $0 < \alpha < \epsilon/(e^{\|A\|} + e^{\|B\|})$ and $\alpha \leq 1$. Then we have

$$\begin{aligned}
e^{\alpha A} - e^{\alpha B} &= \alpha(A - B) + \sum_{n=2}^{\infty} \frac{\alpha^n}{n!}(A^n - B^n) \\
&\geq \alpha\epsilon + \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n) \\
&\geq \alpha\epsilon - \alpha^2 \|\sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n)\| \\
&\geq \alpha\epsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!}(\|A\|^n + \|B\|^n) \\
&\geq \alpha(\epsilon - \alpha(e^{\|A\|} + e^{\|B\|})) > 0.
\end{aligned}$$

Here we should note an interesting characterization of chaotic order recently obtained by Yamazaki and Yanagida [22], which is associated with Kantorovich inequality and consequently Specht's ratio, see [23].

3. Proof.

We begin with the Putnam inequality for p-hyponormal operators; we turn it into the following lemma via the Löwner-Heinz inequality:

Lemma 1. *If T is a q-hyponormal operator, then T satisfies (6) for all $0 < p \le q$. Consequently (4) holds for any q-hyponormal operators T.*

The second half is ensured by the fact that u-$\lim_{t\to 0} \frac{A^t-1}{t} = \log A$ for a positive invertible operator A.

Thus Lemma 1 suggests us to find a family $\{T_q; q>0\}$ of q-hyponormal operators such that $\|T_q - T\| \to 0$ as $q \to 0$ for a given log-hyponormal operator T. In this situation, the Aluthge transform completely responds to our demand. As a matter of fact, Tanahashi prepared the following result in [20]:

Lemma 2. *If T is an invertible log-hyponormal operator with the polar decomposition $T = U|T|$, then the Aluthge transform $\tilde{T} = |T|^q U |T|^{1-q}$ is q-hyponormal for $0 < q \le \frac{1}{2}$.*

To prove Theorem T, we take $T_q = |T|^q U |T|^{1-q}$ for $0 < q < \frac{1}{2}$. Since $\sigma(T_q) = \sigma(T)$ for all q, it is complete.

We finally give a short proof to Lemma 2 via Theorem A

Putting $p = 2q$ and $r = 1-q$ in (1), we have

$$\begin{aligned}(T_q^*T_q)^{1-q} &= (|T|^{1-q}U^*|T|^{2q}U|T|^{1-q})^{1-q}\\ &= U^*(U|T|^{1-q}U^*|T|^{2q}U|T|^{1-q}U^*)^{1-q}U\\ &= U^*(|T^*|^{1-q}|T|^{2q}|T^*|^{1-q})^{1-q}U\\ &\ge U^*|T^*|^{2(1-q)}U\\ &= |T|^{2(1-q)},\end{aligned}$$

so that $(T_q^*T_q)^{2q} \ge |T|^{2q}$ via the Löwner-Heinz inequality by $1-q \ge q$. On the other hand, we have also

$$\begin{aligned}(T_qT_q^*)^q &= (|T|^qU|T|^{2(1-q)}U^*|T|^q)^q\\ &= (|T|^q|T^*|^{2(1-q)}|T|^q)^q\\ &\le |T|^{2q}.\end{aligned}$$

Therefore it follows that

$$(T_q^*T_q)^q \ge |T|^{2q} \ge (T_qT_q^*)^q,$$

as desired.

Remark. (1) (5) is obtained by Putnam [17] for $p = 1$, Xia [21] for $\frac{1}{2} \le p < 1$ and Cho-Itoh [3] for $0 < p < \frac{1}{2}$.

(2) The proof of Lemma 2 is available to show Tanahashi's result [20; Theorem 4].

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