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Characterizations of chaotic order associated with Kantorovich inequality

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Abstract

By using the order preserving operator inequality shown in [11] which is associated with Kantorovich inequality, we shall give some characterizations of chaotic order.

1 Introduction

An operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also, an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible.

$A \geq B \geq 0$ ensures $A^p \geq B^p$ for any $p \in [0,1]$ by well-known Löwner-Heinz theorem. However, it is also well known that $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for any $p > 1$. Related to this result, the following result is given in [5].

Theorem A ([5]). *If $A \geq B > 0$ and $MI \geq B \geq mI > 0$, then*

$$\left(\frac{M}{m}\right)^p A^p \geq B^p \quad \text{for } p \geq 1.$$

Recently, more precise estimation than Theorem A was given in [11] as follows:

Theorem B ([11]). *If $A \geq B > 0$ and $MI \geq B \geq mI > 0$, then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m, M, p) A^p \geq B^p \quad \text{for } p \geq 1, \tag{1.1}$$

where

$$K_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}. \tag{1.2}$$

Theorem B is related to both Hölder-McCarthy inequality [13] and Kantorovich inequality: *If A is an operator on a Hilbert space H such that $MI \geq A \geq mI > 0$, then $(A^{-1}x, x)(Ax, x) \leq (m+M)^2/4mM$ holds for every unit vector x in H.* Many authors investigated a lot of papers on Kantorovich inequality, among others, there is a long research series of Mond-Pečarić, some of them are [14] and [15].

The following Theorem F is an extension of the Löwner-Heinz theorem:

Theorem F (Furuta inequality [7]).

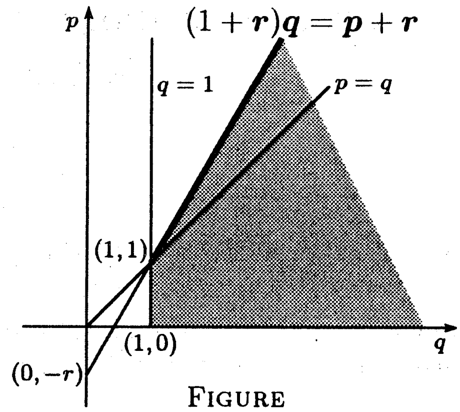
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.



FIGURE

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [3][12] and also an elementary one-page proof in [8]. It is shown in [17] that the domain drawn for p, q and r in the Figure is best possible one for Theorem F.

Ando [1] shows that $\log A \geq \log B$ (so called chaotic order) is equivalent to $(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geq B^p$ for all $p \geq 0$. By using Theorem F, a generalization of Ando's characterization is given as follows:

Theorem C ([4][6][9]). *Let* A *and* B *be positive and invertible operators on a Hilbert space* H. *Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ for all $p \geq 0$ and $r \geq 0$.

In this paper, we shall give some characterizations of chaotic order by applying Theorem B and Theorem C.

2 Results

Theorem 1. *Let* A *and* B *be positive and invertible operators on a Hilbert space* H *satisfying* $\log A \geq \log B$ *and* $MI \geq B \geq mI > 0$. *Then*

$$\left(\frac{M}{m}\right)^p A^p \geq K_+(m, M, p+1)A^p \geq B^p \quad \text{for } p \geq 0, \tag{2.1}$$

where $K_+(m, M, p)$ *is defined in* (1.2).

Theorem 1 can be considered as an extension of Theorem A. Moreover, we obtain a new characterization of chaotic order as follows:

Theorem 2. *Let* A *and* B *be positive and invertible operators on a Hilbert space* H *satisfying* $MI \geq B \geq mI > 0$. *Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $\dfrac{(m^p + M^p)^2}{4m^pM^p}A^p \geq B^p$ for all $p \geq 0$.

As a generalization of both Theorem 1 and (i) $\Longrightarrow$ (ii) of Theorem 2, we show the following result.

Theorem 3. *Let A and B be positive and invertible operators on a Hilbert space H satisfying* $\log A \geq \log B$ *and* $MI \geq B \geq mI > 0$. *Then*

$$K_+\left(m^r, M^r, 1 + \frac{p}{r}\right)A^p \geq B^p \qquad \textit{for } p > 0 \textit{ and } r > 0, \tag{2.2}$$

where $K_+(m, M, p)$ is defined in (1.2).

Theorem 3 implies Theorem 1 when we put $r = 1$ in Theorem 3. And also Theorem 3 yields (i) $\Longrightarrow$ (ii) of Theorem 2 when we put $r = p$ in Theorem 3. Related to $K_+(m, M, p)$ in (1.2), we obtain the following proposition.

Proposition 4. *Let $K_+(m, M, p)$ be defined in* (1.2). *Then*

$$F(p, r, m, M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

is an increasing function of p, r and M, and also a decreasing function of m for $p > 0$, $r > 0$ and $M > m > 0$. And the following inequality holds:

$$\left(\frac{M}{m}\right)^p \geq K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geq 1 \qquad \text{for any } p > 0,\ r > 0 \text{ and } M > m > 0. \tag{2.3}$$

By considering Proposition 4, we obtain a more precise characterization of chaotic order than Theorem 2.

Theorem 5. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geq B \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $M_h(p)A^p \geq B^p$ holds for all $p > 0$, where $h = \frac{M}{m} > 1$ and

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log(h^{\frac{p}{h^p-1}})}. \tag{2.4}$$

We remark that $M_h(1) = \dfrac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$ is called Specht's ratio [2][16].

3 Proof of results

Proof of Theorem 1. Put $r = 1$ in (ii) of Theorem C, then $\log A \geq \log B$ ensures the following inequality:

$$(B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{1}{p+1}} \geq B \quad \text{for } p \geq 0.$$

Put $A_1 = (B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{1}{p+1}}$ and $B_1 = B$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$ and $M \geq B_1 \geq m > 0$. Applying Theorem B to A_1 and B_1, we have

$$\left(\frac{M}{m}\right)^{p_1-1}(B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{p_1}{p+1}} \geq K_+(m,M,p_1)(B^{\frac{1}{2}}A^pB^{\frac{1}{2}})^{\frac{p_1}{p+1}} \geq B^{p_1} \quad \text{for } p \geq 0 \text{ and } p_1 \geq 1. \tag{3.1}$$

Put $p_1 = p+1 \geq 1$ in (3.1) and multiply $B^{\frac{-1}{2}}$ on both sides, then we have

$$\left(\frac{M}{m}\right)^p A^p \geq K_+(m,M,p+1)A^p \geq B^p \quad \text{for } p \geq 0. \tag{2.1}$$

Hence the proof of Theorem 1 is complete. □

In order to give a proof of Theorem 2, we need the following lemma.

Lemma 6. *If $m > 0$ and $M > 0$, then*

$$\lim_{p\to 0}\left\{\frac{(m^p+M^p)^2}{4m^pM^p}\right\}^{\frac{1}{p}} = 1.$$

Proof. Noting that

$$\lim_{p\to 0}\left(\frac{m^p+M^p}{2}\right)^{\frac{1}{p}} = \sqrt{mM},$$

we have

$$\lim_{p\to 0}\left\{\frac{(m^p+M^p)^2}{4m^pM^p}\right\}^{\frac{1}{p}} = \lim_{p\to 0}\frac{1}{mM}\left(\frac{m^p+M^p}{2}\right)^{\frac{2}{p}} = \frac{1}{mM}\left(\sqrt{mM}\right)^2 = 1.$$ □

Proof of Theorem 2.

(a) *Proof of* (i) $\Longrightarrow$ (ii). Put $r = p$ in (ii) of Theorem C, then $\log A \geq \log B$ ensures the following inequality:

$$(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geq B^p \quad \text{for } p \geq 0.$$

Put $A_1 = (B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}}$ and $B_1 = B^p$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$ and $M^p \geq B_1 \geq m^p > 0$. Applying Theorem B to A_1 and B_1, we have

$$K_+(m^p,M^p,p_1)(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{p_1}{2}} \geq (B^p)^{p_1} \quad \text{for } p \geq 0 \text{ and } p_1 \geq 1. \tag{3.2}$$

Put $p_1 = 2 \geq 1$ in (3.2) and multiply $B^{\frac{-p}{2}}$ on both sides, then we have

$$K_+(m^p,M^p,2)A^p \geq B^p \quad \text{for } p \geq 0.$$

Hence the proof of (i) $\Longrightarrow$ (ii) is complete since $K_+(m^p,M^p,2) = \dfrac{(m^p+M^p)^2}{4m^pM^p}$.

(b) *Proof of* (ii) $\Longrightarrow$ (i). Taking logarithm of both sides of (ii) since $\log t$ is an operator monotone function, we have

$$\log\left\{\left(\frac{(m^p+M^p)^2}{4m^pM^p}\right)^{\frac{1}{p}}A\right\}\geq\log B\quad\text{for all } p\geq 0. \tag{3.3}$$

Letting $p\to+0$ in (3.3), we have $\log A\geq\log B$ by Lemma 6. □

Proof of Theorem 3. By Theorem C, $\log A\geq\log B$ is equivalent to the following inequality:

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}\geq B^r\qquad\text{for } p>0 \text{ and } r>0.$$

Put $A_1=(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}$ and $B_1=B^r$, then A_1 and B_1 satisfy $A_1\geq B_1>0$ and $M^r\geq B_1\geq m^r>0$. Applying Theorem B to A_1 and B_1, we have

$$K_+(m^r,M^r,p_1)A_1^{p_1}\geq B_1^{p_1}\qquad\text{for } p_1\geq 1. \tag{3.4}$$

Put $p_1=\frac{p+r}{r}\geq 1$ in (3.4), then we have

$$K_+\left(m^r,M^r,\frac{p+r}{r}\right)B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\geq B^{p+r}. \tag{3.5}$$

By multiplying $B^{\frac{-r}{2}}$ on both sides of (3.5), we have

$$K_+\left(m^r,M^r,1+\frac{p}{r}\right)A^p\geq B^p\qquad\text{for } p>0 \text{ and } r>0. \tag{2.2}$$

Hence the proof of Theorem 3 is complete. □

We prepare the following four lemmas to give a proof of Proposition 4.

Lemma 7. *For each $h>1$,*

$$f(t)=\log\left(\frac{h^t-1}{t}\right) \tag{3.6}$$

is a convex function for $t>0$.

Proof. Put $x(t)=\dfrac{h^t-1}{t}$, then $f(t)=\log\{x(t)\}$ and

$$f''(t)=\frac{x(t)x''(t)-\{x'(t)\}^2}{\{x(t)\}^2},$$

so that $f''(t)\geq 0$ for $t>0$ is equivalent to the following (3.7) since $\{x(t)\}^2\geq 0$:

$$x(t)x''(t)-\{x'(t)\}^2\geq 0\qquad\text{for } t>0. \tag{3.7}$$

By calculation on differential calculus and refinement, we have

$$x(t)x''(t) - \{x'(t)\}^2 = \frac{1}{t^4}(h^t - 1 + th^{\frac{t}{2}}\log h)(h^t - 1 - th^{\frac{t}{2}}\log h),$$

so that (3.7) is equivalent to the following (3.8) because $h^t - 1 + th^{\frac{t}{2}}\log h \geq 0$ for $h > 1$ and $t > 0$:

$$h^t - 1 - th^{\frac{t}{2}}\log h \geq 0 \qquad \text{for } h > 1 \text{ and } t > 0. \tag{3.8}$$

Put $y(t) = h^t - 1 - th^{\frac{t}{2}}\log h$. Then $y(0) = 0$ and

$$y'(t) = h^{\frac{t}{2}}\log h(h^{\frac{t}{2}} - 1 - \log h^{\frac{t}{2}}),$$

so that $y'(t) > 0$ for $h > 1$ and $t > 0$. Therefore $y(t) \geq 0$ for $h > 1$ and $t > 0$, which is equivalent to (3.8). Consequently, the proof of Lemma 7 is complete. □

Lemma 8. *Let $h > 1$. Then*

$$g(p, r, h) = \left(\frac{r}{p+r}\frac{h^{p+r} - 1}{h^r - 1}\right)^{\frac{1}{p}} \tag{3.9}$$

is an increasing function of p and r for $p > 0$ and $r > 0$.

Proof. Define $f(t)$ as in Lemma 7, i.e.,

$$f(t) = \log\left(\frac{h^t - 1}{t}\right). \tag{3.6}$$

Then by (3.9),

$$\log\{g(p, r, h)\} = \frac{\log\left(\frac{h^{p+r} - 1}{p+r}\right) - \log\left(\frac{h^r - 1}{r}\right)}{p} = \frac{f(p+r) - f(r)}{p}. \tag{3.10}$$

(a) *Proof of the result that $g(p, r, h)$ is increasing for $p > 0$.*

Let $p_1 \geq p_2 > 0$ and $r > 0$. Since $f(t)$ is convex for $t > 0$ by Lemma 7,

$$\theta f(t_1) + (1 - \theta)f(t_2) \geq f(\theta t_1 + (1 - \theta)t_2) \tag{3.11}$$

holds for $\theta \in [0, 1]$, $t_1 > 0$ and $t_2 > 0$. Put $\theta = \frac{p_2}{p_1} \in [0, 1]$, $t_1 = p_1 + r > 0$ and $t_2 = r > 0$, then

$$\theta t_1 + (1 - \theta)t_2 = \frac{p_2}{p_1}(p_1 + r) + \left(1 - \frac{p_2}{p_1}\right)r = p_2 + r. \tag{3.12}$$

By (3.11) and (3.12), we have

$$\frac{p_2}{p_1}f(p_1 + r) + \left(1 - \frac{p_2}{p_1}\right)f(r) \geq f(p_2 + r),$$

so that

$$\frac{f(p_1+r)-f(r)}{p_1} \geq \frac{f(p_2+r)-f(r)}{p_2}. \tag{3.13}$$

By (3.10) and (3.13), $g(p,r,h)$ is increasing for $p > 0$.

(b) *Proof of the result that $g(p,r,h)$ is increasing for $r > 0$.*

Let $r_1 \geq r_2 > 0$ and $p > 0$. Since $f(t)$ is convex for $t > 0$ by Lemma 7, $f''(t) \geq 0$, so that $f'(t)$ is increasing, that is, $f'(t+r_1) - f'(t+r_2) \geq 0$. Therefore $s(t) = f(t+r_1) - f(t+r_2)$ is increasing for $t \geq 0$. Then we have $f(p+r_1) - f(p+r_2) = s(p) \geq s(0) = f(r_1) - f(r_2)$, that is,

$$\frac{f(p+r_1)-f(r_1)}{p} \geq \frac{f(p+r_2)-f(r_2)}{p}. \tag{3.14}$$

By (3.10) and (3.14), $g(p,r,h)$ is increasing for $r > 0$.

Consequently the proof of Lemma 8 is complete. □

Lemma 9. *For $p \geq 1$ and $t > 1$,*

$$pt^{p-1} \geq \frac{t^p - 1}{t-1} \geq pt^{\frac{p-1}{2}}. \tag{3.15}$$

Proof. To prove the first inequality of (3.15), define $h(t) = t^p$. Since $h(t)$ is a convex function of t for $p \geq 1$, we have $h'(t) \geq \dfrac{h(t)-h(1)}{t-1}$ for $t > 1$, which is equivalent to the first inequality of (3.15). On the other hand, the second inequality of (3.15) is equivalent to the following:

$$t^p - pt^{\frac{p+1}{2}} + pt^{\frac{p-1}{2}} - 1 \geq 0 \qquad \text{for } p \geq 1 \text{ and } t > 1. \tag{3.16}$$

So we have only to prove (3.16). Put $f(t) = t^p - pt^{\frac{p+1}{2}} + pt^{\frac{p-1}{2}} - 1$. Then $f(1) = 0$ and

$$\begin{aligned} f'(t) &= pt^{p-1} - \frac{p(p+1)}{2}t^{\frac{p-1}{2}} + \frac{p(p-1)}{2}t^{\frac{p-3}{2}} \\ &= pt^{\frac{p-3}{2}}\left(t^{\frac{p+1}{2}} - \frac{p+1}{2}t + \frac{p-1}{2}\right). \end{aligned} \tag{3.17}$$

Put $g(t) = t^{\frac{p+1}{2}} - \frac{p+1}{2}t + \frac{p-1}{2}$, then $g'(t) = \frac{p+1}{2}t^{\frac{p-1}{2}} - \frac{p+1}{2} \geq 0$ for $p \geq 1$ and $t > 1$, and also $g(1) = 0$. Therefore $g(t) \geq 0$ for $p \geq 1$ and $t > 1$, so that $f'(t) = pt^{\frac{p-3}{2}}g(t) \geq 0$ for $p \geq 1$ and $t > 1$ by (3.17). Hence $f(t) \geq 0$ for $p \geq 1$ and $t > 1$, which is equivalent to (3.16). Consequently the proof of Lemma 9 is complete. □

Lemma 10. *For $p > 0$, $r > 0$ and $h > 1$,*

$$h \geq g(p,r,h) \geq h^{\frac{1}{2}}, \tag{3.18}$$

where $g(p,r,h)$ is as in Lemma 8, i.e.,

$$g(p,r,h) = \left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{p}}. \tag{3.9}$$

Proof. Replace p with $\frac{p+r}{r} \geq 1$ in Lemma 9, we have the following inequality.

$$\left(\frac{p+r}{r}\right)t^{\frac{p}{r}} \geq \frac{t^{\frac{p+r}{r}-1}}{t-1} \geq \left(\frac{p+r}{r}\right)t^{\frac{p}{2r}} \quad \text{for } p > 0,\ r > 0 \text{ and } t > 1. \tag{3.19}$$

Put $t = h^r > 1$ in (3.19). Then we have

$$h^p \geq \frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1} \geq h^{\frac{p}{2}} \quad \text{for } p > 0,\ r > 0 \text{ and } h > 1, \tag{3.20}$$

therefore we have (3.18) by taking $\frac{1}{p}$ exponent of each side of (3.20). □

Proof of Proposition 4. Put $h = \frac{M}{m} > 1$ and $g(p,r,h)$ is as in Lemma 8, i.e.,

$$g(p,r,h) = \left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{p}}. \tag{3.9}$$

Then

$$\begin{aligned}
K_+\left(m^r, M^r, \frac{p+r}{r}\right) &= \frac{\left(\frac{p}{r}\right)^{\frac{p}{r}}}{\left(1+\frac{p}{r}\right)^{1+\frac{p}{r}}}\frac{(M^{p+r}-m^{p+r})^{1+\frac{p}{r}}}{(M^r-m^r)(m^rM^{p+r}-M^rm^{p+r})^{\frac{p}{r}}} && \text{by (1.2)} \\
&= \left(\frac{r}{p+r}\right)\left(\frac{p}{p+r}\right)^{\frac{p}{r}}\frac{(h^{p+r}-1)^{1+\frac{p}{r}}}{(h^r-1)(h^{p+r}-h^r)^{\frac{p}{r}}} && \text{by } h = \frac{M}{m} > 1 \\
&= \left\{\frac{1}{h}\left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{p}}\left(\frac{p}{p+r}\frac{h^{p+r}-1}{h^p-1}\right)^{\frac{1}{r}}\right\}^p && (3.21) \\
&= \left\{\frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h)\right\}^p && \text{by (3.9)}.
\end{aligned}$$

By Lemma 10, we have the following (3.22).

$$h \geq \frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h) \geq 1 \quad \text{for } p > 0 \text{ and } r > 0. \tag{3.22}$$

By (3.21) and (3.22), we have (2.3), i.e.,

$$\left(\frac{M}{m}\right)^p \geq K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geq 1 \quad \text{for any } p > 0,\ r > 0 \text{ and } M > m > 0. \tag{2.3}$$

(a) *Proof of the result that* $F(p,r,m,M) = K_+(m^r, M^r, \frac{p+r}{r})$ *is increasing for* $p > 0$ *and* $r > 0$.

By Lemma 8, $g(p,r,h)$ is increasing for $p > 0$ and $r > 0$. Then we obtain that $g(p,r,h)\cdot g(r,p,h)$ is increasing for $p > 0$ and $r > 0$. By (3.21) and (3.22), $F(p,r,m,M) = K_+(m^r, M^r, \frac{p+r}{r})$ is increasing for $p > 0$ and $r > 0$.

(b) *Proof of the result that* $F(p,r,m,M) = K_+(m^r, M^r, \frac{p+r}{r})$ *is an increasing function of* M *and also a decreasing function of* m *for* $M > m > 0$.

Firstly, for $s > 0$,

$$
\begin{aligned}
g\left(\frac{p}{s},\frac{r}{s},h^s\right) &= \left(\frac{\frac{r}{s}}{\frac{p}{s}+\frac{r}{s}}\frac{(h^s)^{\frac{p}{s}+\frac{r}{s}}-1}{(h^s)^{\frac{r}{s}}-1}\right)^{\frac{s}{p}} \\
&= \left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{s}{p}} \\
&= \{g(p,r,h)\}^s,
\end{aligned}
\tag{3.23}
$$

so that $g(p,r,h) = \{g(\frac{p}{s},\frac{r}{s},h^s)\}^{\frac{1}{s}}$ for $s > 0$. Then for $s > 1$, we have

$$
\begin{aligned}
K_+\left(m^r, M^r, \frac{p+r}{r}\right) &= \left\{\frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h)\right\}^p && \text{by (3.21)} \\
&= \left\{\frac{1}{h^s}\cdot g\left(\frac{p}{s},\frac{r}{s},h^s\right)\cdot g\left(\frac{r}{s},\frac{p}{s},h^s\right)\right\}^{\frac{p}{s}} && \text{by (3.23)} \\
&\le \left\{\frac{1}{h^s}\cdot g(p,r,h^s)\cdot g(r,p,h^s)\right\}^p && \text{by the result of (a)} \\
&= K_+\left(m^r, (h^{s-1}M)^r, \frac{p+r}{r}\right) && \text{since } h^s = \frac{h^{s-1}M}{m},
\end{aligned}
\tag{3.24}
$$

so that $K_+(m^r, M^r, \frac{p+r}{r})$ is an increasing function of M for $M > m > 0$ since $h^{s-1}M > M$. On the other hand, by the same way as (3.24) we have

$$
K_+\left(m^r, M^r, \frac{p+r}{r}\right) \le \left\{\frac{1}{h^s}\cdot g(p,r,h^s)\cdot g(r,p,h^s)\right\}^p = K_+\left((h^{1-s}m)^r, M^r, \frac{p+r}{r}\right),
$$

since $h^s = \dfrac{M}{h^{1-s}m}$. Hence $K_+(m^r, M^r, \frac{p+r}{r})$ is a decreasing function of m for $M > m > 0$ since $m > h^{1-s}m$.

By (a) and (b), the proof of Proposition 4 is complete. □

We need the following lemmas to give a proof of Theorem 5.

Lemma 11. *Let* $M > m > 0$, $p > 0$ *and* $K_+(m,M,p)$ *be defined in* (1.2). *Then*

$$
\lim_{r\to+0} K_+\left(m^r, M^r, 1+\frac{p}{r}\right) = M_h(p),
$$

where $h = \frac{M}{m} > 1$, *and* $M_h(p)$ *is defined in* (2.4).

Proof. Define $g(p,r,h)$ as in Lemma 8, i.e.,

$$g(p,r,h) = \left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{p}}. \tag{3.9}$$

As in the proof of Proposition 4, we have

$$K_+\left(m^r, M^r, \frac{p+r}{r}\right) = \left\{\frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h)\right\}^p. \tag{3.21}$$

We define $f(t)$ as follows:

$$f(t) = \log(h^t - 1). \tag{3.25}$$

Then

$$f'(t) = \frac{h^t \log h}{h^t - 1} = \log h^{\frac{h^t}{h^t-1}}, \tag{3.26}$$

so that

$$\begin{aligned}
\lim_{r\to+0}\log\left(\frac{h^{p+r}-1}{h^p-1}\right)^{\frac{1}{r}} &= \lim_{r\to+0}\frac{\log(h^{p+r}-1)-\log(h^p-1)}{r}\\
&= \lim_{r\to+0}\frac{f(p+r)-f(p)}{r} \qquad \text{by (3.25)}\\
&= f'(p)\\
&= \log h^{\frac{h^p}{h^p-1}} \qquad \text{by (3.26)},
\end{aligned}$$

therefore $\lim_{r\to+0}\left(\frac{h^{p+r}-1}{h^p-1}\right)^{\frac{1}{r}} = h^{\frac{h^p}{h^p-1}}$. Since $\lim_{r\to+0}\left(1+\frac{r}{p}\right)^{\frac{p}{r}} = e$ and $\lim_{r\to+0}\frac{h^r-1}{r} = \log h$, we have

$$\lim_{r\to+0} g(p,r,h) = \lim_{r\to+0}\left(\frac{h^{p+r}-1}{p+r}\frac{r}{h^r-1}\right)^{\frac{1}{p}} = \left(\frac{h^p-1}{p\log h}\right)^{\frac{1}{p}} = \left(\frac{1}{\log h^{\frac{p}{h^p-1}}}\right)^{\frac{1}{p}} \tag{3.27}$$

and

$$\lim_{r\to+0} g(r,p,h) = \lim_{r\to+0}\left(\frac{p}{p+r}\right)^{\frac{1}{r}}\left(\frac{h^{p+r}-1}{h^p-1}\right)^{\frac{1}{r}} = \frac{h^{\frac{h^p}{h^p-1}}}{e^{\frac{1}{p}}}. \tag{3.28}$$

Applying (3.27) and (3.28) in (3.21), we have

$$\begin{aligned}
\lim_{r\to+0} K_+\left(m^r, M^r, \frac{p+r}{r}\right) &= \lim_{r\to+0}\left\{\frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h)\right\}^p \qquad \text{by (3.21)}\\
&= \frac{1}{h^p}\cdot\frac{1}{\log h^{\frac{p}{h^p-1}}}\cdot\frac{h^{\frac{ph^p}{h^p-1}}}{e} \qquad \text{by (3.27) and (3.28)}\\
&= \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}}.
\end{aligned}$$

Hence the proof of Lemma 11 is complete. □

Lemma 12. *Let $h > 1$ and $M_h(p)$ be defined in* (2.4). *Then*

$$\lim_{p\to+0}\{M_h(p)\}^{\frac{1}{p}} = 1.$$

Proof. Put $g(p) = h^{\frac{p}{h^p-1}}$, then $M_h(p) = \dfrac{g(p)}{e\log g(p)}$. It is easily obtained that

$$\lim_{p\to+0} g(p) = h^{\frac{1}{\log h}} = e$$

and

$$g'(p) = \left\{\frac{h^p - 1 - ph^p\log h}{(h^p-1)^2}\right\} h^{\frac{p}{h^p-1}}\log h.$$

Then $g'(p)$ is bounded as $p \to +0$ since

$$\begin{aligned}\lim_{p\to+0}\frac{h^p - 1 - ph^p\log h}{(h^p-1)^2} &= \lim_{p\to+0}\frac{-ph^p\{\log h\}^2}{2(h^p-1)h^p\log h} \quad \text{by L'Hospital's theorem}\\ &= \lim_{p\to+0}\frac{-p\log h}{2(h^p-1)}\\ &= \frac{-1}{2}.\end{aligned}$$

Then we have

$$\begin{aligned}\lim_{p\to+0}\log\{M_h(p)\}^{\frac{1}{p}} &= \lim_{p\to+0}\frac{\log g(p) - \log\{\log g(p)\} - 1}{p}\\ &= \lim_{p\to+0}\frac{g'(p)}{g(p)}\left\{1 - \frac{1}{\log g(p)}\right\} \quad \text{by L'Hospital's theorem}\\ &= 0,\end{aligned}$$

so that $\lim_{p\to+0}\{M_h(p)\}^{\frac{1}{p}} = 1$. Hence the proof of Lemma 12 is complete. □

Proof of Theorem 5.

(a) *Proof of* (i) $\Longrightarrow$ (ii). By Theorem 3, $\log A \geq \log B$ implies

$$K_+\left(m^r, M^r, 1 + \frac{p}{r}\right) A^p \geq B^p \quad \text{for } p > 0 \text{ and } r > 0. \tag{2.2}$$

Letting $r \to +0$ in (2.2), we have $M_h(p)A^p \geq B^p$ for $p > 0$ since $K_+(m^r, M^r, 1 + \frac{p}{r}) \to M_h(p)$ as $r \to +0$ by Lemma 11.

(b) *Proof of* (ii) $\Longrightarrow$ (i). By taking logarithm of both sides of (ii), we have

$$\log(\{M_h(p)\}^{\frac{1}{p}}A) \geq \log B \quad \text{for } p > 0. \tag{3.29}$$

Then letting $p \to +0$ in (3.29), we have $\log A \geq \log B$ since $\{M_h(p)\}^{\frac{1}{p}} \to 1$ as $p \to +0$ by Lemma 12.

Hence the proof of Theorem 5 is complete. □

4 Concluding Remarks

Remark 1. Let A and B be positive and invertible operators on a Hilbert space H. We consider an order $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ which interpolates usual order $A \geq B$ and chaotic order $\log A \geq \log B$ continuously. The following result is easily obtained by Theorem B.

Proposition 13. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ and $MI \geq B \geq mI > 0$, then*

$$K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right)A^p \geq B^p \qquad \text{for } p \geq \delta,$$

where $K_+(m,M,p)$ is defined in (1.2).

Proof. Put $A_1 = A^\delta$ and $B_1 = B^\delta$, then $A_1 \geq B_1 > 0$ and $M^\delta \geq B_1 \geq m^\delta$. By applying Theorem B to A_1 and B_1, we have

$$K_+(m^\delta, M^\delta, p_1)A_1^{p_1} \geq B_1^{p_1} \quad \text{for } p_1 \geq 1. \tag{4.1}$$

Put $p_1 = \frac{p}{\delta} \geq 1$ in (4.1), then we have

$$K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right)A^p \geq B^p \quad \text{for } p \geq \delta. \qquad \square$$

We show the following result to consider the relation between Proposition 13 and Theorem 5.

Proposition 14. *Let $K_+(m,M,p)$ and $M_h(p)$ be defined in* (1.2) *and* (2.4), *respectively. Then for $p > 0$ and $M > m > 0$,*

$$\lim_{\delta \to +0} K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right) = M_h(p),$$

where $h = \frac{M}{m} > 1$.

Proposition 14 can be proved by the same way as Lemma 11.

Proof. Define $g(p,r,h)$ as in Lemma 8, i.e.,

$$g(p,r,h) = \left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{p}}. \tag{3.9}$$

By (3.21), we have

$$\begin{aligned} K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right) &= K_+\left(m^\delta, M^\delta, \frac{(p-\delta)+\delta}{\delta}\right) \\ &= \left\{\frac{1}{h}\cdot g(p-\delta,\delta,h)\cdot g(\delta,p-\delta,h)\right\}^{p-\delta} \qquad \text{by (3.21)}. \end{aligned} \tag{4.2}$$

We define $f(t)$ as follows:

$$f(t) = \log(h^t - 1). \tag{3.25}$$

Then

$$f'(t) = \frac{h^t \log h}{h^t - 1} = \log h^{\frac{h^t}{h^t-1}}, \tag{3.26}$$

so that

$$\begin{aligned}
\lim_{\delta\to+0} \log\left(\frac{h^p-1}{h^{p-\delta}-1}\right)^{\frac{1}{\delta}} &= \lim_{\delta\to+0} \frac{\log(h^p-1) - \log(h^{p-\delta}-1)}{\delta} \\
&= \lim_{\delta\to+0} \frac{f(p) - f(p-\delta)}{\delta} \qquad \text{by (3.25)} \\
&= f'(p) \\
&= \log h^{\frac{h^p}{h^p-1}} \qquad \text{by (3.26)},
\end{aligned}$$

therefore $\lim_{\delta\to+0}\left(\frac{h^p-1}{h^{p-\delta}-1}\right)^{\frac{1}{\delta}} = h^{\frac{h^p}{h^p-1}}$. Since $\lim_{\delta\to+0}\left(1-\frac{\delta}{p}\right)^{\frac{p}{\delta}} = \frac{1}{e}$ and $\lim_{\delta\to+0}\frac{h^\delta-1}{\delta} = \log h$, we have

$$\lim_{\delta\to+0} g(p-\delta,\delta,h) = \lim_{\delta\to+0}\left(\frac{h^p-1}{p}\frac{\delta}{h^\delta-1}\right)^{\frac{1}{p-\delta}} = \left(\frac{h^p-1}{p\log h}\right)^{\frac{1}{p}} = \left(\frac{1}{\log h^{\frac{p}{h^p-1}}}\right)^{\frac{1}{p}} \tag{4.3}$$

and

$$\lim_{\delta\to+0} g(\delta,p-\delta,h) = \lim_{\delta\to+0}\left(\frac{p-\delta}{p}\right)^{\frac{1}{\delta}}\left(\frac{h^p-1}{h^{p-\delta}-1}\right)^{\frac{1}{\delta}} = \frac{h^{\frac{h^p}{h^p-1}}}{e^{\frac{1}{p}}}. \tag{4.4}$$

Applying (4.3) and (4.4) in (4.2), we have

$$\begin{aligned}
\lim_{\delta\to+0} K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right) &= \lim_{\delta\to+0}\left\{\frac{1}{h}\cdot g(p-\delta,\delta,h)\cdot g(\delta,p-\delta,h)\right\}^{p-\delta} \qquad \text{by (4.2)} \\
&= \frac{1}{h^p}\cdot\frac{1}{\log h^{\frac{p}{h^p-1}}}\cdot\frac{h^{\frac{ph^p}{h^p-1}}}{e} \qquad \text{by (4.3) and (4.4)} \\
&= \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}}.
\end{aligned}$$

Hence the proof of Proposition 14 is complete. □

Remark 2. We summarize the results which have been obtained as follows: *Let* $A > 0$ *and* $MI \geq B \geq mI > 0$. *Then the following assertions hold:*

(i) $A \geq B$ implies $K_+(m, M, p)A^p \geq B^p$ for $p > 1$,

(ii) for each $\delta \in (0,1]$, $A^\delta \geq B^\delta$ implies $K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right) A^p \geq B^p$ for $p > \delta$,

(iii) $\log A \geq \log B$ implies $M_h(p)A^p \geq B^p$ for $p > 0$,

where $h = \frac{M}{m} > 1$, and $K_+(m, M, p)$ and $M_h(p)$ are defined in (1.2) and (2.4), respectively.

Proposition 14 states that as the order in the assumption of (ii) interpolates the orders of (i) and (iii) continuously, the scalar in the consequence of (ii) also interpolates the scalar of (i) and (iii) continuously. Therefore Theorem 5 can be considered as a natural result which is parallel to Theorem B.

Remark 3. Very recently, the following characterization of chaotic order was obtained.

Theorem D ([6]). *If $A, B > 0$, then $\log A \geq \log B$ if and only if for any $\delta \in (0,1]$ there exists an $\alpha = \alpha_\delta > 0$ such that $(e^\delta A)^\alpha > B^\alpha$.*

On the other hand, Theorem 2 and Theorem 5 can be rewritten in the following form.

Theorem 2'. *If $A, B > 0$, then $\log A \geq \log B$ if and only if for any $p \geq 0$ there exists a $K_p > 1$ such that $K_p \to 1$ as $p \to +0$, and $(K_pA)^p \geq B^p$.*

Also we can obtain Theorem D from Theorem 2 by the almost same way to rewriting Theorem 2 into Theorem 2'. We remark that Theorem 2 is proved by using Theorem C and Theorem C can be proved by using Theorem F and Theorem D, so that Theorem 2 can be considered as a formal extension of Theorem D.

Remark 4. Theorem 2' is a parallel result to the following Theorem E [10].

Theorem E ([10]). *If $A, B > 0$, then $\log A \geq \log B$ if and only if for any $p \geq 0$ there exists the unique unitary operator U_p such that $U_p \to I$ as $p \to +0$, and $(U_pAU_p^*)^p \geq B^p$.*

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