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Equivalence relation between an order preserving operator inequality and related operator functions

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Abstract

This report is based on the following two papers:

[FHI] T.Furuta, M.Hashimoto and M.Ito, *Equivalence relation between generalized Furuta inequality and related operator functions*, Scientiae Mathematicae, **1** (1998), 257–259.

[F] T.Furuta, *Simplified proof of an order preserving operator inequality*, Proc. Japan. Acad., **74** (1998), 114.

In this paper, we shall show equivalence relation between an order preserving operator inequality and related operator functions.

1 Introduction

Chapter 1, 2 and 3 are based on [FHI].

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Theorem F is an extension of the celebrated Löwner-Heinz theorem: $A \geq B \geq 0$ *ensures* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0,1]$.

Theorem F ([5]).

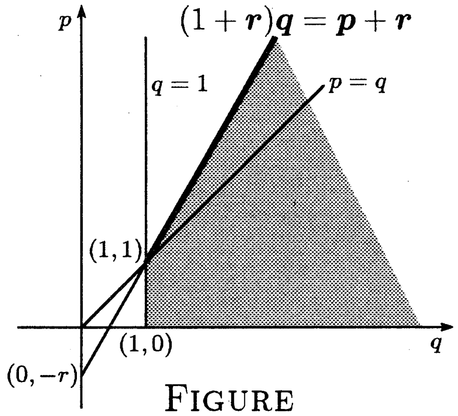
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.



FIGURE

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [2][14] and also an elementary one-page proof in [6]. It is shown in [15] that the domain drawn for p, q and r in the Figure is best possible one for Theorem F.

In [9] we established the following Theorem G as extensions of Theorem F.

Theorem G ([9]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$, and $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$, that is, for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$ and $r \geq t$.

Ando-Hiai[1] established excellent log majorization results and proved the following useful inequality equivalent to the main log majorization theorem:
If $A \geq B \geq 0$ with $A > 0$, then

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geq 1$ and $r \geq 1$.

Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself, and also extends results of [3][7] and [8]. Recently a nice mean theoretic proof of Theorem G is shown in [4] and the result on the best possibility of Theorem G is shown in [16]. Very recently the following Theorem H is obtained in [10] as an extension of Theorem G.

Theorem H ([10]). *Let $A \geq B \geq 0$ with $A > 0$. For each $t \in [0,1]$, $q \geq 0$ and $p \geq max\{q,t\}$,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$. Moreover for each $1 \geq q \geq t \geq 0$ and $p \geq q$, $G_{p,q,t}(A,A,r,s) \geq G_{p,q,t}(A,B,r,s)$, that is,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for any $r \geq t$ and $s \geq 1$.

Alternative proofs of Theorem H are shown in [12][13], and the condition of Theorem H is extended in [11].

2 Result

We obtain the following Theorem 1 associated with Theorem G and Theorem H.

Theorem 1. *The following statements hold and follow from each other;*

(i) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$ and $r \geq t$.

(ii) *If $A \geq B \geq 0$ with $A > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,*

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$ and $r \geq t$.

(iii) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$.

(iv) *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, $q \geq 0$ and $p \geq t$,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

(i) and (iii) in Theorem 1 have been obtained as Theorem G. (ii) and (iv) have been also obtained as Theorem H and an extension of Theorem H. (ii) and (iv) used to be recognized extensions of (i) and (iii) respectively, and (i), (ii), (iii) and (iv) proved sepalately. Theorem 1 asserts that (i), (ii), (iii) and (iv) follow from each other. In other words, if we prove only one of them, then we obtain the others. So, in this report, we shall give a simplified proof of (ii), too.

We need the following lemma to give proofs.

Lemma F. *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

3 Proof of Theorem 1

We may assume that B is invertible without loss of generality. (i), (ii), (iii) and (iv) have been already proved in [9][10][11][12] and [13], so that we have only to prove the equivalence relation among them. We shall show (iv) $\rightarrow$ (iii) $\rightarrow$ (i) $\rightarrow$ (ii) $\rightarrow$ (iv) as follows.

Proof of (iv) $\Longrightarrow$ (iii). Put $q = 1$ and let $p \geq 1$ in (iv), we have (iii).

Proof of (iii) $\Longrightarrow$ (i). In (iii), comparing the functional value and the maximal value of $F_{p,t}(A,B,r,s)$, we have the following inequality: For each $t \in [0,1]$ and $p \geq 1$,

$$\begin{aligned} F_{p,t}(A,B,r,s) &\leq F_{p,t}(A,B,t,1) \\ &= A^{\frac{-t}{2}}BA^{\frac{-t}{2}} \\ &\leq A^{1-t} \qquad \text{by } A \geq B \end{aligned} \tag{3.1}$$

holds for any $r \geq t$ and $s \geq 1$. Multiplying $A^{\frac{r}{2}}$ from the both sides of (3.1), we have (i).

Proof of (i) $\Longrightarrow$ (ii). Since $q \in [0,1]$, $A \geq B \geq 0$ assures $A^q \geq B^q$ by Löwner-Heinz theorem. Put $A_1 = A^q$, $B_1 = B^q$, $t_1 = \frac{t}{q} \in [0,1]$, $p_1 = \frac{p}{q} \geq 1$ and $r_1 = \frac{r}{q} \geq t_1$ in (i). Then

$$A_1^{1-t_1+r_1} = A^{q-t+r},\ A_1^{\frac{r_1}{2}} = A^{\frac{r}{2}},\ A_1^{\frac{-t_1}{2}} = A^{\frac{-t}{2}},\ B_1^{p_1} = B^p$$

and

$$\frac{1-t_1+r_1}{(p_1-t_1)s+r_1} = \frac{q-t+r}{(p-t)s+r},$$

that is, we have the following (3.2): For each $1 \geq q \geq t \geq 0$ and $p \geq q$,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \tag{3.2}$$

holds for any $r \geq t$ and $s \geq 1$. Hence the proof of (i) $\Longrightarrow$ (ii) is complete.

Proof of (ii) $\Longrightarrow$ (iv). Put $D = A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}$ and $q = t$ in (ii), we hava the following (3.3): For each $t \in [0,1]$ and $p \geq t$,

$$A^r \geq (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{r}{(p-t)s+r}} \tag{3.3}$$

holds for any $r \geq t$ and $s \geq 1$. (3.3) is equivalent to the following (3.4) by Lemma F.

$$(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)s}{(p-t)s+r}} \geq D^s. \tag{3.4}$$

Applying Löwner-Heinz theorem to (3.3) and (3.4), we have the following (3.5) and (3.6):

$$A^u \geq (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}} \quad \text{for } r \geq u \geq 0. \tag{3.5}$$

$$(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}} \geq D^w \quad \text{for } s \geq w \geq 0. \tag{3.6}$$

In order to show that $G_{p,q,t}(A,B,r,s)$ is a decreasing function of both r and s, we rewrite $G_{p,q,t}(A,B,r,s)$ as follows:

$$\begin{aligned} G_{p,q,t}(A,B,r,s) &= A^{\frac{-r}{2}}(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}} \\ &= A^{\frac{-r}{2}}f(s)A^{\frac{-r}{2}} \\ &= D^{\frac{s}{2}}(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}}D^{\frac{s}{2}} \quad \text{by Lemma F} \\ &= D^{\frac{s}{2}}g(r)D^{\frac{s}{2}} \end{aligned}$$

where

$$f(s) \equiv (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}}$$

and

$$g(r) \equiv (D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}}.$$

So we have only to prove that $f(s)$ and $g(r)$ are decreasing for r and s respectively.

(a) Proof of the result that $G_{p,q,t}(A,B,r,s)$ is decreasing for $s \geq 1$ such that $(p-t)s \geq q-t$.

$$\begin{aligned} f(s) &= (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}} \\ &= \{(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{(p-t)(s+w)+r}{(p-t)s+r}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}} \quad \text{for } s \geq w \geq 0 \\ &= \{A^{\frac{r}{2}}D^{\frac{s}{2}}(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}}D^{\frac{s}{2}}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}} \quad \text{by Lemma F} \\ &\geq (A^{\frac{r}{2}}D^{\frac{s}{2}}D^wD^{\frac{s}{2}}A^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)(s+w)+r}} \\ &= (A^{\frac{r}{2}}D^{s+w}A^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)(s+w)+r}} \\ &= f(s+w). \end{aligned}$$

The last inequality holds by (3.6) and Löwner-Heinz theorem since $\frac{q-t+r}{(p-t)(s+w)+r} \in [0,1]$ holds by the condition of (ii). Hence $G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}} f(s) A^{\frac{-r}{2}}$ is decreasing for $s \geq 1$ such that $(p-t)s \geq q-t$.

(b) Proof of the result that $G_{p,q,t}(A,B,r,s)$ is decreasing for $r \geq t$.

$$\begin{aligned}
g(r) &= D^{\frac{s}{2}}(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}}D^{\frac{s}{2}} \\
&= D^{\frac{s}{2}}\{(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)s+r+u}{(p-t)s+r}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}} && \text{for } r \geq u \geq 0 \\
&= D^{\frac{s}{2}}\{D^{\frac{s}{2}}A^{\frac{r}{2}}(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}}A^{\frac{r}{2}}D^{\frac{s}{2}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}} && \text{by Lemma F} \\
&\geq D^{\frac{s}{2}}\{D^{\frac{s}{2}}A^{\frac{r}{2}}A^uA^{\frac{r}{2}}D^{\frac{s}{2}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}} \\
&= D^{\frac{s}{2}}(D^{\frac{s}{2}}A^{r+u}D^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}} \\
&= g(r+u).
\end{aligned}$$

The last inequality holds by (3.5), Löwner-Heinz theorem and taking inverses since $\frac{q-t-(p-t)s}{(p-t)s+r+u} \in [-1,0]$ holds by the condition of (ii). Hence $G_{p,q,t}(A,B,r,s) = D^{\frac{s}{2}}g(r)D^{\frac{s}{2}}$ is decreasing for $r \geq t$.

So the proof of (ii) $\Longrightarrow$ (iv) is complete by (a) and (b).

Consequently we have finished the proof of Theorem 1.

4 Simplified proof of (ii)

This chapter is based on [F].

Proof of (ii) *in Theorem 1.* Firstly we shall show the following (4.1) which is the case of $r = t$ in (ii): If $A \geq B \geq 0$ with $A > 0$, then

$$A^q \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} \tag{4.1}$$

holds for $1 \geq q \geq t \geq 0$, $p \geq q$ and $s \geq 1$.

(1st) In case $2 \geq s \geq 1$. $s-1 \in [0,1]$, $\frac{q}{(p-t)s+t} \in [0,1]$ and $A^t \geq B^t \cdots (**)$ hold by the condition of (ii) and Löwner-Heinz theorem. Then we have

$$\begin{aligned}
B_1 &= \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} \\
&= \{B^{\frac{p}{2}}(B^{\frac{p}{2}}A^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{(p-t)s+t}} && \text{by Lemma F} \\
&\leq \{B^{\frac{p}{2}}(B^{\frac{p}{2}}B^{-t}B^{\frac{p}{2}})^{s-1}B^{\frac{p}{2}}\}^{\frac{q}{(p-t)s+t}} && \text{by } (**) \\
&= B^q \leq A^q = A_1
\end{aligned} \tag{4.2}$$

for $1 \geq q \geq t \geq 0$, $p \geq q$ and $2 \geq s \geq 1$.

(2nd) Repeating (4.2) for $A_1 \geq B_1 \geq 0$, we obtain the following (4.3):

$$A_1^{q_1} \geq \{A_1^{\frac{t_1}{2}}(A_1^{\frac{-t_1}{2}}B_1^{p_1}A_1^{\frac{-t_1}{2}})^{s_1}A_1^{\frac{t_1}{2}}\}^{\frac{q_1}{(p_1-t_1)s_1+t_1}} \tag{4.3}$$

holds for $1 \geq q_1 \geq t_1 \geq 0$, $p_1 \geq q_1$ and $2 \geq s_1 \geq 1$.

By putting $1 = q_1 \geq t_1 = \frac{t}{q} \geq 0$, $p_1 = \frac{(p-t)s+t}{q} \geq q_1 = 1$ and restoring A_1 and B_1 in (4.3), we obtain the following (4.4):

$$A^q \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{ss_1}A^{\frac{t}{2}}\}^{\frac{q}{(p-t)ss_1+t}} \tag{4.4}$$

holds for $1 \geq q \geq t \geq 0$, $p \geq q$ and $4 \geq ss_1 \geq 1$. By repeating this process from (4.2) to (4.4), (4.1) holds for any $s \geq 1$.

Lastly let

$$A_2 = A^q \text{ and } B_2 = \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}}.$$

Then (4.1) assures $A_2 \geq B_2 \geq 0$. By Theorem F

$$A_2{}^{1+r_2} \geq (A_2{}^{\frac{r_2}{2}}B_2{}^{p_2}A_2{}^{\frac{r_2}{2}})^{\frac{1+r_2}{p_2+r_2}} \tag{4.5}$$

holds for $p_2 \geq 1$ and $r_2 \geq 0$.
Put $p_2 = \frac{(p-t)s+t}{q} \geq 1$, $r_2 = \frac{r-t}{q} \geq 0$ and restore A_2 and B_2 in (4.5), then we obtain the following (4.6): For each $1 \geq q \geq t \geq 0$ and $p \geq q$,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \tag{4.6}$$

holds for any $r \geq t$ and $s \geq 1$. It is just (ii).

Consequently the proof of (ii) is complete.

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