数理解析研究所講究録
1080 巻 1999 年 67-70

An Example of a p-Quasihyponormal Operator

東北大学大学院理学研究科　内山 敦 (Atsushi Uchiyama)

Introduction. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called p-hyponormal if $(T^{*}T)^{p}\geq(TT^{*})^{p}$ for $p>0$, and T is called p-quasihyponormal if $T^{*}\{(T^{*}T)^{p}-(TT^{*})^{p}\}T\geq 0$ for $p>0$. T is called paranormal[8] if $\|Tx\|^{2}\leq\|T^{2}x\|\|x\|$ for all $x\in \mathcal{H}$. It is well-known by Ando[3] that every p-hyponormal operator is paranormal. M. Lee and S. Lee showed that every p-quasihyponormal operator for $0<p\leq 1$ is paranormal. It is well-known that every p-hyponormal operator $T=U|T|$ is q-hyponormal for all $q\in(0,p)$ by Heinz's inequality and its generalized Aluthge transform $T(s,t)=|T|^{s}U|T|^{t}$ for $s,t>0$ is a q-hyponormal for some $q=q(s,t,p)>0$. (See [1],[2],[6],[7] and [13]). But the assertions that p-quasihyponormal is q-quasihyponormal if $0<q<p$ and the generalized Aluthge transform $T(s,t)=|T|^{s}U|T|^{t}$ for $s,t>0$ of a p-quasihyponormal operator $T=U|T|$ is a q-quasihyponormal for some $q=q(s,t,p)>0$ are not true.

In this paper, we give a p-quasihyponormal operator $T=U|T|$ such that (i) T is not q-quasihyponormal for all $q\in(0,p)$, (ii) $|T|^{s}U|T|^{t}$ for $s,t>0$ is not q-quasihyponormal for all $q\in(0,\infty)$ and (iii) T is a p-quasihyponormal for a $p>1$, but is not paranormal.

Lemma 1. (Hölder-McCarthy Inequality[9]) For any positive operator A and $x\in \mathcal{H}$,

$$(1)\quad (A^{r}x,x)\leq\|x\|^{2(1-r)}(Ax,x)^{r}\quad (\text{if } 0<r\leq 1),$$

$$(2)\quad (A^{r}x,x)\geq\|x\|^{2(1-r)}(Ax,x)^{r}\quad (\text{if } r\geq 1).$$

Using above lemma, M. Lee and H. Lee obtained the following.

Theorem 1. (M. Lee and H. Lee[10]) If T is a p-quasihyponormal operator such as $0 < p \leq 1$, then T is paranormal.

Here, we construct an example of p-quasihyponormal operator which satisfies the conditions(i)-(iii) in the introduction.

Let $\{\varepsilon_n;\ n \in \mathbb{Z}\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{Z})$ and p_n the projection of $\ell^2(\mathbb{Z})$ to $\mathbb{C}\varepsilon_n$. Using the shift operator S on $\ell^2(\mathbb{Z})$ with $S\varepsilon_n = \varepsilon_{n+1}$ and positive 2×2 Hermitian matrices A and B, we define operators H and T on $\mathbb{C}^2 \otimes \ell^2(\mathbb{Z})$ by

$$H = \sum_{n<0} A \otimes p_n + \sum_{n \geq 0} B \otimes p_n$$

and

$$T = (1 \otimes S)H.$$

$T = U|T|$, where $U = 1 \otimes S$ and $|T| = H$. Since $|T^*| = U|T|U^* = \Sigma_{n \leq 0} A \otimes p_n + \Sigma_{n>0} B \otimes p_n$, it is easy to see that

$$T^*(|T|^{2p} - |T^*|^{2p})T = A(B^{2p} - A^{2p})A \otimes p_{-1}$$

for $p > 0$. Hence we have the following.

Lemma 2. T is p-quasihyponormal if and only if $A(B^{2p} - A^{2p})A \geq 0$.

In what follows we assume that A and B are of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

respectively, here $\alpha > 0$. Let f be a function on the half interval $(0, \infty)$ defined by

$$f(p) = \left(\frac{9^p + 1}{2}\right)^{\frac{1}{2p}}.$$

Then it is strictly increasing.

Theorem 2. (1) T is p-quasihyponormal if and only if $\alpha \leq f(p)$.

(2) If $\alpha = f(p)$, then T is not q-quasihyponormal for $q \in (0, p)$, but q-quasihyponormal for $q \in [p, \infty)$. Hence T satisfies the condition(i).

Proof. (1) Since

$$B^{2p} = \frac{1}{2}\begin{pmatrix} 9^p + 1 & 9^p - 1 \\ 9^p - 1 & 9^p + 1 \end{pmatrix},$$

it is easy to see that T is p-quasihyponormal if and only if $(9^p+1)/2-\alpha^{2p}\geq 0$.

(2) It is immediate from (1). QED

Theorem 3. Let $T(s,t)=|T|^sU|T|^t$ for $s,t>0$.

(1) If $T(s,t)$ is p-quasihyponormal, then $\alpha\leq f(s)$.

(2) If $\alpha=f(p)$ and $s\in(0,p)$, then $T(s,t)$ is not q-quasihyponormal for all $q>0$. Hence T satisfies the condition(ii).

Proof. (1) Since

$$\begin{aligned}T(s,t)^*(|T(s,t)|^{2p}-|T(s,t)^*|^{2p})T(s,t)\\ =&A^{s+t}\{(A^tB^{2s}A^t)^p-A^{2(s+t)p}\}A^{s+t}\otimes p_{-2}\\ &+A^tB^s\{B^{2(s+t)p}-(B^sA^{2t}B^s)^p\}B^sA^t\otimes p_{-1},\end{aligned}$$

$T(s,t)$ is p-quasihyponormal if and only if

$$(A^tB^{2s}A^t)^p-A^{2(s+t)p}\geq 0, \quad \text{and} \quad A^tB^s\{B^{2(s+t)p}-(B^sA^{2t}B^s)^p\}B^sA^t\geq 0.$$

The former inequality implies that $\alpha\leq f(s)$.

(2) It is immediate from (1). QED

Theorem 4. T is paranormal if and only if $\alpha\leq\sqrt{5}=f(1)$.

Proof. It is well-known by Ando[3] that an operator S is a paranormal if and only if $S^{*2}S^2-2kS^*S+k^2\geq 0$ for all $k\in\mathbb{R}$.

Since

$$\begin{aligned}T^{*2}T^2-2kT^*T+k^2=&\sum_{n<-1}(A^2-k)^2\otimes p_n+(AB^2A-2kA^2+k^2)\otimes p_{-1}\\ &+\sum_{n\geq 0}(B^2-k)^2\otimes p_n.\end{aligned}$$

$$\begin{aligned}T \text{ is a paranormal} &\Leftrightarrow AB^2A-2kA^2+k^2\geq 0 \quad \forall k\in\mathbb{R}\\ &\Leftrightarrow 5\alpha^2-2k\alpha^2+k^2\geq 0 \quad \forall k\in\mathbb{R}\\ &\Leftrightarrow \alpha^4-5\alpha^2\leq 0\\ &\Leftrightarrow \alpha\leq\sqrt{5}=f(1) \text{ (since } \alpha>0). \quad \text{QED}\end{aligned}$$

Remark. If $\alpha=f(p)$ for $p>1$, then T is a p-quasihyponormal by Theorem 2, but T is not paranormal by Theorem 4. Hence T satisfies the condition(iii).

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Mathematical Institute, Tohoku University,
Sendai 980-8578 Japan
uchiyama@math.tohoku.ac.jp