数理解析研究所講究録
1083 巻 1999 年 125-133

Remarks on Controllability of Membranes Systems

埼玉大学理工学研究科 馬 建平 (Ma JianPing)

January 21, 1999

1 Introduction

In 1970s, J.P. Quinnn and D.L. Russell solved a control problem of a membrane system where the control area is limited in a rectangle. In their case, they put two control functions upon the segments of the rectangle for one membrane. It seems that the condition which they apply is little strong. In this paper we would like to consider a control system of a coupled vibrating membranes system with a single control function applied at one segment. You can see the condition which we use is the weakest in such a case. We define Ω, Ω_1 and Ω_2 with their boundaries Γ, Γ_1 and Γ_2, respectively, as follows:

$$\begin{aligned}
&\Omega = \{(x,y) \in R^2 |\ \ 0 < x < 1, 0 < y < 1\} \\
&\Omega_1 = \{(x,y) \in R^2 |\ \ 0 < x < c, 0 < y < 1\} \\
&\Omega_2 = \{(x,y) \in R^2 |\ \ c < x < 1, 0 < y < 1\} \\
&\Gamma = \partial\Omega, \Gamma_1 = \partial\Omega_1, \Gamma_2 = \partial\Omega_2.
\end{aligned}$$

The evolution of the system is given by the following coupled equations:

$$\begin{aligned}
u^1_{tt}(x,y,t) - \alpha^2(u^1_{xx}(x,y,t) + u^1_{yy}(x,y,t)) = 0 \quad &\text{on } \Omega_1 \times (0,T) \\
u^2_{tt}(x,y,t) - \alpha^2(u^2_{xx}(x,y,t) + u^2_{yy}(x,y,t)) = 0 \quad &\text{on } \Omega_2 \times (0,T)
\end{aligned} \tag{1}$$

with boundary conditions

$$u^1|_{\Gamma_1\cap\Gamma} = u^2|_{\Gamma_2\cap\Gamma} = 0 \tag{2}$$

and initial conditions

$$\begin{aligned}
u^1(x,y,0) = u^1_0(x,y) \quad (x,y) \in \Omega_1, \quad u^2(x,y,0) = u^2_0(x,y) \quad (x,y) \in \Omega_2, \\
u^1_t(x,y,0) = v^1_0(x,y) \quad (x,y) \in \Omega_1, \quad u^2_t(x,y,0) = v^2_0(x,y) \quad (x,y) \in \Omega_2.
\end{aligned} \tag{3}$$

In (1) $\alpha = \sqrt{\dfrac{\tau_i}{\sigma_i}} (i = 1,2)$ is a positive constant, where σ_i and τ_i are mass density and tension, respectively. We call $\Gamma_3 := \Gamma_1 \cap \Gamma_2$ the coupled segment. On Γ_3 an admissible control $f(y,t) \in L^2(0,T;L^2(\Gamma_3))$ is applied as follows:

$$\begin{aligned}
u^1|_{\Gamma_3} - u^2|_{\Gamma_3} = 0 \\
u^1_x|_{\Gamma_3} - u^2_x|_{\Gamma_3} = f(y,t).
\end{aligned} \tag{4}$$

We assume that $H = L^2(\Omega_1) \oplus L^2(\Omega_2)$ with an inner product

$$\left\langle \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right\rangle_H = \int_0^1 \int_0^c u^1(x,y)v^1(x,y)dxdy + \int_0^1 \int_c^1 u^2(x,y)v^2(x,y)dxdy.$$

We identify $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in H$ with a function

$$u(x,y) = \begin{cases} u^1(x,y) & (x,y) \in \Omega_1 \\ u^2(x,y) & (x,y) \in \Omega_2. \end{cases}$$

We will consider exact controllability for the system (1)-(4) in a Hilbert space $U \times V \subset V \times H$, where U and V are also Hilbert spaces with a dense and continuous imbedding $U \subset V$ such that for $T > 0$ and any initial data $(u_0, v_0) \in U \times V$ and $f \in L^2(0,T;L^2(\Gamma_3))$ there exists a solution $u(x,y,t)$ of the system (1)-(4) with $(u, u_t) \in C([0,T]; U \times V)$. The definition of the exact controllability is given as follows:

Definition 1 *Let $T > 0$. The system (1)-(4) is called exactly controllable in $U \times V$ if for every initial data $(u_0, v_0) \in U \times V$ there exists a control $f \in \mathcal{V} := L^2(0,T;L^2(\Gamma_3))$ such that the corresponding solution of (1)-(4) satisfies:*

$$u(x,y,T) \equiv u_t(x,y,T) \equiv 0. \tag{5}$$

Control problems for the partial differential equations have been studied by many researchers since 1970s (see [2],[3],[4],[5],[8],[9],[11]). In recent years the coupled vibrating systems have been treated by several authors but mainly in strings and beams (see S.Ohnari [6][7]; J.P.Ma [10]; L.F.Ho [12] and G.Chen, M.C.Delfour, A.M.Krall, G.Payre [13]). According to our knowledge it is the first time for us to deal with the control problem of the coupled vibrating membranes system. The main theorem of this paper will give us a sufficient condition for the system (1)-(4) to be exactly controllable in $U \times V$.

2 Solutions of the system (1)-(4).

In this section we discuss the solutions of the system (1)-(4). First we introduce a strict solution for the system (1)-(4) and we try to define a mild solution based on it. Now, let us define an operator A with domain $D(A)$ in H as follows:

$$\begin{aligned} D(A) &= \left\{ \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in H^2(\Omega_1) \times H^2(\Omega_2) |\ u|_\Gamma = 0;\ u^1|_{\Gamma_3} = u^2|_{\Gamma_3},\ u^1_x|_{\Gamma_3} = u^2_x|_{\Gamma_3} \right\} \\ A\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} &= \begin{pmatrix} -\alpha^2 \Delta u^1 \\ -\alpha^2 \Delta u^2 \end{pmatrix}. \end{aligned} \tag{6}$$

Obviously the operator A is positive and self-adjoint in H with a compact resolvent. Let

$$\Lambda = \{m^2 + n^2; m \in N\ n \in N\}. \tag{7}$$

Then, the eigenvalues of the operator A and the corresponding eigenspaces are given by

$$\lambda_k = k\alpha^2\pi^2 \quad (k \in \Lambda) \tag{8}$$

and

$$Z_k = span\{\ \Phi_{n,m}(x,y) = \begin{pmatrix} 2\sin n\pi x \sin m\pi y \\ 2\sin n\pi x \sin m\pi y \end{pmatrix};\ n^2+m^2=k,\ n,m \in N\}, \tag{9}$$

respectively.
We see that the eigenfunctions $\{\Phi_{n,m};\ n,m \in N\}$ of the operator A satisfy

$$\Phi_{n,m} \perp \Phi_{n',m'} \quad \text{for } (n,m) \neq (n',m') \tag{10}$$

and $\Phi_{n,m}$ form a complete orthonormal base in H, in particular,

$$Z = \cup Z_k \quad k \in \Lambda \tag{11}$$

is dense in $D(A)$.
From (6), we see that $D(A)$ is a closed subspace of $H^2(\Omega_1) \oplus H^2(\Omega_2)$ and becomes a Hilbert space U with the inner product

$$(u,v)_{D(A)} = (Au, Av)_H \quad \text{for } u,v \in D(A).$$

Since A is self-adjoint and positive, A has a unique positive square root $A^{\frac{1}{2}}$, so that $D(A) \subset D(A^{\frac{1}{2}})$, $A^{\frac{1}{2}}z \in D(A^{\frac{1}{2}})$ for all $z \in D(A)$, and $A^{\frac{1}{2}}A^{\frac{1}{2}}z = Az$ for $z \in D(A)$.
Also $D(A^{\frac{1}{2}})$ becomes a Hilbert space V with the inner product

$$(u,v)_{D(A^{\frac{1}{2}})} = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v)_H \quad \text{for } u,v \in D(A^{\frac{1}{2}}).$$

For the purpose of seeking a strict solution for the system (1)-(4) we first assume $f(y,t)$ be given in $H_0^2[0,T; H^2(\Gamma_3) \cap H_0^1(\Gamma_3)]$ for fixed $T > 0$ and $(u_0, v_0) \in U \times V$. Then, the solution $u(x,y,t)$ of the system (1)-(4) can be expressed formally as follows:

$$\begin{aligned} u^1(x,y,t) &= y^1(x,y,t) + p^1(x)f(y,t) \quad (0<x<c, 0<y<1, 0<t<T), \\ u^2(x,y,t) &= y^2(x,y,t) + p^2(x)f(y,t) \quad (c<x<1, 0<y<1, 0<t<T). \end{aligned} \tag{12}$$

Here

$$p^1(x) = -cx + x, \qquad p^2(x) = -cx + c$$

and $y^i(x,y,t)(i=1,2)$ are solutions of the following system:

$$\begin{aligned} &y^i_{tt}(x,y,t) - \alpha^2\triangle y^i(x,y,t) = -p^i(x)f_{tt}(y,t) + \alpha^2 f_{yy}(y,t) \\ &((x,y) \in \Omega_1 \text{ for } i=1,\ \ (x,y) \in \Omega_2 \text{ for } i=2,\ \ 0<t<T), \\ &y|_\Gamma = 0, \\ &y^1(c,y,t) = y^2(c,y,t), \quad y^1_x(c,y,t) = y^2_x(c,y,t), \end{aligned} \tag{13}$$

with

$$y(x,y,0) = u_0(x,y), \qquad y_t(x,y,0) = v_0(x,y). \tag{14}$$

We see that the system (1)-(4) can be replaced by the system (12)-(14). If we put

$$Y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, Y_t = \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix}, P = \begin{pmatrix} p^1 \\ p^2 \end{pmatrix} \tag{15}$$

with

$$Y_0 = \begin{pmatrix} u_0^1 \\ u_0^2 \end{pmatrix}, Y_{t0} = \begin{pmatrix} v_0^1 \\ v_0^2 \end{pmatrix},$$

then (12)-(14) become

$$\begin{pmatrix} Y_t \\ Y_{tt} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} Y \\ Y_t \end{pmatrix} + \begin{pmatrix} 0 \\ Bf(t) \end{pmatrix} \tag{16}$$

with

$$\begin{pmatrix} Y(0) \\ Y_t(0) \end{pmatrix} = \begin{pmatrix} Y_0 \\ Y_{t0} \end{pmatrix}, \tag{17}$$

where $Bf(t) = -f_{tt}P + \alpha^2 f_{yy}$.
Thus, we consider (16)-(17) in a Hilbert space $V \times H$ with an inner product

$$\left\langle \begin{pmatrix} U_1 \\ V_1 \end{pmatrix}, \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} \right\rangle_{V \times H} = (A^{\frac{1}{2}}U_1, A^{\frac{1}{2}}U_2)_H + (V_1, V_2)_H \tag{18}$$

for $\binom{U_1}{V_1}, \binom{U_2}{V_2} \in V \times H$.
Let $\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$ be an operator in $V \times H$ with domain $D(\mathcal{A})$ defined by

$$D(\mathcal{A}) = U \times V, \qquad \mathcal{A}\begin{pmatrix} Y \\ Y_t \end{pmatrix} = \begin{pmatrix} Y_t \\ -AY \end{pmatrix} \tag{19}$$

for $\binom{Y}{Y_t} \in D(\mathcal{A})$.
It is easy to prove that $\mathcal{A}$ is an infinitesimal generator of a semigroup in $V \times H$. Now we give the definition of *strict solutions* for the system (16)-(17) as follows:

Definition 2 *For* $\binom{Y_0}{Y_{t0}} \in D(\mathcal{A})$ *and* $Bf(t) \in C([0,T]; V \times H)$*, we say that*

$$\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} : [0,T] \to V \times H$$

is a strict solution to (16)-(17) if and only if
(1) $\binom{Y_1(t)}{Y_2(t)} \in C^1([0,T]; V \times H)$.
(2) For any $t \in [0,T]$, $\binom{Y_1(t)}{Y_2(t)} \in U \times V$; $\mathcal{A}\binom{Y_1(t)}{Y_2(t)} \in C([0,T]; V \times H)$.
(3) $\binom{Y_1(t)}{Y_2(t)}$ *satisfies (16)-(17) on* $[0,T]$.

From semigroup theory, we know that, for any $\binom{Y_0}{Y_{t0}} \in U \times V$ and given

$$f(y,t) \in H_0^4[0,T; H^4(\Gamma_3) \cap H_0^1(\Gamma_3)]$$

for fixed $T > 0$, there exists a unique strict solution $\binom{Y(t)}{Y_t(t)}$ of (16)-(17) expressed as

$$\begin{pmatrix} Y(t) \\ Y_t(t) \end{pmatrix} = e^{t\mathcal{A}} \begin{pmatrix} Y_0 \\ Y_{t0} \end{pmatrix} - \int_0^t e^{(t-s)\mathcal{A}} \begin{pmatrix} 0 \\ Bf \end{pmatrix} ds. \tag{20}$$

Now, let us consider the following homogeneous problem which correponds to the system (1)-(4):

$$w_{tt}^i - \alpha^2 \Delta w^i = 0 \quad \text{on } \Omega_i \times [0,T] \ (i = 1,2) \tag{21}$$

with homogeneous boundary conditions

$$\begin{aligned} w^1|_{\Gamma_1 \cap \Gamma} &= w^2|_{\Gamma_2 \cap \Gamma} = 0 \\ w^1|_{\Gamma_3} &= w^2|_{\Gamma_3} \\ w_x^1|_{\Gamma_3} &= w_x^2|_{\Gamma_3} \end{aligned} \tag{22}$$

and initial conditions:

$$w(0) = w_0, \ \ w_t(0) = w_1. \tag{23}$$

For any fixed $(w_0, w_1) \in Z \times Z \subset U \times V$, where

$$w_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \Phi_{nm}, \quad w_1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \Phi_{nm},$$

we know that there exists a unique solution

$$\begin{aligned} w(x,y,t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \sqrt{\lambda_{n,m}} t + b_{nm} \frac{1}{\sqrt{\lambda_{n,m}}} \sin \sqrt{\lambda_{n,m}} t) \Phi_{nm} \\ (\lambda_{n,m} &= \alpha^2 (n^2 + m^2) \pi^2; \ n, m = 1, 2, 3 \cdots) \end{aligned} \tag{24}$$

for the system (21)-(23).

Now, let u be a strict solution of the system (1)-(4). We multiply the corresponding solution of (21)-(23) by u. Integrating by parts we obtain for fixed $T > 0$ the equality

$$\begin{aligned} 0 = \int_0^T \int_\Omega (w_{tt} - \alpha^2 \Delta w) u \, dxdydt = \int_\Omega (w_t(T) u(T) - w(T) u_t(T) + w_0 v_0 - w_1 u_0) \, dxdy \\ + \int_0^T \int_0^1 \alpha^2 f(y,t) w^1(c,y,t) dydt. \end{aligned} \tag{25}$$

Hence, we have

$$\int_\Omega (w_t(T) u(T) - w(T) u_t(T) dxdy = \int_\Omega (-w_0 v_0 + w_1 u_0) dxdy - \int_0^T \int_0^1 \alpha^2 f(y,t) w^1(c,y,t) dydt. \tag{26}$$

This permits us to define a mild solution as follows:

Definition 3 *For any given $(u_0, v_0) \in U \times V$ and $f(y,t) \in L^2(0,T; L^2(\Gamma_3))$, we say that (u, u_t) is a mild solution of the system (1)-(4) if $(u, u_t) \in C([0,T]; U \times V)$ and if (26) is satisfied for every $(w_0, w_1) \in Z \times Z$.*

The definiton is justified by

Lemma 1 *For any $(u_0, v_0) \in U \times V$ and $f \in L^2(0,T; L^2(\Gamma_3))$, there is a unique mild solution for the system (1)-(4).*

3 The Main Result

Now, under any initial data $(u_0, v_0) \in U \times V$, we assume $u(x,y,T) = u_t(x,y,T) = 0$ and try to seek an $f \in \mathcal{V}$ in the form of

$$f(y,t) = \sum_{m=1}^{\infty} f_m(t) \sin m\pi y, \tag{27}$$

where

$$f_m(t) = \int_0^1 f(y,t) \sin m\pi y dy \quad \text{for } t \in [0,T]. \tag{28}$$

We expand $u_0(x,y), v_0(x,y)$ in the following ways:

$$u_0(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{n,m}\Phi_{n,m}(x,y), \quad v_0(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta_{n,m}\Phi_{n,m}(x,y)$$

with

$$\begin{aligned} \alpha_{n,m} &= \int_0^1\int_0^1 2u_0(x,y) \sin n\pi x \sin m\pi y dx dy = (u_0, \Phi_{n,m})_H \\ \beta_{n,m} &= \int_0^1\int_0^1 2v_0(x,y) \sin n\pi x \sin m\pi y dx dy = (v_0, \Phi_{n,m})_H. \end{aligned}$$

Substituting (24) into (26) and making use of the above expansions, we have:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-a_{n,m}\beta_{n,m} + b_{n,m}\alpha_{n,m}) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^T \gamma_n f_m(t)(a_{n,m}\cos\sqrt{\lambda_{n,m}}t + \frac{b_{n,m}}{\sqrt{\lambda_{n,m}}}\sin\sqrt{\lambda_{n,m}}t)dt \tag{29}$$

where

$$\gamma_n = 2\alpha^2 \sin n\pi c \quad \text{for } n = 1,2,\cdots. \tag{30}$$

Thus, (33) reduces to an infinite collection of moment problems for functions $f_m(t) \in L^2(0,T)$:

$$\begin{aligned} -a_{n,m}\beta_{n,m} + b_{n,m}\alpha_{n,m} &= \int_0^T \gamma_n f_m(t)(a_{n,m}\cos\sqrt{\lambda_{n,m}}t + \frac{b_{n,m}}{\sqrt{\lambda_{n,m}}}\sin\sqrt{\lambda_{n,m}}t)dt \\ &(n,m = 1,2,3,\cdots). \end{aligned} \tag{31}$$

If we assume $\gamma_n \neq 0$, then we can rewrite (35) in the following manner:

$$\begin{aligned} &-\frac{\beta_{n,m}}{\gamma_n} = \int_0^T \cos\sqrt{\lambda_{n,m}}t\ f_m(t)\ dt \\ &\frac{\sqrt{\lambda_{n,m}}\alpha_{n,m}}{\gamma_n} = \int_0^T \sin\sqrt{\lambda_{n,m}}t\ f_m(t)\ dt \\ &\text{for } n,m = 1,2,3,\cdots,. \end{aligned} \tag{32}$$

This is a double moment problem; one moment problem for each fixed m.
We see that $\gamma_n \neq 0$ becomes a necessary condition for the system (1)-(4) to be exactly controllable. By (34) we have $\gamma_n \neq 0$ if and only if c is irrational.
From now we try to use a well-known result from [3] (K.D.Graham, D.L.Russell, page 189)

to deal with the moment problem (36).
Let

$$c_{n,m} = -\frac{\beta_{n,m}}{\gamma_n}; \quad d_{n,m} = \frac{\sqrt{\lambda_{n,m}}\alpha_{n,m}}{\gamma_n}.$$

For every fixed m, we note that the sequence $\{\sqrt{\lambda_{n,m}}\}$ possesses an *asymptotic gap* Θ such as

$$\Theta = \liminf_{n\to\infty}(\lambda_{n+1,m}^{\frac{1}{2}} - \lambda_{n,m}^{\frac{1}{2}}) = \alpha\pi. \tag{33}$$

Also the *density* D of the same sequence $\{\sqrt{\lambda_{n,m}}\}$ is given by

$$D = \lim_{n\to\infty}\frac{n}{\sqrt{\lambda_{n,m}}} = \frac{1}{\alpha\pi},$$

which satisfies $\Theta = \dfrac{1}{D}$.
Using the result from Lemma 6.3 in [3] (K.D.Graham and D.L.Russell page 189), we know that, if any $T > 2\pi D$ and if any sequences $\{c_{n,m}\}$; $\{d_{n,m}\}(n = 1, 2, 3 \cdots$ for every fixed $m)$ satisfy

$$\sum_{n=1}^{\infty}|c_{n,m}|^2 < +\infty \quad \sum_{n=1}^{\infty}|d_{n,m}|^2 < +\infty,$$

then the moment problem (36) has a solution $f_m(t) \in L^2(0,T)$ with constant K_1 and K_2 independently of n, m satisfying

$$K_1(\sum_{n=1}^{\infty}|c_{n,m}|^2 + \sum_{n=1}^{\infty}|d_{n,m}|^2) \le ||f_m(t)||^2_{L^2(0,T)} \le K_2(\sum_{n=1}^{\infty}|c_{n,m}|^2 + \sum_{n=1}^{\infty}|d_{n,m}|^2)$$

for every fixed m $(m = 1, 2, \cdots)$. Here K_1, K_2 are determined by the gap Θ and the positive number $T - 2\pi D$.
Now, we introduce a set E which plays a very important role in our control problem (see [9], M.Tucsnak, page 923). For a real number ρ, we denote by $|||\rho|||$ the distance from Z;

$$|||\rho||| = \min_{n\in Z}|\rho - n|.$$

Let us denote by E the set of all irrationals $\rho \in (0,1)$ such that if $[0, a_1, a_2, \cdots, a_n, \cdots]$ is the expansion of ρ as a continued fraction, then $\{a_n\}_{n=1}^{\infty}$ is bounded. From a proposition in [9],(M.Tucsnak, page 923) we have that an irrational number $\rho \in (0,1)$ is in E if and only if there is a constant $C > 0$ such that $|||q\rho||| \ge \frac{C}{q}$ for any positive integer q.
As a consequence of these results, we obtain the following theorem concerning the exact controllability of the system (1)-(4):

Theorem 1 *Let $c \in E$. Assume $T > \dfrac{2}{\alpha}$ and $(u_0, v_0) \in U \times V$, then the system (1)-(4) is exactly controllable in $U \times V$.*

Proof.

Note $c \in E$. For any n, there exists $k \in Z$, such that $|nc-k| \le \frac{1}{2}$.

Since $|nc-k|\pi \le \frac{\pi}{2}$, we have

$$|\sin n\pi c| = |\sin \pi(nc-k)| \ge \frac{2}{\pi}|\pi(nc-k)| = 2|nc-k| = 2|||nc||| \ge \frac{2C}{n},$$

for a constant $C > 0$. Thus, we obtain that

$$|\frac{1}{\gamma_n}| \le \frac{n}{4C\alpha^2} \quad (n = 1, 2, 3, \cdots).$$

On the other hand, for any $(u_0, v_0) \in U \times V$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{n,m}^2 |\alpha_{n,m}|^2 < \infty \\ &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{n,m} |\beta_{n,m}|^2 < \infty. \end{aligned} \tag{34}$$

Note

$$|c_{n,m}|^2 = |\frac{\beta_{n,m}}{\gamma_n}|^2 \text{ and } |d_{n,m}|^2 = |\frac{\sqrt{\lambda_{n,m}}\alpha_{n,m}}{\gamma_n}|^2.$$

Hence, we have, for any fixed m,

$$\begin{aligned} &\sum_{n=1}^{\infty} |c_{n,m}|^2 \le M_1(\alpha, C)(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{n,m}|\beta_{n,m}|^2) < +\infty \\ &\sum_{n=1}^{\infty} |d_{n,m}|^2 \le M_2(\alpha, C)(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{n,m}^2|\alpha_{n,m}|^2) < +\infty, \end{aligned} \tag{35}$$

for positive constants $M_1(\alpha, C)$ and $M_2(\alpha, C)$.

Thus, if $T > \frac{2}{\alpha}$, then there exist functions $f_m(t) \in L^2(0,T)(m = 1, 2, 3, \cdots,)$ which are solutions of the moment problems (36). For each m, we also know there is a $K > 0$, which is independent of m, such that

$$||f_m||_{L^2(0,T)}^2 \le K(\sum_{n=1}^{\infty} |c_{n,m}|^2 + \sum_{n=1}^{\infty} |d_{n,m}|^2), \tag{36}$$

that is, for a constant $M_3(K, C, \alpha) > 0$, we get

$$||f_m||_{L^2(0,T)}^2 \le M_3(K, C, \alpha)\sum_{n=1}^{\infty}(\lambda_{n,m}^2|\alpha_{n,m}|^2 + \lambda_{n,m}|\beta_{n,m}|^2). \tag{37}$$

Therefore, we finally obtain

$$\begin{aligned} &||\sum_{m=k_1}^{k_2} f_m(t)\sin m\pi y||_{L^2(0,T;L^2(\Gamma_3))}^2 = \frac{1}{2}\sum_{m=k_1}^{k_2} ||f_m(t)||_{L^2(0,T)}^2 \\ &\le \frac{1}{2}\sum_{m=k_1}^{k_2} M_3(K, C, \alpha)\sum_{n=1}^{\infty}(\lambda_{n,m}^2|\alpha_{n,m}|^2 + \lambda_{n,m}|\beta_{n,m}|^2). \end{aligned} \tag{38}$$

The left-hand side of (42) converges to 0 as $k_1 \to \infty, k_2 \to \infty$ by using (38). This completes our proof of Theorem 1.

References

[1] J.L. Lions, E. Magenes, *Non-homogenous Boundary Value Problems and Applications I, Springer-Verlag, 1970.*

[2] V. Komornik, *Exact Controllablity and Stabilization, Masson, RMA 36, 1994.*

[3] K.D.Graham, D.L.Russell, *Boundary value control of the wave equation in a spherical region*, SIAM J. Control,13, pp.174-196, 1975.

[4] K.Tsujioka, *On controllability of Euler-Bernoulli equations with a singular boundary condition,* Saitama Math. J.,13, pp.59-67, 1996.

[5] K.Tsujioka, *Remarks on controllability of second order evolution equations in Hilbert spaces,* SIAM J. Control,8, pp.90-99, 1970.

[6] S.Ohnari, *Approximate controllability of coupled vibrating strings with coupled point control,* Adv. Math. Sci. Appl., 6, pp.641-651, 1995.

[7] S.Ohnari, *Controllability of coupled vibrating beams with control applied at coupled point*, Saitama Math.J.,13, pp.69-86, 1995.

[8] D.L.Russell, *Non-harmonic Fourier series in control theory of distributed parameter systems,* J. Math. Anal., 18, pp.542-560, 1967.

[9] M.Tucsnak, *Regularity and exact controllability for a beam with piezoelectric actuator,* SIAM J, Control and Optimization,34, pp.922-930, 1996.

[10] J.P. Ma, *Controllability of coupled flexible beams with control applied at coupled point,* to appear in Adv. Math. Sci. Appl. volume 9 of 1.

[11] R.F.Curtain, H.J.Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory,* TAM 21, Springer-Verlag, 1995.

[12] L.F. Ho, *Controllability and stabilizability of coupled strings with control applied at the coupled points,* SIAM J. Control Optim., 31, pp.1416-1437, 1993.

[13] G.Chen, M.C.Delfour, A.M.Krall and G.Payre, *Modelling, stabilization and control of serially connected beams,* SIAM J. Control Optim., 25, pp.526-546, 1987.