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THE UNIQUE EXISTENCE OF A CLOSED ORBIT OF FITZHUGH–NAGUMO SYSTEM

日本大学理工学部・林 誠 (Makoto Hayashi)

1. Introduction

Our purpose is to give a parameter range for which the ordinary differential equations governing the FitzHugh–Nagumo system have a unique non-trivial closed orbit. It is wider than those already known.

To explicate the ion mechanism for the excitation and the conduction of nerve Hodgkin and Huxley([H–H]) developed the system of four nonlinear ordinary differential equations as a model of nerve conduction in the squid giant axon(Loligo). R. FitzHugh([Fi]) and J. Nagumo et al.([Na]) simplified the system by introducing the following two dimensional autonomous system of ordinary differential equations:

$$\begin{cases} \dot{w} = v - \frac{1}{3}w^3 + w + I \\ \dot{v} = \rho(a - w - bv), \end{cases} \tag{FHN}$$

where the dot ($\dot{}$) denotes differetiation and a, ρ, b are real constants such that

[C1] $\qquad a \in \mathbb{R}, \qquad \rho > 0, \qquad 0 < b < 1.$

The variable w corresponds to the potential difference through the axon membrane and v represents the potassim activation. The quantity I is the current through the membrane. The system (FHN) for special values of I has been investigated by using numerical methods and phase space analysis in [Fi] or [Na].

The system (FHN) has a unique equilibrium point (x_I, y_I) for each $I \in \mathbb{R}$, where

$$x_I = \sqrt[3]{\{3(I + a/b) + \sqrt{9(I + a/b)^2 + 4(1/b - 1)^3}\}/2}$$
$$+\sqrt[3]{\{3(I + a/b) - \sqrt{9(I + a/b)^2 + 4(1/b - 1)^3}\}/2}$$

and

$$y_I = (a - x_I)/b.$$

Instead of the parameter I we introduce a new parameter η. By the transformation $\eta = x_I$, $x = w - \eta$ and $y = v - a/b + \eta/b + \rho b(w - \eta)$, the system (FHN) is transformed to the following system:

$$\begin{cases} \dot{x} = y - \left\{ \dfrac{1}{3}x^3 + \eta x^2 + (\eta^2 + \rho b - 1)x \right\} \\ \dot{y} = -\dfrac{\rho b}{3} \left\{ x^3 + 3\eta x^2 + 3(\eta^2 + \dfrac{1}{b} - 1)x \right\} \end{cases} \tag{FNS}$$

The system (FNS) is called the FitzHugh nerve system and has a unique equilibruim point $E(0,0)$. We shall give some results for the system (FNS) is equivalent to the system (FHN).

Note that, if $\rho b \geq 1$, then the system (FNS) has no non-trivial closed orbits. Thus instead of [C1], we can assume the condition

[C2] $\qquad a \in \mathbb{R}, \qquad 0 < b < 1, \qquad 0 < \rho < 1/b.$

It was studied in such papers as [H1], [K–S] and [Su] for the system (FNS) with the condition [C2]. Let $\eta_0 = \sqrt{1 - \rho b}$. The following is our main result.

Theorem. *The system* (FNS) *has a unique non-trivial closed orbit if* $\eta^2 < \eta_0^2$.

This result improves those given in [K–S] and [H1]. In fact, the result that 'If either $\eta^2 \leq 4^{-1}\eta_0^2$ or $\{\rho b^2 - 7b + 6 < 0$ and $\eta^2 < \eta_0^2\}$, then the system (FNS) has a unique non-trivial closed orbit' was given in [K–S]. In [H1] the result that 'There is a positive constant $\eta_1 \leq \eta_0$ such that the system (FNS) has a unique non-trivial closed orbit for $|\eta| \leq \eta_1$' was given. Therefore the result of the above theorem is clearly stronger than those in [K–S] and [H1].

2. Lemmas

In this section we prepare some lemmas to be used in the next section to prove the Theorem. We consider the Liénard system of the following form

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x), \end{cases} \tag{1}$$

where F is continuously differentiable and g is continuous. We assume the following conditions for the system (1):

[C3] $xg(x) > 0$ if $x \neq 0$,

[C4] There exist $a_1 < 0$ and $a_2 > 0$ such that $F(a_1) = F(a_2) = 0$,

$xF(x) \leq 0$ for $a_1 < x < a_2$, $F(x)$ is nondecreasing for $x < a_1$ and $x > a_2$,

[C5] $\lim_{x \to \pm\infty} \int_0^x \left\{ F'(\xi) + |g(\xi)| \right\} d\xi = \pm\infty.$

To prove the Theorem we shall use the following three lemmas.

Lemma 1. *Assume that the system* (1) *satisfies the conditions* [C3], [C4], [C5] *and besides*

[C6] $G(a_1) > G(a_2)$ *and there exists a constant* $\alpha \geq 0$ *such that* $\dfrac{F(x)}{G^\alpha(x)}$ *is nondecreasing for* $x \in (a_1, x_1) \cup (a_2, +\infty)$*; moreover, there exists a constant* $\delta > 0$ *such that* $\dfrac{F(x)}{G^\alpha(x)}$ *is strictly increasing in* x *with* $0 < |x| < \delta$,

where $G(x) = \int_0^x g(\xi) d\xi$ *and* $x_1 < 0$ *is a number satisfying the equation* $G(a_2) = G(x_1)$.

Then the system (1) *has a unique non-trivial closed orbit.*

Proof of Lemma 1. Under the conditions [C3], [C4] and [C5] the system (1) has at least one non-trivial closed orbit. See [H1]. Moreover, by [Ze], under the conditions [C3], [C4] and [C6] the system (1) has at most one non-trivial closed orbit. □

We can assume that the condition $\eta^2 < \eta_0^2$ in the Theorem holds with $\eta \geq 0$. The proof for the case $\eta < 0$ is essentially the same.

Lemma 2. *Let*

$$\Gamma(x) = \{2\eta_0^2 + 3(\frac{1}{b} - 1)\}x^2 + \eta\{2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1)\}x - 3(\eta^2 - \eta_0^2)(\eta^2 + \frac{1}{b} - 1).$$

Then $\Gamma(\epsilon\,(\eta - \eta_0)) > 0$ *if* $\eta^2 < \eta_0^2$, *where* $\epsilon = \dfrac{3(\eta + \eta_0)(\eta^2 + \frac{1}{b} - 1)}{2\eta_0\{2(\eta_0^2 - \eta^2) + 3(\eta_0^2 + \frac{1}{b} - 1)\}}$.

Proof of Lemma 2. If $\eta^2 < \eta_0^2$, we have

$$\begin{aligned}\Gamma(\epsilon(\eta-\eta_0)) &= \epsilon^2(\eta-\eta_0)^2\{2\eta_0^2+3(\frac{1}{b}-1)\}\\ &\quad+\epsilon\eta\{2(\eta_0^2-\eta^2)+3(\eta_0^2+\frac{1}{b}-1)\}(\eta-\eta_0)-3(\eta^2-\eta_0^2)(\eta^2+\frac{1}{b}-1)\\ &= \epsilon^2(\eta-\eta_0)^2\{2\eta_0^2+3(\frac{1}{b}-1)\}+3(\frac{\eta}{2\eta_0}-1)(\eta^2-\eta_0^2)(\eta^2+\frac{1}{b}-1)>0. \quad \square\end{aligned}$$

Lemma 3. *Let*

$$g(x)=\frac{\rho b}{3}\left\{x^3+3\eta x^2+3(\eta^2+\frac{1}{b}-1)x\right\} \text{ and } G(x)=\int_0^x g(\xi)d\xi.$$

Then $G(a)-G(-a)\geq 0$ for every $a>0$.

Proof of Lemma 3. Since $G(a)-G(-a)=\frac{2}{3}\rho b\eta a^3 \geq 0$, the proof is completed. □

3. Proof of Theorem

We shall prove the Theorem by using the above three lemmas. We set $F(x) = (1/3)x^3+\eta x^2+(\eta^2-\eta_0^2)x$. Then, if $\eta^2<\eta_0^2$, we see easily that the system (FNS) satisfies the conditions [C3], [C4] and [C5]. We shall check the condition [C6] in Lemma 1. We have for $\eta^2<\eta_0^2$

$$a_1=\frac{-3\eta-\sqrt{12\eta_0^2-3\eta^2}}{2}<0 \quad\text{and}\quad a_2=\frac{-3\eta+\sqrt{12\eta_0^2-3\eta^2}}{2}>0.$$

Then we get

$$G(a_1)-G(a_2)=\frac{\rho b}{4}\{\eta^3+2\eta_0^2+6(\frac{1}{b}-1)\}\sqrt{12\eta_0^2-3\eta^2}>0.$$

If $\eta^2<\eta_0^2$, since $0<\epsilon<1$, we have $a_2>\frac{3}{2}(\eta_0-\eta)>\epsilon(\eta_0-\eta)$. Let $x_1<0$ be a number satisfying the equation $G(a_2)=G(x_1)$. From the above fact, the monotonicity of G and Lemma 3, it follows that

$$G(x_1)=G(a_2)>G(\epsilon(\eta_0-\eta))\geq G(\epsilon(\eta-\eta_0)).$$

Using the fact that $a_1<x_1<\epsilon(\eta-\eta_0)<0$, we shall show that $F(x)/G^\alpha(x)$ is nondecreasing for $x\in(a_1,\epsilon(\eta-\eta_0))\cup(a_2,+\infty)$. This means that $F'(x)G(x)-\alpha F(x)g(x)\geq 0$ for $x\in(a_1,\epsilon(\eta-\eta_0))\cup(a_2,+\infty)$.

From the calculation in [H1] we see that the above claim means that

$$
\begin{aligned}
&\Phi(x,\alpha)\\
&= (3-4\alpha)x^4 + 6\eta(3-4\alpha)x^3 + 3\left\{5(3-4\alpha)\eta^2 - (1-4\alpha)\eta_0^2 + 2(3-2\alpha)(\frac{1}{b}-1)\right\}x^2\\
&\quad + 12\eta\left\{2(2-3\alpha)\eta^2 - (1-3\alpha)\eta_0^2 + 3(1-\alpha)(\frac{1}{b}-1)\right\}x + 18(\eta^2-\eta_0^2)(\eta^2+\frac{1}{b}-1)(1-2\alpha)\\
&\geq 0
\end{aligned}
$$

for $x \in (a_1, \epsilon(\eta-\eta_0)) \cup (a_2, +\infty)$.

Let $\alpha = \frac{3}{4}$. Thus we get the following expression which is of degree 2 in x.

$$
\begin{aligned}
&\Phi(x,\frac{3}{4})\\
&= 3\left[\{2\eta_0^2 + 3(\frac{1}{b}-1)\}x^2 + \eta\{2(\eta_0^2-\eta^2) + 3(\eta_0^2+\frac{1}{b}-1)\}x - 3(\eta^2-\eta_0^2)(\eta^2+\frac{1}{b}-1)\right]\\
&= 3\Gamma(x).
\end{aligned}
$$

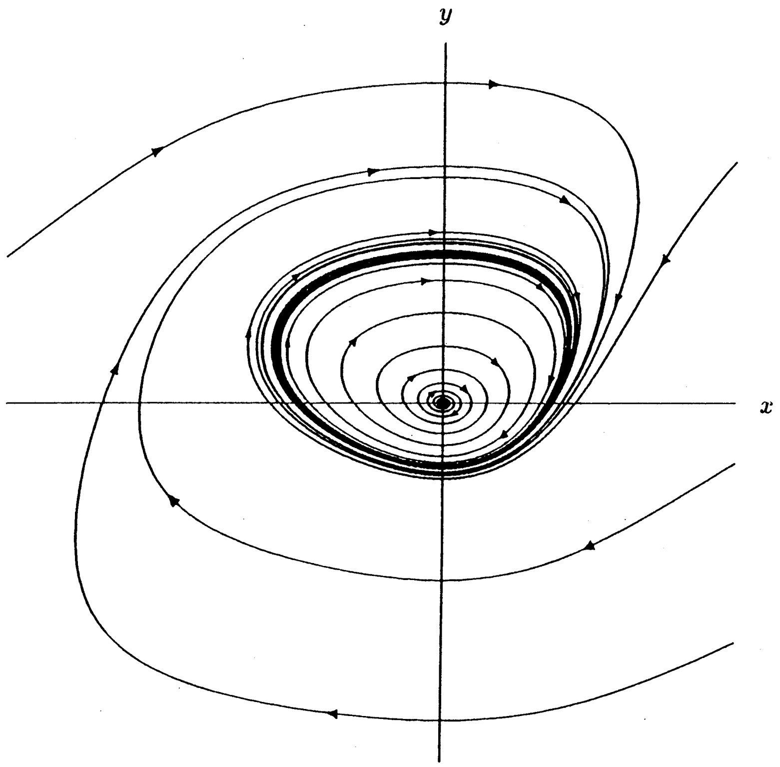
If $\eta^2 < \eta_0^2$, from the fact that Γ is a function of the degree 2, the inequality $\Gamma(0) > 0$ and Lemma 2, we conclude that $\Phi(x,\frac{3}{4}) \geq 0$ for $x \in (a_1, \epsilon(\eta-\eta_0)) \cup (a_2, +\infty)$. Therefore the condition [C6] in Lemma 1 is satisfied. □

4. A numerical example

We shall present the phase portrait of the following system as an example illustrating the application of the Theorem. We consider the system (FNS) with $b = 1/2$, $\rho = 1$ and $\eta^2 = 3/8$:

$$
\begin{cases}
\dot{x} = y - \left(\frac{1}{3}x^3 + \frac{\sqrt{6}}{4}x^2 - \frac{1}{8}x\right)\\
\dot{y} = -\frac{1}{6}\left(x^3 + \frac{3\sqrt{6}}{4}x^2 + \frac{33}{8}x\right)
\end{cases}
\tag{2}
$$

In this case, since $\eta_0^2 = 1 - \rho b = 1/2 > \eta^2$, the system (2) satisfies the condition in the Theorem. Thus we see that the system (2) has a unique non-trivial closed orbit(see the Figure below). We note that this system does not satisfy the condition in [H1] nor that of [K–S], either.



Figure

5. Appendix

Recently, in [H–T] the result that the system (FNS) has a unique non-trivial closed orbit if $\eta^2 = \eta_0^2 > \frac{1}{b} - 1$ was given by using some Lyapunov-type functions.

In [Su] it was shown that the system (FNS) has no non-trivial closed orbits if it satisfies the condition

$$\eta^2 \geq \eta_0^2 \text{ and } \eta^4 - 4\eta^2\eta_0^2 + \eta_0^4 + 2(\frac{1}{2} - 1)\eta^2 - 4(\frac{1}{b} - 1)\eta_0^2 + 4(\frac{1}{b} - 1)^2 \geq 0$$

or

$$2\{\eta_0^2 + (\frac{1}{b} - 1)\}^3 < \eta^2\{\eta^2 + 3(\frac{1}{b} - 1)\}^2.$$

We do not know yet what happens in the region in the (η, η_0)-plane in which $\eta^2 > \eta_0^2$, but the condition of [Su] is not satisfied. But some numerical experiments tell us that the system may have two non-trivial closed orbits if (η, η_0) is in the above mentioned region. Thus we have a **conjecture:**

'The system (FNS) has either exactly two non-trivial closed orbits or no non-trivial closed orbits in the above mentioned region.'

REFERENCES

[Fi] R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophy. J., **1** (1961), 445–466.

[H1] M. Hayashi, *On the uniqueness of the closed orbit of FitzHugh–Nagumo system*, Math. Japon., **46** (2) (1997), 331–336.

[H–T] M. Hayashi and M. Tsukada, *A uniqueness theorem on the Liénard system with a non-hyperbolic equilibrium point*, submitted.

[K–S] E. Kaumann and U. Staude, *Uniqueness and nonexistence of limit cycles for the FitzHugh equation*, Lecture Notes in Math. **1017**, Springer–Verlag, New York, (1983), p. 313–321.

[Na] J. Nagumo, S. Arimoto and S. Yoshizawa, *An active pulse transmission line simulating nerve axon,*, Proc. IRE., **50** (1962), 2061–2070.

[Su] J. Sugie, *Non-existence of periodic solutions for the FitzHugh nerve system*, Quart. Appl. Math., **49** (1991), 543–554.

[Ze] X. Zeng and others, *On the uniqueness of the limit cycle of the generalized Liénard equation*, Bull. London Math. Soc., **26** (1994), 213–247.

Department of Mathematics,
College of Science and Technology, Nihon University,
Narashinodai, Funabashi, 274-8501, Japan

E-mail address: mhayashi@penta.ge.cst.nihon-u.ac.jp