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On Stability of Solutions in Linear Autonomous Functional Differential Equations

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1 Introduction

Let R denote the real line and E a Banach space with norm $|\cdot|$. If $x:(-\infty,b)\to E$, then the function $x_t:(-\infty,0]\to E, t\in(-\infty,b)$, is defined by $x_t(\theta)=x(t+\theta),\theta\in(-\infty,0]$. We deal with the linear autonomous functional differential equation with infinite delay in the Banach space E:

$$\text{(AL)} \qquad \frac{dx(t)}{dt}=Ax(t)+L(x_t).$$

The phase space $\mathcal{B}$ and some hypotheses in Eq.(AL) are demonstrated in Section 3.

The aim of this paper is to investigate stability properties of the zero solution for Eq.(AL). Recently, spectral properties of the generator of the solution semigroup to Eq.(AL) were studied in [4](see [5] for the complete proof of these). Moreover, using these properties, the authors analyzed the stability of the zero solution of some integro-differential equation. Note that the manner employed in [4] is based upon a concrete equation. Also, the stability of solutions for a special case of Eq.(AL) with finite delay was discussed in [9]. Since solution operators of Eq.(AL) form a C_0-semigroup on $\mathcal{B}$, we will find stability criteria in the general setting of C_0-semigroups. Our approach follows closely that in [9].

On the other hand, there are two concepts for a general C_0-semigroup $T(t)$ on E :

(1) $T(t)$ is asymptotically stable if $|T(t)x|\to 0$ as $t\to\infty$ for every $x\in E$;

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(2) $T(t)$ is exponentially asymptotically stable if $\|T(t)\| \to 0$ as $t \to \infty$.
If E is of finite dimension, then these two concepts are equivalent. In general, if E is of infinite dimension, these two concepts are different (see [2, 6]).

In the present paper we shall give conditions to ensure the equivalence of these two concepts when E is of infinite dimensional.

In Section 2, we will get results on asymptotic behavior of a general C_0-semigroup. In particular, we show that the asymptotic stability and the exponential asymptotic stability of a C_0-semigroup are equivalent under a condition on the growth bound and the essential growth bound of the C_0-semigroup.

In Section 3, we will apply the results obtained in Section 2 to find stability criteria of the zero solution for Eq.(AL). Moreover, we give a sufficient condition for the asynchronous exponential growth of the solution semigroup to Eq.(AL) (see [11]).

In Section 4, the results obtained in Section 3 are illustrated in an integro-differential equation.

2 Asymptotic behavior of a C_0-semigroup $T(t)$

Let $T(t)$ be a C_0-semigroup on E and A its infinitesimal generator throughout this section. We consider an asymptotic behavior of a $T(t)$, which is concerned with informations about its essential spectrum. To do so, we will prepare the following lemma, which is easily shown by induction.

Lemma 2.1 *If x belongs to the null space $M := N((A-\lambda I)^m), I$ being the identity, then*

$$T(t)x = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (A - \lambda I)^k x.$$

The growth bound $\omega_s(T)$, and the essential growth bound $\omega_e(T)$ of $T(t)$ are defined by

$$\omega_s(T) := \lim_{t\to\infty} \frac{\log \|T(t)\|}{t} = \inf_{t>0} \frac{\log \|T(t)\|}{t}, \quad \omega_e(T) := \lim_{t\to\infty} \frac{\log \alpha(T(t))}{t} = \inf_{t>0} \frac{\log \alpha(T(t))}{t},$$

where $\|T(t)\|$ stands for the operator norm of $T(t)$ and $\alpha(T(t))$ is the measure of noncompactness of $T(t)$ which is described by the Kuratowskii measure of noncompactness of bounded sets in E(cf.[10]). Then the spectral radius $r_s(T(t))$ and the essential spectral radius $r_e(T(t))$ are given as $r_s(T(t)) = \exp(t\omega_s(T))$ and $r_e(T(t)) = \exp(t\omega_e(T))$. Let $\rho(A)$ be the resolvent set of A, $\sigma(A)$ the spectrum of A, $P_\sigma(A)$ the point spectrum of A, $E_\sigma(A)$ the essential spectrum of A, and set $N_\sigma(A) := \sigma(A) \setminus E_\sigma(A)$. The points in $N_\sigma(A)$ are called normal eigenvalues of A. Recall that, by definition, λ is a normal eigenvalue of A if it is an isolated point of $\sigma(A)$, the range $R(A - \lambda I)$ is closed and the generalized eigenspace $M_\lambda(A)$ is of finite dimension.

Then the following relations hold (cf.[10]):

$$\sup\{\Re\lambda : \lambda \in \sigma(A)\} \leq \omega_s(T), \quad \sup\{\Re\lambda : \lambda \in E_\sigma(A)\} \leq \omega_e(T) \tag{1}$$

and

$$\omega_s(T) = \max\{\omega_e(T), s(A)\},$$

where

$$s(A) := \sup\{\Re\lambda : \lambda \in N_\sigma(A)\}.$$

Put $P_0(A) = \{\lambda \in N_\sigma(A) : \Re\lambda = 0\}$ and $R(\mu, A) = (\mu I - A)^{-1}$.

Proposition 2.2 *The following results hold true.*

1) *The following statements are equivalent.*

(i) $\omega_s(T) < 0$.

(ii) *There are an* $\omega_0 > 0$ *and a* $K \geq 1$ *such that*

$$|T(t)x| \leq Ke^{-\omega_0 t}|x|, t \geq 0, x \in E.$$

(iii) $\|T(t)\| \to 0$ *as* $t \to \infty$.

2) *Assume that* $\omega_e(T) < \omega_s(T)$. *Then there exist a* $\lambda \in C$ *and an* $x_0 \in N(A - \lambda I), x_0 \neq 0$ *such that* $\Re\lambda = \omega_s(T)$ *and*

$$|T(t)x_0| = e^{\omega_s(T)t}|x_0|, t \geq 0.$$

3) *Assume that* $\omega_e(T) < \omega_s(T) = 0$. *Then the following statements hold.*

(i) *If* $R(\mu, A)$ *has a pole of order* 1 *at* $\mu = \lambda$ *for all* $\lambda \in P_0(A)$, *then there is a constant* $H > 0$ *such that*

$$|T(t)x| \leq H|x|, t \geq 0, x \in E.$$

(ii) *If there is a* $\lambda_0 \in P_0(A)$ *such that* $R(\mu, A)$ *has a pole of order* $m \geq 2$ *at* $\mu = \lambda_0$, *then there exists an* $x_0 \in E$ *such that*

$$|T(t)x_0| \to \infty \quad as \quad t \to +\infty.$$

Proof. 1) Since the inequality $e^{\omega_s(T)t} \leq \|T(t)\|$ holds for $t > 0$, it is easy to show the assertion 1).

2) Since $\omega_e(T) < \omega_s(T)$, it follows from Theorem 2 in [4] that there is a $\lambda_0 \in N_\sigma(A)$ such that $\Re\lambda_0 = \omega_s(T)$. Take an $x_0 \in N(A - \lambda_0 I), x_0 \neq 0$. By Lemma 2.1 we have $T(t)x_0 = e^{\lambda_0 t}x_0$, and hence $|T(t)x_0| = e^{\omega_s(T)t}|x_0|$.

3) From Theorem 2 in [4] we see that $P_0(A)$ is finite ; denote it by $\{\lambda_1, \lambda_2, \cdots, \lambda_q\}$. Let m_j be the index of λ_j, and set $M_j = N((A - \lambda_j I)^{m_j}), j = 1, 2, \cdots, q$. Then $E = M \oplus M_0$, where $M = M_1 \oplus M_2 \oplus, \cdots, \oplus M_q$, and $M_0 = \cap_{j=1}^q R((A - \lambda_j I)^{m_j})$. Let P_j be the projection

in E such that $P_jE = M_j, (I-P_j)E = R((A-\lambda_j I)^{m_j})$; and set $P = P_1+P_2+\cdots+P_q, P_0 = I - P$.

(i) By Theorem 3 in [4] we have

$$|T(t)P_0x| \leq Ke^{-\varepsilon t}\|P_0\||x| \tag{2}$$

for some $K \geq 1$ and $\varepsilon > 0$. Since $P_jx \in M_j$ and $m_j = 1, 1 \leq j \leq q$, it follows from Lemma 2.1 that $T(t)P_jx = e^{\lambda_j t}P_jx$. Noting that $\Re\lambda_j = 0$, we see that

$$|T(t)P_jx| \leq |P_jx| \leq \|P_j\||x|, t \geq 0, x \in E. \tag{3}$$

Combining(2) and (3) we obtain the following inequality

$$\begin{aligned} |T(t)x| &\leq |T(t)Px| + |T(t)P_0x| \\ &\leq \sum_{j=1}^{q} |T(t)P_jx| + |T(t)P_0x| \\ &\leq H|x|, \end{aligned}$$

where $H = \sum_{j=1}^{q} \|P_j\| + K\|P_0\|$.

(ii) Let j be an index such that $R(\mu, A)$ has a pole of order $m \geq 2$ at $\mu = \lambda_j$. From the definition of M_j we see that there is a $x_0 \in M_j$ such that $(A - \lambda_j I)^{m-1}x_0 \neq 0$. Since $m \geq 2$, it follows from Lemma 2.1 that

$$\begin{aligned} |T(t)x_0| &= |\sum_{k=0}^{m-1} \frac{t^k}{k!}(A-\lambda_j I)^k x_0| \\ &= t^{m-1}|\sum_{k=0}^{m-1} \frac{1}{k!t^{m-1-k}}(A-\lambda_j I)^k x_0| \\ &\to \infty (t \to \infty). \end{aligned}$$

Therefore the proof is complete.

We note that $T(t)x_0, x_0 \in D(A)$, is a solution of Eq.(AL) with $L = 0$; that is,

$$\frac{dx(t)}{dt} = Ax(t). \tag{A}$$

Definition

1) $T(t)$ (or the zero solution of Eq.(A)) is said to be stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x| < \delta, x \in E$, then $|T(t)x| < \varepsilon$ for all $t \in [0, \infty)$.

2) $T(t)$ (or the zero solution of Eq.(A)) is said to be asymptotically stable if for any $x \in E, |T(t)x| \to 0$ as $t \to +\infty$.

3) $T(t)$ (or the zero solution of Eq.(A)) is said to be exponentially asymptotically stable if there are an $\omega_0 > 0$ and a $K \geq 1$ such that

$$|T(t)x| \leq Ke^{-\omega_0 t}|x|, \quad t \geq 0, \quad x \in E.$$

Theorem 2.3 *Assume that $\omega_e(T) < \omega_s(T)$. Then the following statements are equivalent.*

1) *$T(t)$ is asymptotically stable.*
2) *$T(t)$ is exponentially asymptotically stable.*
3) *For any $\lambda \in P_\sigma(A)$, $\Re\lambda$ is negative.*

Proof. We will derive the assertion 3) from the assertion 1), because the other parts are obvious. To do so, it suffices to show that the inequality $\omega_s(T) < 0$ holds. Assume, for a contradiction, that $\omega_s(T) \geq 0$. Since $\omega_e(T) < \omega_s(T)$, from the assertion 2) in Proposition 2.2 it follows that there exist a $\lambda \in C$ and an $x_0 \in N(A - \lambda I), x_0 \neq 0$, such that $\Re\lambda = \omega_s(T)$ and

$$|T(t)x_0| = \begin{cases} e^{\omega_s(T)t}|x_0|, & \text{if } \omega_s(T) > 0 \\ |x_0|, & \text{if } \omega_s(T) = 0. \end{cases}$$

This contradicts the assertion 1). Hence the proof is complete.

Remark If $\omega_e(T) = \omega_s(T)$, then, in general, the two concepts of asymptotic stability and exponential asymptotic stability for C_0-semigroup $T(t)$ are different. Of course, if $T(t)$ is a C_0-compact semigroup, the these two concepts are equivalent.

Theorem 2.4 *Assume that $\omega_e(T) < \omega_s(T) = 0$. Then the following statements are equivalent.*

1) *$T(t)$ is stable.*
2) *$R(\mu, A)$ has a pole of order 1 at $\mu = \lambda$ for all $\lambda \in P_0(A)$.*
3) *There is an $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \in [0, \infty)$.*

Proof. The statements 1) and 3) are equivalent by definition of the stability. From the assumption we see that $P_0(A)$ consists of finite normal eigenvalues of A. Hence if $\lambda_0 \in P_0(A)$, λ_0 is a pole of $R(\mu, A)$([10]). We will prove the assertion 2) from the assertion 1). Now, assume that the assertion 2) does not hold. Then there is $\lambda_0 \in P_0(A)$ such that $R(\mu, A)$ has a pole of order $m \geq 2$ at $\mu = \lambda_0$. From the assertion 3) in Proposition 2.2 there is an $x_0 \in E$ such that $|T(t)x_0| \to \infty$ as $t \to \infty$, which contradicts the assumption. The remaining parts are easily shown by Proposition 2.2.

3 Stability of solutions in Eq.(AL)

In this section we will apply the results obtained in the previous section to Eq.(AL). Let $\mathcal{B}$ be a Banach space consisting of some functions mapping $(-\infty, 0]$ into E ; the norm in $\mathcal{B}$ is denoted by $|\cdot|$. For a complex number λ and for an $x \in E$ we define a function $\varepsilon_\lambda \otimes x : (-\infty, 0] \to E$ by $(\varepsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta}x, \theta \in (-\infty, 0]$. We assume that the phase space $\mathcal{B}$ satisfies the following axioms (B-1,2,3).

(B-1) If a function $x : (-\infty, \sigma + a) \to E$ is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then
(i) $x_t \in \mathcal{B}$ for all $t \in [\sigma, \sigma + a)$ and x_t is continuous in $t \in [\sigma, \sigma + a)$;
(ii) $H^{-1}|x(t)| \leq |x_t| \leq K(t - \sigma)\sup\{|x(s)| : \sigma \leq s \leq t\} + M(t - \sigma)|x_\sigma|$
for all $t \in [\sigma, \sigma + a)$, where $H > 0$ is constant, $K : [0, \infty) \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and they are independent of x.

(B-2) If $\{\phi^n\}$ is a Cauchy sequence in $\mathcal{B}$, and if the sequence $\{\phi^n(\theta)\}$ converges to a function $\phi(\theta)$ uniformly on every compact interval of $(-\infty, 0]$, then ϕ lies in $\mathcal{B}$ and $\lim_{n\to\infty} |\phi^n - \phi| = 0$.

We introduce the trivial C_0-semigroup $S(t)$ on $\mathcal{B}$ defined as

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) & t + \theta \geq 0 \\ \phi(t + \theta) & t + \theta < 0. \end{cases}$$

Let $S_0(t)$ be the restriction of $S(t)$ to the subspace $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$. Then the equality $\omega_e(S_0) = \omega_s(S_0)$ holds(cf.[4]).

(B-3) There exists a constant γ_0 such that $\varepsilon_\lambda \otimes x \in \mathcal{B}$ for $\Re\lambda > \gamma_0$ and $x \in E$, and that

$$\|\varepsilon_\lambda\| := \sup\{\|\varepsilon_\lambda \otimes x\| : x \in E,\ |x| \leq 1\}$$

is finite for each λ with $\Re\lambda > \gamma_0$, and bounded for $\Re\lambda > \gamma_1$ for some $\gamma_1 \geq \gamma_0$.

We call the constant γ_0 in (B-3) the abscissa of the exponent of the space $\mathcal{B}$. For the exponential functions as elements of $\mathcal{B}$, we have the following result : If $\Re\lambda > \omega_s(S_0)$, then ε_λ lies in $\mathcal{L}(E, \mathcal{B})$,the Banach space of bounded linear operators on E into $\mathcal{B}$, and it is holomorphic for λ. Hence $\gamma_0 \leq \omega_s(S_0) = \omega_e(S_0)$ (cf.[4],[5]).

Let C_{00} be the set of continuous functions mapping $(-\infty, 0]$ into E, with compact support. The following axiom for $\mathcal{B}$ is used later on.

(C) If a uniformly bounded sequence $\{\phi^n(\theta)\}$ in C_{00} converges to a function $\phi(\theta)$ uniformly on every compact set of $(-\infty, 0]$, then $\phi \in \mathcal{B}$ and $\lim_{n\to\infty} |\phi^n - \phi| = 0$.

Let $\mathcal{B} = E \times \mathcal{L}_\gamma$ be the quotient space with respect to the seminorm defined by

$$\|\phi\| = |\phi(0)| + \int_{-\infty}^{0} e^{-\gamma\theta}|\phi(\theta)|\, d\theta.$$

for a measurable function $\phi : (-\infty, 0] \to E$ such that $e^{-\gamma\theta}|\phi(\theta)|$ is integrable on $(-\infty, 0]$.

Then this space is a Banach space satisfying (B-1,2,3), γ is the abscissa of the exponent, and $\omega_e(S_0) = \omega_s(S_0) = \gamma$. If $\gamma < 0$, then $\mathcal{B}$ is a uniform fading memory space (cf.[1]).

According to [1], we have the following facts for the phase space $\mathcal{B}$ with an additional axiom (C):

(1) $\mathcal{B}$ is a fading memory space if and only if $S_0(t)$ is asymptotically stable.

(2) $\mathcal{B}$ is a uniform fading memory space if and only if $S_0(t)$ is exponentially asymptotically stable.

From the above facts we see that two concepts of the asymptotic stability and the exponential asymptotic stability of a C_0-semigroup are, in general, different.

We assume that Eq.(AL) always satisfies the following hypotheses :

(H-1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-compact semigroup $T(t), t \geq 0$, on E;

(H-2) $L : \mathcal{B} \to E$ is a bounded linear operator.

From the hypotheses (H-1)and (H-2) we see that a mild solution of Eq.(AL) through $(0,\phi), \phi \in \mathcal{B}$, exists uniquely on $[0,\infty)$ (cf.[5],[7]). Denote by $u(t)$ or $u(t,\phi)$ this mild solution. It has the following properties:

(i) $u_0 = \phi \in \mathcal{B}$ and u is continuous on $[0,\infty)$.

(ii)
$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)L(u_s)\, ds, \qquad t \geq 0.$$

Define the solution operator $U_L(t)$ on $\mathcal{B}$ by

$$(U_L(t)\phi)(\theta) = u(t+\theta,\phi), \qquad \theta \in (-\infty, 0].$$

By using the axioms of the phase space, we see that $U_L(t)$ is a C_0-semigroup on $\mathcal{B}$. In the particular case that $L \equiv 0$, $U_0(t)$ is given by

$$(U_0(t)\phi)(\theta) = \begin{cases} T(t+\theta)\phi(0) & -t < \theta \leq 0 \\ \phi(t+\theta) & \theta \leq -t. \end{cases}$$

Denote by A_L and A_0 the infinitesimal generators of $U_L(t)$ and $U_0(t)$, respectively, and set $K_L(t) = U_L(t) - U_0(t)$. To characterize the point spectrum $P_\sigma(A_L)$ of A_L, we need the characteristic operator $\Delta(\lambda)$ given as

$$\Delta(\lambda)x = (\lambda I - A - L_\lambda)x, \qquad x \in D(A), \tag{4}$$

where $L_\lambda x = L(\varepsilon_\lambda \otimes x)$. It is a closed linear operator which is well defined for $\Re\lambda > \gamma_0$.

The following two lemmas can be found in [4],[5],[8].

Lemma 3.1 *If $\Re\lambda > \gamma_0$, then $\lambda \in P_\sigma(A_L)$ if and only if the null space $N(\Delta(\lambda))$ is nontrivial.*

If the original semigroup $T(t)$ is a C_0-compact semigroup on E or L is a compact operator, then $K_L(t)$ is a compact operator for $t > 0$. This implies that $\alpha(U_L(t)) = \alpha(U_0(t))$ for $t > 0$. From this we can obtain the the following estimate of the essential growth bound of $U_L(t)$.

Lemma 3.2 *If $T(t)$ is a C_0-compact semigroup on E and if $\mathcal{B}$ satisfies the axiom (C), then*

$$\omega_e(U_0) = \omega_e(U_L) \le \hat{\omega}_e(S_0) := \lim_{t\to\infty} \frac{1}{t} \log \overline{\lim}_{s\to t-0} \alpha(S_0(s)).$$

We remark that $\omega_e(S_0) \le \hat{\omega}_e(S_0)$. In fact, if $0 < \epsilon < t$, then

$$\alpha(S_0(t)) \le \alpha(S_0(t-\epsilon))\alpha(S_0(\epsilon)) \le \sup_{t-\epsilon\le s<t} \alpha(S_0(s))\|S_0(\epsilon)\|.$$

Since $\|S_0(t)\|$ is locally bounded for $t \ge 0$, we have the remark.

Using Lemma 3.2, we have the following result.

Theorem 3.3 *Let $T(t)$ be a C_0-compact semigroup on E and $\mathcal{B}$ a uniform fading memory space. Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$. Then the following statements are equivalent.*

1) *The zero solution of Eq.(AL) is asymptotically stable.*
2) *The zero solution of Eq.(AL) is exponentially asymptotically stable.*
3) *For any λ such that $\Re\lambda > \gamma_0$ and $N(\Delta(\lambda))$ is nontrivial, $\Re\lambda$ is negative.*

Proof. From the assumptions in the theorem and Lemma 3.2 it follows that $\omega_e(U_0) = \omega_e(U_L) \le \hat{\omega}_e(S_0) < \omega_s(U_L)$. Hence $\omega_e(U_L) < \omega_s(U_L)$. Therefore, Theorem 3.3 follows from Theorem 2.3.

Put $P_\sigma(\Delta) = \{\lambda \in C : N(\Delta(\lambda)) \text{ is nontrivial}\}, P_0(\Delta) = \{\lambda \in P_\sigma(\Delta) : \Re\lambda = 0\}, D = \{\mu \in C : \Re\mu > \hat{\omega}_e(S_0)\}$ and $D_- = D \setminus (P_\sigma(\Delta) \cup P_\sigma(A))$.

Lemma 3.4 *Let $T(t)$ be a C_0-compact semigroup on E and let $\mathcal{B}$ satisfy the axiom (C). Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$.*

1) *For every $\lambda \in D_-$ the following relation holds :*

$$(5) \qquad R(\lambda, A_L)\phi = R(\lambda, A_0)\phi + \varepsilon_\lambda \otimes \Delta(\lambda)^{-1} L(R(\lambda, A_0)\phi), \qquad \phi \in \mathcal{B}.$$

2) *If $\Delta(\lambda)^{-1}$ has a pole of order n at λ_0 in $(P_\sigma(\Delta) \setminus P_\sigma(A)) \cap D$, then $R(\lambda, A_L)$ has a pole of at most order n at λ_0.*

Proof. From the proof of Theorem 3.3 we have $\omega_e(U_L) \le \hat{\omega}_e(S_0) < \omega_s(U_L)$. Moreover, the abscissa γ_0 of the exponent of the phase space $\mathcal{B}$, $\omega_e(S_0)$ and $\omega_s(S_0)$ are related as $\gamma_0 \le \omega_e(S_0) = \omega_s(S_0)$. Let ε be any number such that $0 < \varepsilon < \omega_s(U_L) - \hat{\omega}_e(S_0)$. Put $D(\varepsilon) = \{\mu \in C : \Re\mu > \hat{\omega}_e(S_0) + \varepsilon\}$ and $D_-(\varepsilon) = D(\varepsilon) \setminus (P_\sigma(\Delta) \cup P_\sigma(A))$. In view of the relation (1), we see that $P_\sigma(A_0)$ and $P_\sigma(A_L)$ consist of at most finite normal eigenvalues in $D(\varepsilon)$. Of course, $P_\sigma(A) = P_\sigma(A_0)$ and $P_\sigma(A_L) = P_\sigma(\Delta)$, because of Lemma 3.1. Hence, $R(\lambda, A), R(\lambda, A_0)$ and $R(\lambda, A_L)$ are holomorphic in $D_-(\varepsilon)$,and so $LR(\lambda, A_0)$ and $LR(\lambda, A_L)$ are also holomorphic in $D_-(\varepsilon)$. Notice that $\Delta(\lambda)$ is well defined on $D_-(\varepsilon)$. Using the above facts and the proof in [5, Theorem 4.2], we have

$$(6) \qquad R(\lambda, A_L)\phi = R(\lambda, A_0)\phi + \varepsilon_\lambda \otimes R(\lambda, A)L(R(\lambda, A_L)\phi), \qquad \phi \in \mathcal{B}, \text{ on } D_-(\varepsilon)$$

and

$$(7) \qquad \Delta(\lambda)R(\lambda, A)L(R(\lambda, A_L)\phi) = L(R(\lambda, A_0)\phi), \qquad \phi \in \mathcal{B}, \text{ on } D_-(\varepsilon)$$

If $\lambda \in D(\varepsilon) \setminus P_\sigma(\Delta)$, then $\Delta(\lambda)^{-1}$ exists. Hence for $\lambda \in D_-(\varepsilon)$ the relation (7) becomes

$$(8) \qquad R(\lambda, A)L(R(\lambda, A_L)\phi) = \Delta(\lambda)^{-1}L(R(\lambda, A_0)\phi), \qquad \phi \in \mathcal{B}.$$

Therefore, combining the relation (6) with (8), the relation (5) holds for all $\lambda \in D_-(\varepsilon)$. Since ε is arbitrary, the proof of the assertion 1) is completed. The assertion 2) is easily shown by the relation (5), because λ_0 is a pole of $R(\lambda, A_L)$(cf.[10, Proposition 4.11]).

Using the above lemma and Theorem 2.4, we have the following result.

Theorem 3.5 *Let $T(t)$ be a C_0-compact semigroup on E and $\mathcal{B}$ a uniform fading memory space. Assume that $P_\sigma(A) \cap P_0(\Delta) = \emptyset$, and $\omega_s(U_L) = 0$. If $\Delta(\mu)^{-1}$ has a pole of order 1 at $\mu = \lambda$ for all $\lambda \in P_0(\Delta)$, then the zero solution of Eq.(AL) is stable.*

Recall that a C_0-semigroup $H(t)$ on E is said to have *asynchronous exponential growth with intrinsic growth constant $\lambda_0 \in R$* provided there exists a nonzero finite rank operator P_0 in E such that $\lim_{t\to\infty} e^{-\lambda_0 t}H(t) = P_0$(see [11]). Put $P\sigma_0(\Delta) = \{\mu \in P_\sigma(\Delta) : \Re\mu = \sup\{\Re\lambda : \lambda \in P_\sigma(\Delta)\}\}$. Then the following result holds.

Theorem 3.6 *Let $T(t)$ be a C_0-compact semigroup on E and $\mathcal{B}$ satisfies the axiom (C). Assume that $\hat{\omega}_e(S_0) < \omega_s(U_L)$ and that λ_0 in R satisfies the following conditions :*

(1)$P\sigma_0(\Delta) = \{\lambda_0\}$ and λ_0 belongs to the resolvent set of A.

(2)λ_0 is a simple pole of $\Delta(\lambda)^{-1}$.

Then the solution semigroup $U_L(t)$ of Eq.(AL) has asynchronous exponential growth with intrinsic growth constant $\lambda_0 \in R$.

The proof follows immediately from Proposition 2.3 in [11], Lemma 3.1, Lemma 3.2 and Lemma 3.4.

4 An example

Let $E = L^2([0,\pi], C)$, the set of square integrable functions on $[0,\pi]$. Consider the equation

$$(9) \qquad u'(t) = Au(t) + b\int_{-\infty}^{t} e^{-c(t-s)}u(s)\,ds,$$

where A is defined as $Af = f''$ for $f \in E$ such that f is continuously differentiable, the derivative f' is absolutely continuous, $f'' \in E$, and that $f(0) = f(\pi) = 0$. It is well known that A is a closed linear operator with dense domain. It is self adjoint, the spectrum of A consists of only point spectrum $\lambda = -n^2, n = 1, 2, \cdots$, $R(\lambda, A)$ has a pole of order 1 at these points, and $|R(\lambda, A)| \leq 1/|\lambda + 1|$ for $\Re\lambda > -1$. Hence A is the infinitesimal generator of a C_0-semigroup $T(t)$ such that $|T(t)| \leq e^{-t}$ for $t \geq 0$. Furthermore, $T(t)$ is a C_0-compact semigroup.

Set

$$L(\phi) = b \int_{-\infty}^{0} e^{c\theta}\phi(\theta)\, d\theta.$$

Then we have that $|L(\phi)| \leq |b|\|\phi\|$ for $\phi \in E \times \mathcal{L}_{-c}$. Hence, we will take $\mathcal{B} = E \times \mathcal{L}_{-c}$ as the phase space of Eq.(9). We note that if $c > 0$, then $\mathcal{B}$ is a uniform fading memory space. The solution semigroup of Eq.(9) and its infinitesimal generator are denoted by $U_L(t)$ and A_L, respectively.

The characteristic operator $\Delta(\lambda)$, given by (4), now becomes

$$\Delta(\lambda)f = \lambda f - Af - b\int_{-\infty}^{0} e^{(c+\lambda)\theta}\, d\theta f \qquad f \in D(A), \quad \Re\lambda > -c.$$

If we set $h(\lambda) = \lambda - b/(c+\lambda)$ for $\Re\lambda > -c$, then we can write $\Delta(\lambda) = h(\lambda)I - A$. Moreover, the equation $h(\lambda) = -n^2$ has the roots $\kappa_n = [-(c+n^2) - \sqrt{D}]/2, \ \lambda_n = [-(c+n^2) + \sqrt{D}]/2$, where $D = (c+n^2)^2 - 4(cn^2 - b)$.

Define a curve $b = \chi(c)$ in the c-b plane by

$$\chi(c) = \begin{cases} 0 & \text{for } c \leq 1 \\ -(c-1)^2/4 & \text{for } c > 1, \end{cases}$$

and divide the c-b plane into the subregions as

$$\Pi_1 : b > \chi(c), -\infty < c < \infty, \quad \Pi_2 : b \leq \chi(c), c > 1, \quad \Pi_3 : b \leq \chi(c), c \leq 1.$$

The following result can be found in [4].

Lemma 4.1 *The abscissa of the exponent of the phase space $\mathcal{B}$, and the growth bound and the essential growth bound of U_L become as follows:*

$$\hat{\omega}_e(S_0) = \omega_e(S_0) = \omega_s(S_0) = \omega_e(U_L) = -c = \gamma_0,$$

$$\omega_s(U_L) = \begin{cases} \lambda_1 & \text{if} \quad (c, b) \in \Pi_1 \\ -(c+1)/2 & \text{if} \quad (c, b) \in \Pi_2 \\ -c & \text{if} \quad (c, b) \in \Pi_3. \end{cases}$$

Proposition 4.2 1) *The zero solution of Eq.(9) is exponentially asymptotically stable if and only if* $c > 0$ *and* $b < c$.

2) *Assume that* $(c, b) \in \Pi_1$ *and* $c > 0$. *The following statements are equivalent.*

(i) *The zero solution of Eq.(9) is asymptotically stable.*

(ii) *The zero solution of Eq.(9) is exponentially asymptotically stable.*

(iii) $b < c$.

3) *Assume that* $(c, b) \in \Pi_2$. *Then, the zero solution of Eq.(9) is asymptotically stable if and only if it is exponentially asymptotically stable.*

Proof. The assertion 1) was obtained in [4]. If $(c, b) \in \Pi_1$, then $-c < \lambda_1$. This implies the inequality $\omega_e(U_L) < \omega_s(U_L)$, because of Lemma 4.1. If $b < c$, then λ_1 is negative. Hence the assertion 2) follows from Lemma 4.1 and Theorem 3.3. Since $(c, b) \in \Pi_2$, we have $c > 1$, from which it follows that the inequality $\omega_e(U_L) < \omega_s(U_L)$ holds. Thus Theorem 3.3 implies the assertion 3).

The following result was already shown in [4] and its proof is strongly dependent on the concrete form of Eq.(9). In contrast with it, our manner is based on Theorem 3.5.

Proposition 4.3 *Assume that* $c > 0$. *Then the zero solution of Eq.(9) is stable but not asymptotically stable if and only if* $b = c$.

Proof. Assume that the zero solution of Eq.(9) is stable but not asymptotically stable. Since $c > 0$, it follows from 1) in Proposition 4.2 that $b \geq c$. If $b > c$, then $\lambda_1 > 0$. Thus $e^{\lambda_1 t} \leq \|U_L(t)\| \to \infty$ as $t \to \infty$. Hence $b = c$.

Next, we assume that $b = c$. Then $\lambda_1 = 0$. Thus we have $-c = \omega_e(U_L) < \omega_s(U_L) = 0$ from Lemma 4.1. Of course, $P_\sigma(A) \cap P_0(A_L) = \emptyset$. For the characteristic operator $\Delta(\lambda) = h(\lambda)I - A$, we see that $N(\Delta(\lambda_1)) \neq \{0\}$, and $P_0(A_L) = \{\lambda_1\}$. Moreover, since $T(t)$ is a C_0-compact semigroup, there exists an $\varepsilon_0, 0 < \varepsilon_0 < \omega_s(U_L) - \hat{\omega}_e(S_0)$ such that λ_1 is a unique pole of $\Delta(\mu)^{-1}$ within $H(\varepsilon_0) := \{\mu \in C : \Re\mu \geq -\varepsilon_0\}$.

Finally, we shall show that λ_1 is a simple pole of $\Delta(\mu)^{-1}$. Set $g(w) = R(w, A)$. Then $g(h(z)) = \Delta(z)^{-1}$ and $h(0) = -1$. Since $g(w)$ has a pole of order 1 at $w = -1$, we have $g(w) = \hat{g}(w)/(w + 1)$, where $\hat{g}(w)$ is holomorphic at $w = -1$ and $\hat{g}(-1) \neq 0$. Hence we have

$$g(h(z)) = \frac{\hat{g}(h(z))}{h(z) + 1} = \frac{\hat{g}(h(z))(z + c)}{z(z + c + 1)},$$

from which $g(h(z))$ has a pole of order 1 at $z = 0$. Thus $\lambda_1 (= z = 0)$ is a simple pole of $\Delta(z)^{-1}$. Therefore the proof of the theorem follows from Theorem 3.5.

Proposition 4.4 *Assume that* $b = c > 0$. *Then the solution semigroup* $U_L(t)$ *of Eq.(9) has asynchronous exponential growth with intrinsic growth constant* $0 \in R$.

The proof follows immediately from Theorem 3.6 and the proof of Proposition 4.3.

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