数理解析研究所講究録
1083 巻 1999 年 20-31

Numerical Solution of Two Dimensional Sine-Gordon Equations Based on FEM

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1 Introduction

In this paper, we use the finite element method to numerically solve two dimensional sine-Gordon equations. For theoretical and numerical treatments of weak solutions of sine-Gordon equations we refer to [1], [6], [2] and [3]. The FEM of non-linear two dimensional problems is complicated as one has to assemble these equations over the domain which needs a lot of numerical integrations techniques. For simplicity of calculations, we divide the domain Ω into triangular elements that will be used as subdomains to find the solution over them. After the assembly, we impose the initial and boundary conditions, solve the systems of equations, postprocess the solutions and show the results graphically.

2 Two dimensional sine-Gordon equation

Let $\Omega=(0,1)\times(0,1)$. Consider the problem of finding the numerical solutions to the two dimensional sine-Gordon partial differential equations :

$$\frac{\partial^2 u(x,y,t)}{\partial t^2}+\alpha\frac{\partial u(x,y,t)}{\partial t}-\beta\Delta u(x,y,t)+\gamma\sin u(x,y,t)=f(x,y,t), \tag{2.1}$$

with initial and boundary conditions given as

$$\begin{aligned} &u(x,y,0)=\sin(\pi x)\sin(\pi y),\quad \frac{\partial}{\partial t}u(x,y,0)=0, && (x,y)\in\Omega \\ &u(x,y,t)\Big|_{\partial\Omega}=0, && (x,y,t)\in\partial\Omega\times(0,T), \end{aligned}$$

where, $\alpha,\beta>0,$ and γ are physical constants, and $f\in L^2\big(\Omega\times(0,T)\big)$.

3 Steps of the numerical solutions

We study the sine-Gordon equations based on FEM discussed in [4] and [5]. The major steps of the FEM numerical solutions of the above equations are as follows:

1. Discretization of the domain.
2. Weak formulation.
3. Development of the FEM using the weak form.
4. Assembly of finite element to get the global system.
5. Impositions of the boundary conditions.
6. Numerical solutions of the equations.
7. Post-computation of numerical solutions.

In what follows, we will explain about each steps.

- **Discretization of the domain.** We devide the domain Ω into equally spaced subdomains, that will be used to approximate the solution over them.

- **Weak formulation.** Let Ω^e be an arbitrary element in Ω. We develop FEM on it. Multiplying (2.1) by an arbitrary function w and integrating

$$0 = \int_{\Omega^e} w\,[\ddot{u} + \alpha\dot{u} - \beta\Delta u + \gamma\sin(u) - f\,]\,dx\,dy \tag{3.1}$$

Using integration by parts and divergence theorem, (3.1) becomes

$$0 = \int_{\Omega^e} [w\ddot{u} + \alpha w\dot{u} + \beta\nabla w\nabla u + \gamma w\sin(u) - w\,f\,]\,dx\,dy - \beta\oint_{\Gamma^e} wq\;ds. \tag{3.2}$$

Here, $q \equiv \left[n_x\dfrac{\partial u}{\partial x} + n_y\dfrac{\partial u}{\partial y}\right]$ and n_x , n_y are the components of the unit normal vector and ds is the arclength of an infinitesimal line element along the boundary. We shall call $u \equiv$ primary variable, and $q \equiv$ secondary variable.

- **FEM models.** Suppose that u is approximated over a typical element Ω^e at time t by the expression

$$u(x,y,t) \approx U^e(x,y,t) = \sum_{j=1}^{n} \xi_j^e(t)\psi_j^e(x,y), \tag{3.3}$$

Here, ξ_j^e is the value of U^e at jth node of the element at time t, and ψ_j^e is the Lagrange interpolation functions, with the property $\psi_i^e(x_j,y_j) = \delta_{ij}$. By substituting (3.3) in (3.2), we have $\sum\limits_{i=1}^{m} \ddot{\xi}_i^e\psi_{ij}^e + \alpha\sum\limits_{i=1}^{m} \dot{\xi}_i^e\psi_{ij}^e + \beta\sum\limits_{i=1}^{m} \xi_i^e\phi_{ij}^e + \gamma s_j^e - f_j^e = \beta q_j^e$, where $\psi_{ij}^e = (\psi_i^e,\psi_j^e)$, $\phi_{ij}^e = (\phi_i^e,\phi_j^e)$, $s_j^e = ((\sin(\sum\limits_{i=1}^{m}\xi_i^e\psi_i^e),\psi_j^e)$, $f_j^e = (f^e,\psi_j^e)$ and $q_j^e = (q^e,\psi_j)$.

- **Interpolation functions.** The linear interpolation functions for the triangular element are

$$\psi_i^e = \frac{1}{2A_e}(\alpha_i^e + \beta_i^e x + \gamma_i^e y) \qquad (i = 1, 2, 3), \tag{3.4}$$

where A_e is the area of the triangle, and α_i^e, β_i^e and γ_i^e are the geometric constants

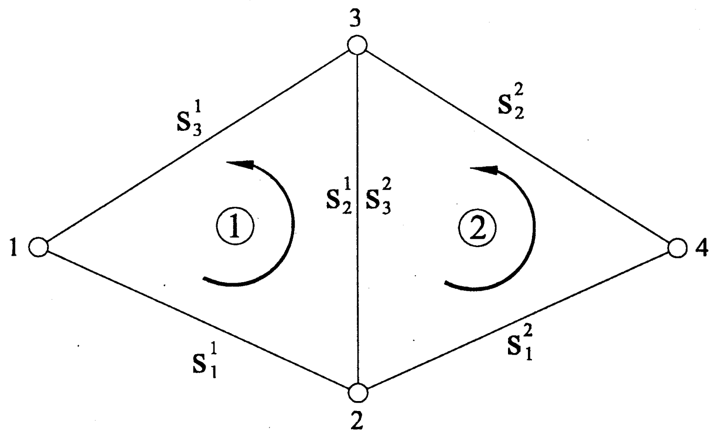
$$\left\{ \begin{array}{lll} \alpha_i^e = & x_j^e y_k^e - x_k^e y_j^e & \qquad i \neq j \neq k, \\ \beta_i^e = & y_j^e - y_k^e & \qquad i, j \ \& \ k \\ \gamma_i^e = & -(x_j^e - x_k^e) & \qquad \text{permute in natural order.} \end{array} \right.$$

The linear interpolation functions have the properties

$$\psi_i^e(x_j^e, y_j^e) = \delta_{ij} \qquad (i, j = 1, 2, 3)$$

$$\sum_{i=1}^{3} \psi_i^e = 1, \quad \sum_{i=1}^{3} \frac{\partial \psi_i^e}{\partial x} = 0, \quad \sum_{i=1}^{3} \frac{\partial \psi_i^e}{\partial y} = 0.$$

- **Assembly.** The assembly of finite element equations is based on the following two principles (cf. 5):
  1. *Continuity of primary variables.*
  2. *"Equilibrium" of secondary variables.*



$S_j^i \equiv$ side j of element i, $(i = 1, 2)$ and $(j = 1, 2, 3)$.

図 1: Assembly process of two elements

The element equations of the two elements are :

1. *First triangular element*

$$\begin{cases} (\psi^1_{11}\ddot{\xi}^1_1 + \psi^1_{12}\ddot{\xi}^1_2 + \psi^1_{13}\ddot{\xi}^1_3) + \alpha(\psi^1_{11}\dot{\xi}^1_1 + \psi^1_{12}\dot{\xi}^1_2 + \psi^1_{13}\dot{\xi}^1_3) \\ +\beta(\phi^1_{11}\xi^1_1 + \phi^1_{12}\xi^1_2 + \phi^1_{13}\xi^1_3) + \gamma\Big(\sin(\psi^1_1\xi^1_1 + \psi^1_2\xi^1_2 + \psi^1_3\xi^1_3), \psi^1_1\Big) = f^1_1 + \beta q^1_1, \\ (\psi^1_{21}\ddot{\xi}^1_1 + \psi^1_{22}\ddot{\xi}^1_2 + \psi^1_{23}\ddot{\xi}^1_3) + \alpha(\psi^1_{21}\dot{\xi}^1_1 + \psi^1_{22}\dot{\xi}^1_2 + \psi^1_{23}\dot{\xi}^1_3) \\ +\beta(\phi^1_{21}\xi^1_1 + \phi^1_{22}\xi^1_2 + \phi^1_{23}\xi^1_3) + \gamma\Big(\sin(\psi^1_1\xi^1_1 + \psi^1_2\xi^1_2 + \psi^1_3\xi^1_3), \psi^1_2\Big) = f^1_2 + \beta q^1_2, \\ (\psi^1_{31}\ddot{\xi}^1_1 + \psi^1_{32}\ddot{\xi}^1_2 + \psi^1_{33}\ddot{\xi}^1_3) + \alpha(\psi^1_{31}\dot{\xi}^1_1 + \psi^1_{32}\dot{\xi}^1_2 + \psi^1_{33}\dot{\xi}^1_3) \\ +\beta(\phi^1_{31}\xi^1_1 + \phi^1_{32}\xi^1_2 + \phi^1_{33}\xi^1_3) + \gamma\Big(\sin(\psi^1_1\xi^1_1 + \psi^1_2\xi^1_2 + \psi^1_3\xi^1_3), \psi^1_3\Big) = f^1_3 + \beta q^1_3. \end{cases} \tag{3.5}$$

2. *Second triangle*

$$\begin{cases} (\psi^2_{11}\ddot{\xi}^2_1 + \psi^2_{12}\ddot{\xi}^2_2 + \psi^2_{13}\ddot{\xi}^2_3) + \alpha(\psi^2_{11}\dot{\xi}^2_1 + \psi^2_{12}\dot{\xi}^2_2 + \psi^2_{13}\dot{\xi}^2_3) \\ +\beta(\phi^2_{11}\xi^2_1 + \phi^2_{12}\xi^2_2 + \phi^2_{13}\xi^2_3) + \gamma\Big(\sin(\psi^2_1\xi^2_1 + \psi^2_2\xi^2_2 + \psi^2_3\xi^2_3), \psi^2_1\Big) = f^2_1 + \beta q^2_1, \\ (\psi^2_{21}\ddot{\xi}^2_1 + \psi^2_{22}\ddot{\xi}^2_2 + \psi^2_{23}\ddot{\xi}^2_3) + \alpha(\psi^2_{21}\dot{\xi}^2_1 + \psi^2_{22}\dot{\xi}^2_2 + \psi^2_{23}\dot{\xi}^2_3) + \\ \beta(\phi^2_{21}\xi^2_1 + \phi^2_{22}\xi^2_2 + \phi^2_{23}\xi^2_3) + \gamma\Big(\sin(\psi^2_1\xi^2_1 + \psi^2_2\xi^2_2 + \psi^2_3\xi^2_3), \psi^2_2\Big) = f^2_2 + \beta q^2_2, \\ (\psi^2_{31}\ddot{\xi}^2_1 + \psi^2_{32}\ddot{\xi}^2_2 + \psi^2_{33}\ddot{\xi}^2_3) + \alpha(\psi^2_{31}\dot{\xi}^2_1 + \psi^2_{32}\dot{\xi}^2_2 + \psi^2_{33}\dot{\xi}^2_3) \\ +\beta(\phi^2_{31}\xi^2_1 + \phi^2_{32}\xi^2_2 + \phi^2_{33}\xi^2_3) + \gamma\Big(\sin(\psi^2_1\xi^2_1 + \psi^2_2\xi^2_2 + \psi^2_3\xi^2_3), \psi^2_3\Big) = f^2_3 + \beta q^2_3. \end{cases} \tag{3.6}$$

From the connectivity of primary variables (ξ^i_j, $i = 1, 2$ and $j = 1, 2, 3$), we have the correspondence between the local and global nodal values is (see Fig. 1) :

$$u^1_1 = U_1, \quad u^1_2 = u^2_1 = U_2, \quad u^2_2 = U_4, \quad u^1_3 = u^2_3 = U_3. \tag{3.7}$$

From the balance of the secondary variables (q), we have:

$$(q^1)_{2-3} = (q^2)_{3-1} \quad \text{or} \quad (q^1)_{2-3} = (-q^2)_{3-1}$$

i.e.,

$$q^1_{22} + q^2_{13} = 0, \qquad q^1_{32} + q^2_{33} = 0. \tag{3.8}$$

From (3.8), we must add the second equation of element 1 to the first equation of element 2, and also the third equation of element 1 to the third equation of element 2 :

$$
\begin{cases}
\Big[(\psi^1_{21}\ddot{\xi}^1_1 + \psi^1_{22}\ddot{\xi}^1_2 + \psi^1_{23}\ddot{\xi}^1_3) + (\psi^2_{11}\ddot{\xi}^2_1 + \psi^2_{12}\ddot{\xi}^2_2 + \psi^2_{13}\ddot{\xi}^2_3)\Big] + \alpha\Big[(\psi^1_{21}\dot{\xi}^1_1 + \psi^1_{22}\dot{\xi}^1_2 + \psi^1_{23}\dot{\xi}^1_3) + (\psi^2_{11}\dot{\xi}^2_1 + \psi^2_{12}\dot{\xi}^2_2 + \psi^2_{13}\dot{\xi}^2_3)\Big] \\
+\beta\Big[(\phi^1_{21}\xi^1_1 + \phi^1_{22}\xi^1_2 + \phi^1_{23}\xi^1_3) + (\phi^2_{11}\xi^2_1 + \phi^2_{12}\xi^2_2 + \phi^2_{13}\xi^2_3)\Big] + \gamma\Big[\big(\sin(\psi^1_1\xi^1_1 + \psi^1_2\xi^1_2 + \psi^1_3\xi^1_3), \psi^1_2\big) \\
+\big(\sin(\psi^2_1\xi^2_1 + \psi^2_2\xi^2_2 + \psi^2_3\xi^2_3), \psi^2_1\big)\Big] = (f^1_2 + \beta q^1_2) + (f^2_1 + \beta q^2_1), \\
\\
\Big[(\psi^1_{31}\ddot{\xi}^1_1 + \psi^1_{32}\ddot{\xi}^1_2 + \psi^1_{33}\ddot{\xi}^1_3) + (\psi^2_{31}\ddot{\xi}^2_1 + \psi^2_{32}\ddot{\xi}^2_2 + \psi^2_{33}\ddot{\xi}^2_3)\Big] + \alpha\Big[(\psi^1_{31}\dot{\xi}^1_1 + \psi^1_{32}\dot{\xi}^1_2 + \psi^1_{33}\dot{\xi}^1_3) + (\psi^2_{31}\dot{\xi}^2_1 + \psi^2_{32}\dot{\xi}^2_2 + \psi^2_{33}\dot{\xi}^2_3)\Big] \\
+\beta\Big[(\phi^1_{31}\xi^1_1 + \phi^1_{32}\xi^1_2 + \phi^1_{33}\xi^1_3) + (\phi^2_{31}\xi^2_1 + \phi^2_{32}\xi^2_2 + \phi^2_{33}\xi^2_3)\Big] + \gamma\Big[\big(\sin(\psi^1_1\xi^1_1 + \psi^1_2\xi^1_2 + \psi^1_3\xi^1_3), \psi^1_3\big) \\
+\big(\sin(\psi^2_1\xi^2_1 + \psi^2_2\xi^2_2 + \psi^2_3\xi^2_3), \psi^2_3\big)\Big] = (f^1_3 + \beta q^1_3) + (f^2_3 + \beta q^2_3).
\end{cases}
$$

Using eq.(3.7) equation the above equation becomes

$$
\begin{cases}
\Big[\psi^1_{21}\ddot{\xi}_1 + (\psi^1_{22} + \psi^2_{11})\ddot{\xi}_2 + (\psi^1_{23} + \psi^2_{13})\ddot{\xi}_3 + \psi^2_{12}\ddot{\xi}_4\Big] + \alpha\Big[\psi^1_{21}\dot{\xi}_1 + (\psi^1_{22} + \psi^2_{11})\dot{\xi}_2 + (\psi^1_{23} + \psi^2_{13})\dot{\xi}_3 + \psi^2_{12}\dot{\xi}_4\Big] \\
+\beta\Big[\phi^1_{21}\xi_1 + (\phi^1_{22} + \phi^2_{11})\xi_2 + (\phi^1_{23} + \phi^2_{13})\xi_3 + \phi^2_{12}\xi_4\Big] + \gamma\Big[\big(\sin(\psi^1_1\xi_1 + \psi^1_2\xi_2 + \psi^1_3\xi_3), \psi^1_1\big) \\
+\big(\sin(\psi^2_1\xi_2 + \psi^2_2\xi_4 + \psi^2_3\xi_3), \psi^2_3\big)\Big] = (f^1_1 + \beta f^2_3) + (q^1_1 + \beta q^2_3), \\
\\
\Big[\psi^1_{31}\ddot{\xi}_1 + (\psi^1_{32} + \psi^2_{31})\ddot{\xi}_2 + (\psi^1_{33} + \psi^2_{33})\ddot{\xi}_3 + \psi^2_{32}\ddot{\xi}_4\Big] + \alpha\Big[\psi^1_{31}\dot{\xi}_1 + (\psi^1_{32} + \psi^2_{31})\dot{\xi}_2 + (\psi^1_{33} + \psi^2_{33})\dot{\xi}_3 + \psi^2_{32}\dot{\xi}_4\Big] \\
+\beta\Big[\phi^1_{31}\xi_1 + (\phi^1_{32} + \phi^2_{31})\xi_2 + (\phi^1_{33} + \phi^2_{33})\xi_3 + \phi^2_{32}\xi_4\Big] + \gamma\Big[\big(\sin(\psi^1_1\xi_1 + \psi^1_2\xi_2 + \psi^1_3\xi_3), \psi^1_3\big) \\
+\big(\sin(\psi^2_1\xi_2 + \psi^2_2\xi_4 + \psi^2_3\xi_3), \psi^2_1\big)\Big] = (f^1_3 + f^2_1) + (\beta q^1_3 + \beta q^2_1),
\end{cases}
$$

where ξ_i denotes the global variables.

For the 2-element mesh in fig.1, the assembled equations are

$$
\begin{cases}
\begin{bmatrix} \psi^1_{11} & \psi^1_{12} & \psi^1_{13} & 0 \\ \psi^1_{21} & \psi^1_{22} + \psi^2_{11} & \psi^1_{23} + \psi^2_{13} & \psi^2_{12} \\ \psi^1_{31} & \psi^1_{32} + \psi^2_{31} & \psi^1_{33} + \psi^2_{33} & \psi^2_{32} \\ 0 & \psi^2_{31} & \psi^2_{33} & \psi^2_{32} \end{bmatrix} \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ \ddot{\xi}_3 \\ \ddot{\xi}_4 \end{pmatrix} + \alpha \begin{bmatrix} \psi^1_{11} & \psi^1_{12} & \psi^1_{13} & 0 \\ \psi^1_{21} & \psi^1_{22} + \psi^2_{11} & \psi^1_{23} + \psi^2_{13} & \psi^2_{12} \\ \psi^1_{31} & \psi^1_{32} + \psi^2_{31} & \psi^1_{33} + \psi^2_{33} & \psi^2_{32} \\ 0 & \psi^2_{31} & \psi^2_{33} & \psi^2_{32} \end{bmatrix} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} \\
+\beta \begin{bmatrix} \phi^1_{11} & \phi^1_{12} & \phi^1_{13} & 0 \\ \phi^1_{21} & \phi^1_{22} + \phi^2_{11} & \phi^1_{23} + \phi^2_{13} & \phi^2_{12} \\ \phi^1_{31} & \phi^1_{32} + \phi^2_{31} & \phi^1_{33} + \phi^2_{33} & \phi^2_{32} \\ 0 & \phi^2_{31} & \phi^2_{33} & \phi^2_{32} \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \\
+\gamma \begin{pmatrix} \big(\sin(\psi^1_1\xi_1 + \psi^1_2\xi_2 + \psi^1_3\xi_3), \psi^1_1\big) \\ \big(\sin(\psi^1_1\xi_1 + \psi^1_2\xi_2 + \psi^1_3\xi_3), \psi^1_2\big) + \big(\sin(\psi^2_1\xi_2 + \psi^2_2\xi_4 + \psi^2_3\xi_3), \psi^2_1\big) \\ \big(\sin(\psi^1_1\xi_1 + \psi^1_2\xi_2 + \psi^1_3\xi_3), \psi^1_3\big) + \big(\sin(\psi^2_1\xi_2 + \psi^2_2\xi_4 + \psi^2_3\xi_3), \psi^2_3\big) \\ \big(\sin(\psi^2_1\xi_2 + \psi^2_2\xi_4 + \psi^2_3\xi_3), \psi^2_2\big) \end{pmatrix} = \begin{pmatrix} F^1_1 \\ F^1_2 + F^2_1 \\ F^1_3 + F^2_3 \\ F^2_2 \end{pmatrix}.
\end{cases}
\tag{3.9}
$$

Here, $F^i_j \equiv f^i_j + \beta q^i_j, \quad i = 1, 2, \quad j = 1, 2, 3.$

• Evaluation of element matrices and vectors.

1. Area coordinates. For triangular elements, it is possible to construct 3-nondimensionalized coordinates $L_i, (i = 1, 2, 3)$,

$$
L_i = \frac{A_i}{A}, \qquad A = \sum_{i=1}^{3} A_i, \qquad L_1 = A_1/A = s/h,
$$

Hence $\psi_i = L_i$ represents the interpolation function for linear triangular element.

2. Coordinate transformation. For rectangular Ω_e, we can define a transformation $\Omega_e \to \hat{\Omega}_e$, $\hat{\Omega}_e \equiv$ master element $= (\mu, \eta) : 0 \leq \mu \leq \eta \leq 1$. The relation between (x, y) coord. system and (μ, η) coord. system is:

$$x = \sum_{j=1}^{3} x_j^e \hat{\psi}_j^e(\mu, \eta), \qquad y = \sum_{j=1}^{3} y_j^e \hat{\psi}_j^e(\mu, \eta). \tag{3.10}$$

3. Evaluation of the integral in the (μ, η). After the transformation, integrals on $\hat{\Omega}_T$ have the form

$$\int_{\hat{\Omega}_T} G(\mu, \eta) d\mu d\eta = \int_{\hat{\Omega}_T} \hat{G}(L_1, L_2, L_3) dL_1 dL_2,$$

which can be approximated by the quadrature formula

$$\int_{\hat{\Omega}_T} \hat{G}(L_1, L_2, L_3) dL_1 dL_2 \approx \frac{1}{2} \sum_{i=1}^{N} W_I \hat{G}(S_I).$$

Here W_I and S_I denote the weights and integration points of the quadrature rule.

• **Integrating the sine functions.** To evaluate the integral

$$I = \int_0^h \int_0^h G(x, y) dx dy,$$

where $G(x, y) = (\sin(\sum_{i=1}^{3} \xi_i^e \psi_i^e), \psi_j^e)$ $j = 1, 2, 3$, we use *Gauss-Legendre* method of order m. We start by deviding the interval $[x_i, x_{i+1}]$ in x dirction into m points and $[y_i, y_{i+1}]$ in y direction, then

$$I \approx \sum_{k=1}^{m} \sum_{l=1}^{m} \overline{W}_{kl} G(x_k, y_l), \quad \overline{W}_{kl} = W_k W_l,$$

where $(x_k, y_l) \equiv$ quadrature points and $\overline{W}_{kl} \equiv$ weights.
Let us denote

$$s_j^e(x, y) \equiv \Big(\sin(\psi_i^e(x, y) \xi_i^e), \psi_j^e(x, y)\Big),$$

where $\psi^e(x, y)$ is given by equation (3.4). Applying *G-L* scheme of order m, the j-th component of s_j^e becomes

$$\begin{aligned} s_j^e(\xi) &= \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{1}{2A_e} \overline{W}_{kl} \Big(\sin\big(\frac{1}{2A_e}(\alpha_i^e + \beta_i^e x_k^e + \gamma_i^e y_l^e)\, \xi_i^e\big)(\alpha_j^e + \beta_j^e x_k^e + \gamma_j^e y_l^e)\Big) \\ &= \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{1}{2A_e} \overline{W}_{kl} \Big(\sin\big(\frac{1}{2A_e}(\alpha_i^e + \beta_i^e x_k^e + \gamma_i^e y_l^e)\, \xi_i^e\big)(\alpha_j^e + \beta_j^e x_k^e + \gamma_j^e y_l^e)\Big), \\ & \qquad\qquad (j = 1, 2, 3). \end{aligned}$$

Then, the sine functions in equation (3.9) are approximated by

$$S(\xi) \approx \tilde{S}(\xi) = \begin{Bmatrix} s_1^1 \\ s_2^1 + s_1^2 \\ s_3^1 + s_3^2 \\ s_2^2 \end{Bmatrix}.$$

Hence, equation (3.9) becomes

$$\begin{pmatrix} \Xi \\ \dot{\Xi} \end{pmatrix}' + \begin{bmatrix} 0 & -1 \\ \beta\Psi^{-1}\Phi & \alpha \end{bmatrix} \begin{pmatrix} \Xi \\ \dot{\Xi} \end{pmatrix} = \begin{pmatrix} 0 \\ F - \gamma\tilde{S}(\Xi) \end{pmatrix}, \tag{3.11}$$

which is a first-order differential equation to be solved by using *Runge-Kutta* method.

• **Postprocessing the solution.** After the solving the equations, we get the solutions array (variables values at all nodes) of dimension $n_x \times n_y$, where n_x is the number of elements in the x-direction and also n_y in the y-dirction. This array should be partition according to the number of nodes in x-direction repeatedly n_y times.

4 Numerical simulation results.

We use a rectangular mesh consists of 15 elements in the x-direction and 15 elements in the y-direction, one element of length 0.07, hence we get 456 triangular elements consists the domain solution moreover the gaussian quadrature pointes are chosen to be 6 points for the element and the intial condition is $y_0(x, y) = \sin(\pi x)\ \sin(\pi y)$. The simulation results shown here are taken every 10 times due to the size of each run, i.e., we skip 10 graph after each. The results are shown on the next pages.

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$\alpha = 0.001,\ \beta = 0.0001, \gamma = 1$