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Sturm-Liouville eigenvalue problems for half-linear ordinary differential equations

福岡大・理　草野 尚 (Kusano Takaŝi)
愛媛大・理　内藤 学 (Manabu Naito)

1. Introduction. In this paper the second order half-linear ordinary differential equation

$$(1.1)\qquad (p(t)|x'|^{\alpha-1}x')' + \lambda q(t)|x|^{\alpha-1}x = 0, \quad a \le t \le b,$$

is considered together with the boundary conditions

$$(1.2)\qquad Ax(a) - A'x'(a) = 0, \quad Bx(b) + B'x'(b) = 0.$$

In equation (1.1) we assume that $\alpha > 0$ is a positive constant, p and q are real-valued continuous functions for $a \le t \le b$, and $p(t) > 0$ $(a \le t \le b)$, and $\lambda \in \boldsymbol{R}$ is a real parameter. In the boundary conditions (1.2), A, A', B and B' are given real numbers such that $A^2 + A'^2 \ne 0$ and $B^2 + B'^2 \ne 0$.

If $\alpha = 1$, then equation (1.1) reduces to the linear equation

$$(1.3)\qquad (p(t)x')' + \lambda q(t)x = 0, \quad a \le t \le b,$$

and the reduced problem is the Sturm-Liouville eigenvalue problem. This topic is one of the most important subjects in the theory of second order linear equations. In the special case that $q(t) > 0$ $(a \le t \le b)$, very complete treatments can be developed ([7, 9, 16, 17]), and it is well known that there exists a sequence of real numbers (which are called eigenvalues) $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ with $\lim_{n\to\infty} \lambda_n = +\infty$ such that (i) (1.3)–(1.2) has a nontrivial solution (which is called eigenfunction) if and only if $\lambda = \lambda_n$ for some $n = 0, 1, 2, \cdots$; (ii) the eigenfunction $x = x(t; \lambda_n)$ associated with $\lambda = \lambda_n$ has exactly n zeros in the open interval (a, b).

Moreover, in the case where $q(t)$ changes signs in (a, b), the following is known ([9, 16]): Let $AA' \ge 0$, $BB' \ge 0$ and $A^2 + B^2 \ne 0$. Then there exist a sequence of positive eigenvalues $\lambda_0^+ < \lambda_1^+ < \cdots < \lambda_n^+ < \cdots$ with $\lim_{n\to\infty} \lambda_n^+ = +\infty$ and a sequence of negative eigenvalues $\lambda_0^- > \lambda_1^- > \cdots > \lambda_n^- > \cdots$ with $\lim_{n\to\infty} \lambda_n^- = -\infty$ such that (i) (1.3)–(1.2) has a nontrivial solution (eigenfunction) if and only if $\lambda = \lambda_n^+$ or λ_n^- for some

$n = 0, 1, 2, \cdots$; (ii) the eigenfunctions $x = x(t; \lambda_n^+)$ and $x(t; \lambda_n^-)$ associated with $\lambda = \lambda_n^+$ and λ_n^- have exactly n zeros in (a, b).

The main purpose of this paper is to extend the above rerults for (1.3) in a natural way to the more general equation (1.1). By a solution x of (1.1) it is meant a real-valued function x such that $x \in C^1[a, b]$, $p|x'|^{\alpha-1}x' \in C^1[a, b]$, and x satisfies (1.1) at every point of $[a, b]$. A local solution of (1.1) is similarly defined, and it is known [15] that all local solutions of (1.1) can be continued on the whole interval $[a, b]$. Equation (1.1) has the half-linear property in the sense that if $x(t)$ is a solution of (1.1), then, for any constant c, $cx(t)$ is also a solution of (1.1). Equation (1.1) always has the trivial solution $x(t) \equiv 0$ for all $\lambda \in \boldsymbol{R}$. As in the linear case, if there is a nontrivial solution x of (1.1) satisfying (1.2) for a certain value of $\lambda \in \boldsymbol{R}$, then λ is called an *eigenvalue* of the problem (1.1)–(1.2), and the solution x is called an *eigenfunction* associated with λ.

Qualitative properties of solutions of the half-linear equation (1.1) were studied first by Mirzov [15] and Elbert [3]. Further analysis on (1.1) was made by several authors including Del Pino et al. [2], Elbert [4, 5], Hoshino et al. [8], Kusano et al. [10, 11, 13] and Li and Yeh [14]. Their study shows that most of the basic results for the linear equation (1.3) can be completely extended to the half-linear equation (1.1).

For eigenvalue problems of the form (1.1)–(1.2), a natural extension of the results for (1.3)–(1.2) is given by Elbert [3] and Kusano, Naito and Tanigawa [12]. However, in [3, 12], the case where $q(t) > 0$ $(a \le t \le b)$ is considered. Therefore in this paper we pay our attention to the case where $q(t)$ may change signs in (a, b), and give a complete generalization of the results for the Sturm-Liouville eigenvalue problem (1.3)–(1.2).

The main theorem is as follows:

Theorem 1.1. *Consider the problem* (1.1)–(1.2). *Let* $AA' \ge 0, BB' \ge 0$ *and* $A^2 + B^2 \ne 0$, *and suppose that* $q(t)$ *has a positive value at some point* $t \in [a, b]$. *Then the totality of the positive eigenvalues of* (1.1)–(1.2) *is composed of a sequence* $\{\lambda_n^+\}_{n=0}^\infty$ *such that*

$$\lambda_0^+ < \lambda_1^+ < \cdots < \lambda_n^+ < \cdots, \qquad \lim_{n\to\infty} \lambda_n^+ = +\infty.$$

The eigenfunction $x = x(t; \lambda_n^+)$ *associated with* $\lambda = \lambda_n^+$ *has exactly* n *zeros in* (a, b), *where* $n = 0, 1, 2, \cdots$.

It should be noticed that, in the above theorem, the positive property of $q(t)$ on the whole interval $[a, b]$ is not assumed.

Equation (1.1) can be rewritten as

$$(p(t)|x'|^{\alpha-1}x')' + (-\lambda)(-q(t))|x|^{\alpha-1}x = 0, \qquad a \le t \le b. \tag{1.4}$$

Therefore we get the next result corresponding to Theorem 1.1.

Theorem 1.2. *Consider the problem* (1.1)–(1.2). *Let* $AA' \ge 0, BB' \ge 0$ *and* $A^2 + B^2 \ne 0$, *and suppose that* $q(t)$ *has a negative value at some point* $t \in [a, b]$. *Then the* *totality of the negative eigenvalues of* (1.1)–(1.2) *is composed of a sequence* $\{\lambda_n^-\}_{n=0}^{\infty}$ *such that*

$$\lambda_0^- > \lambda_1^- > \cdots > \lambda_n^- > \cdots, \qquad \lim_{n\to\infty} \lambda_n^- = -\infty.$$

The eigenfunction $x = x(t; \lambda_n^-)$ *associated with* $\lambda = \lambda_n^-$ *has exactly* n *zeros in* (a, b), *where* $n = 0, 1, 2, \cdots$.

We can show without difficulty that the value $\lambda = 0$ is not an eigenvalue of (1.1)–(1.2). Thus Theorems 1.1 and 1.2 yield the following theorem.

Theorem 1.3. *Consider the problem* (1.1)–(1.2). *Let* $AA' \geq 0, BB' \geq 0$ *and* $A^2 + B^2 \neq 0$, *and suppose that* $q(t)$ *takes both a positive value and a negative value on* $[a, b]$. *Then the totality of eigenvalues of* (1.1)–(1.2) *consists of two sequences* $\{\lambda_n^+\}_{n=0}^{\infty}$ *and* $\{\lambda_n^-\}_{n=0}^{\infty}$ *such that*

$$\cdots < \lambda_n^- < \cdots < \lambda_1^- < \lambda_0^- < 0 < \lambda_0^+ < \lambda_1^+ < \cdots < \lambda_n^+ < \cdots$$

and

$$\lim_{n\to\infty} \lambda_n^+ = +\infty, \qquad \lim_{n\to\infty} \lambda_n^- = -\infty.$$

The eigenfunctions $x = x(t; \lambda_n^+)$ *and* $x(t; \lambda_n^-)$ *associated with* $\lambda = \lambda_n^+$ *and* λ_n^- *have exactly* n *zeros in* (a, b), *where* $n = 0, 1, 2, \cdots$.

Remark. In Theorems 1.1, 1.2 and 1.3, the eigenvalues of (1.1)–(1.2) are simple, i.e., for each eigenvalue, the associated eigenfunction is unique up to a multicative constant.

For the proof of Theorems 1.1, a variant of the generalized Prüfer transformation for the half-linear equation (1.1) plays a crucial role. This transformation involves the *generalized sine function* and the *generalized cosine function.* The definition and the basic properties of these generalized trigonometric functions are briefly stated in the next Section 2.

The fundamental theorems (such as the existence, uniqueness and continuous dependence on papràmeters of solutions) and the Sturmian theorems (such as comparison and separation properties concerning the zeros of solutions) are also important tools in this paper. These are also formulated in Section 2. The proof of Theorem 1.1 is given in Section 3.

2. Preparatory Results. We begin by formulating a fundamental theorem on existence, uniqueness and continuous dependence on parameters for solutions of the half-linear equation (1.1).

It is easy to see that $x(t)$ is a solution of (1.1) if and only if

$$(u_1(t), u_2(t)) = (x(t), p(t)|x'(t)|^{\alpha-1}x'(t))$$

is a solution of the first order system

$$\begin{cases} u_1' = r_1(t)|u_2|^{\lambda_1-1}u_2, \\ u_2' = r_2(t)|u_1|^{\lambda_2-1}u_1, \end{cases} \tag{2.1}$$

where $\lambda_1 = 1/\alpha, \lambda_2 = \alpha, r_1(t) = 1/(p(t))^{1/\alpha}$ and $r_2(t) = -\lambda q(t)$. In this sense, the second order equation (1.1) and the first order system (2.1) is the same.

Fundamental theorems on the initial value problem for equation (1.1) or system (2.1) are given in the papers of Mirzov [15] and Elbert [3]. By a result in [15], we have the following theorem.

Lemma 2.1. *Let $c \in [a,b]$, $\xi \in \boldsymbol{R}$, $\eta \in \boldsymbol{R}$ and $\lambda \in \boldsymbol{R}$ be any given constants, and consider equation* (1.1) *under the initial condition*

$$x(c) = \xi, \quad x'(c) = \eta. \tag{2.2}$$

Then the solution $x(t) = x(t; c, \xi, \eta, \lambda)$ of the initial value problem (1.1)–(2.2) *exists on $[a,b]$ and is unique.*

Since the initial value problem (1.1)–(2.2) has a unique solution, we find that, for each $\lambda \in \boldsymbol{R}$, every nontrivial solution of (1.1) has at most a finite number of zeros in $[a,b]$. Further we find that, for each $\lambda \in \boldsymbol{R}$, the solution of (1.1) which satisfies

$$Ax(a) - A'x'(a) = 0 \quad [\text{resp. } Bx(b) + B'x'(b) = 0]$$

is uniquely determined up to a multicative constant.

Applying a standard continuous dependence result (e.g., [1, pp. 18–19]) in the theory of ordinary differential equations, we see that the solution $x(t; c, \xi, \eta, \lambda)$ in Lemma 2.1 is a continuous function of $(t, c, \xi, \eta, \lambda) \in [a,b] \times [a,b] \times \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$. Further, if a sequence $\{(c_i, \xi_i, \eta_i, \lambda_i)\}$ tends to (c, ξ, η, λ) as $i \to \infty$, then the corresponding sequence of solutions $\{x(t; c_i, \xi_i, \eta_i, \lambda_i)\}$ tends to $x(t; c, \xi, \eta, \lambda)$ uniformly for $a \le t \le b$ as $i \to \infty$.

For the half-linear equation (1.1), the Sturm comparison theorem is still valid as follows:

Lemma 2.2. *Consider the two equations*

$$(p_1(t)|x'|^{\alpha-1}x')' + q_1(t)|x|^{\alpha-1}x = 0, \quad a \le t \le b, \tag{2.3}$$

and

$$(p_2(t)|x'|^{\alpha-1}x')' + q_2(t)|x|^{\alpha-1}x = 0, \quad a \le t \le b, \tag{2.4}$$

where $p_i(t)$ and $q_i(t)$ are continuous functions on $[a,b]$ and $p_i(t) > 0$ $(a \le t \le b), i = 1,2$. Suppose that

$$p_1(t) \ge p_2(t) \quad \text{and} \quad q_1(t) \le q_2(t) \quad \text{for} \quad a \le t \le b. \tag{2.5}$$

If a nontrivial solution of (2.3) *has two zeros* t_1 *and* t_2 $(a \le t_1 < t_2 \le b)$, *then every nontrivial solution of* (2.4) *has at least one zero in* $[t_1, t_2]$.

The proof of Lemma 2.2 is found in [3, 15]. As an immediate corollary of Lemma 2.2 we get the following result: Suppose that (2.5) holds, and let $x_1(t)$ and $x_2(t)$ be nontrivial solutions of (2.3) and (2.4), respectively. If $x_1(t)$ has zeros at $t = t_1$ and t_2, then either $x_2(t)$ has a zero in (t_1, t_2) or $x_2(t)$ is a constant multiple of $x_1(t)$. A further corollary is the following extension of the well-known Sturm separation theorem: The zeros of linearly independent solutions of the same equation (2.3) separate each other.

Now let us define the generalized trigonometric functions $S(\tau)$, $C(\tau)$ and $T(\tau)$ which generalize the classical trigonometric functions $\sin\tau$, $\cos\tau$ and $\tan\tau$, respectively. The generalized trigonometric functions are used to extend in a natural way the notion of the Prüfer tarnsformation, known for the Sturm-Liouville equation (1.3), to the half-linear equation (1.1). These generalized functions are introduced by Elbert [3]. For the properties stated below, see [3].

The *generalized sine function* $S = S(\tau)$ is defined as the solution of the specific half-linear equation

$$(|\dot S|^{\alpha-1}\dot S)^{\cdot} + \alpha|S|^{\alpha-1}S = 0 \qquad \left(\cdot = \frac{d}{d\tau}\right) \tag{2.6}$$

satisfying the initial condition

$$S(0) = 0, \quad \dot S(0) = 1. \tag{2.7}$$

The generalized sine function $S(\tau)$ has the same properties as the classical sine function $\sin\tau$. First of all it is defined on $\boldsymbol{R}$ and is periodic with period $2\pi_\alpha$, where

$$\pi_\alpha = \frac{2\pi}{\alpha+1}\Big/\sin\frac{\pi}{\alpha+1}. \tag{2.8}$$

Further, $S(\tau)$ is an odd function having zeros at $\tau = j\pi_\alpha$, $j \in \boldsymbol{Z}$; it is positive on the intervals $2j\pi_\alpha < \tau < (2j+1)\pi_\alpha$, $j \in \boldsymbol{Z}$, and negative on the intervals $(2j+1)\pi_\alpha < \tau < 2(j+1)\pi_\alpha$, $j \in \boldsymbol{Z}$.

The *generalized cosine function* $C(\tau)$ is the derivative $\dot S(\tau)$ of $S(\tau)$: $C(\tau) = \dot S(\tau)$. The $C(\tau)$ is periodic with period $2\pi_\alpha$, and is an even function. It has zeros at $\tau = \left(j + \frac{1}{2}\right)\pi_\alpha$, $j \in \boldsymbol{Z}$, and is positive for $\left(2j - \frac{1}{2}\right)\pi_\alpha < \tau < \left(2j + \frac{1}{2}\right)\pi_\alpha$, $j \in \boldsymbol{Z}$, and negative for $\left(2j + \frac{1}{2}\right)\pi_\alpha < \tau < \left(2j + \frac{3}{2}\right)\pi_\alpha$, $j \in \boldsymbol{Z}$.

We have

$$S(\tau + \pi_\alpha) = -S(\tau) \quad \text{and} \quad C(\tau + \pi_\alpha) = -C(\tau) \quad \text{for all} \quad \tau \in \boldsymbol{R}.$$

Moreover, the generalized Pythagorean theorem holds for $S(\tau)$ and $C(\tau)$:

$$|S(\tau)|^{\alpha+1} + |C(\tau)|^{\alpha+1} = 1 \quad \text{for all} \ \ \tau. \tag{2.9}$$

The *generalized tangent function* $T(\tau)$ is defined by

$$T(\tau) = \frac{S(\tau)}{C(\tau)} \quad \text{for} \quad \tau \neq \left(j + \frac{1}{2}\right)\pi_\alpha, \quad j \in \boldsymbol{Z}. \tag{2.10}$$

It is periodic with period π_α and satisfies

$$\dot{T} = 1 + |T|^{\alpha+1} > 0 \quad \text{for} \quad \tau \neq \left(j + \frac{1}{2}\right)\pi_\alpha, \quad j \in \boldsymbol{Z}, \tag{2.11}$$

so that $T(\tau)$ is strictly increasing for $\left(j - \frac{1}{2}\right)\pi_\alpha < \tau < \left(j + \frac{1}{2}\right)\pi_\alpha$, $j \in \boldsymbol{Z}$. We have

$$\lim T(\tau) = -\infty \quad \text{as} \quad \tau \to \left(j - \frac{1}{2}\right)\pi_\alpha + 0, \quad \text{and}$$

$$\lim T(\tau) = +\infty \quad \text{as} \quad \tau \to \left(j + \frac{1}{2}\right)\pi_\alpha - 0.$$

There exists the inverse function $T^{-1}(\tau)$ of $T(\tau)$, which is multivalued and the principal value, denoted by $T_p^{-1}(\tau)$, can be taken as

$$-\frac{1}{2}\pi_\alpha < T_p^{-1}(\tau) < \frac{1}{2}\pi_\alpha \quad \text{for all} \quad \tau.$$

Then, any value $T^{-1}(\tau)$ is expressed as $T^{-1}(\tau) = T_p^{-1}(\tau) + j\pi_\alpha$ for some $j \in \boldsymbol{Z}$. It is easy to see that $T^{-1}(\tau)$ is strictly increasing for $\tau \in \boldsymbol{R}$ and

$$\lim_{\tau\to-\infty} T^{-1}(\tau) = \left(j - \frac{1}{2}\right)\pi_\alpha, \qquad \lim_{\tau\to+\infty} T^{-1}(\tau) = \left(j + \frac{1}{2}\right)\pi_\alpha.$$

3. Proof of the Main Theorem. In this section we give the proof of Theorem 1.1. We assume throughout this section that $AA' \geq 0, BB' \geq 0$ and $A^2 + B^2 \neq 0$, and that $q(t)$ takes a positive value at some point $t \in [a, b]$.

For each $\lambda \in \boldsymbol{R}$, let $x(t; \lambda)$ be the solution of (1.1) satisfying the initial condition

$$x(a) = A', \quad x'(a) = A. \tag{3.1}$$

By the underlying hypothesis $A^2 + A'^2 \neq 0$, this solution $x(t; \lambda)$ is nontrivial. Note that $x(t; \lambda)$ satisfies the first part of the boundary conditions (1.2) :

$$Ax(a) - A'x'(a) = 0. \tag{3.2}$$

As mentioned in the preceding section, $x(t; \lambda)$ exists on $[a, b]$ and is continuous for $(t, \lambda) \in [a, b] \times \boldsymbol{R}$. In addition, if $\{\lambda_i\}_{i=1}^{\infty}$ tends to $\lambda \in \boldsymbol{R}$ as $i \to \infty$, then the corresponding sequence of solutions $\{x(t; \lambda_i)\}$ tends to $x(t; \lambda)$ uniformly on $[a, b]$ as $i \to \infty$.

It is clear that if $x(t;\lambda)$ satisfies the second part of the boundary conditions (1.2) :

$$Bx(b) + B'x'(b) = 0 \tag{3.3}$$

for some $\lambda \in \boldsymbol{R}$, then the λ is an eigenvalue and $x(t;\lambda)$ is an eigenfunction for the problem (1.1)–(1.2).

For $\lambda = 0$, $x(t;0)$ is explicitly given by

$$x(t;0) = A' + (p(a))^{1/\alpha} A \int_a^t \frac{ds}{(p(s))^{1/\alpha}}, \qquad a \le t \le b. \tag{3.4}$$

Using the conditions on A, A', B and B', we easily see that the value $\lambda = 0$ is not an eigenvalue of (1.1)–(1.2).

In what follows we discuss the case $\lambda > 0$. For the solution $x(t;\lambda), \lambda > 0$, we perform the next transformation, which consists in associating with $x(t;\lambda)$ the polar functions $\rho(t;\lambda)$ and $\theta(t;\lambda)$ defined by

$$x(t;\lambda) = \rho(t;\lambda)S(\theta(t;\lambda)), \qquad (p(t))^{1/\alpha}x'(t;\lambda) = \lambda^{1/\alpha}\rho(t;\lambda)C(\theta(t;\lambda)). \tag{3.5}$$

Here, $S(\tau)$ and $C(\tau)$ are the generalized sine and cosine functions, respectively, which are introduced in Section 2. The transformation (3.5) is a variant of the generalized Prüfer transformation. Note that (3.5) is slightly different from the standard generalized Prüfer transformation, in which the polar functions $\tilde{\rho}(t;\lambda)$ and $\tilde{\theta}(t;\lambda)$ are defined by $x(t;\lambda) = \tilde{\rho}(t;\lambda)S(\tilde{\theta}(t;\lambda)), (p(t))^{1/\alpha}x'(t;\lambda) = \tilde{\rho}(t;\lambda)C(\tilde{\theta}(t;\lambda))$.

In view of (2.9) we have

$$\rho(t;\lambda) = \left\{ |x(t;\lambda)|^{\alpha+1} + \left(\frac{p(t)}{\lambda}\right)^{(\alpha+1)/\alpha} |x'(t;\lambda)|^{\alpha+1} \right\}^{1/(\alpha+1)}.$$

Therefore the nontrivial property of $x(t;\lambda)$ implies

$$\rho(t;\lambda) > 0 \quad \text{for } a \le t \le b \text{ and } \lambda > 0.$$

It can be shown that $(\rho, \theta) = (\rho(t;\lambda), \theta(t;\lambda))$ is determined as the solution of the system of differential equations

$$\rho' = \rho \left\{ \left(\frac{\lambda}{p(t)}\right)^{1/\alpha} - \frac{q(t)}{\alpha} \right\} |S(\theta)|^{\alpha-1} S(\theta) C(\theta), \tag{3.6}$$

$$\theta' = \left(\frac{\lambda}{p(t)}\right)^{1/\alpha} |C(\theta)|^{\alpha+1} + \frac{q(t)}{\alpha} |S(\theta)|^{\alpha+1}, \tag{3.7}$$

under the initial condition

$$\rho(a;\lambda) = \left\{ |A'|^{\alpha+1} + \left(\frac{p(a)}{\lambda}\right)^{(\alpha+1)/\alpha} |A|^{\alpha+1} \right\}^{1/(\alpha+1)}, \tag{3.8}$$

$$\theta(a;\lambda) = T^{-1}\left(\left(\frac{\lambda}{p(a)}\right)^{1/\alpha} \frac{A'}{A}\right), \tag{3.9}$$

where T^{-1} denotes the inverse of the generalized tangent function $T = S/C$. We have $\rho(a;\lambda) > 0$. Further, since $AA' \geq 0$, we may assume with no loss of generality that

$$0 \leq \theta(a;\lambda) < \frac{\pi_\alpha}{2} \quad \text{for the case } A \neq 0, \text{ and} \tag{3.10}$$

$$\theta(a;\lambda) = \frac{\pi_\alpha}{2} \quad \text{for the case } A = 0. \tag{3.11}$$

Observe that $\theta = \theta(t;\lambda)$ can be solved, independently of $\rho = \rho(t;\lambda)$, as the solution of the initial value problem (3.7)–(3.9), and that if $\theta(t;\lambda)$ is known, then $\rho(t;\lambda)$ can be explicitly determined as the initial value problem (3.6)–(3.8):

$$\rho(t;\lambda) = \rho(a;\lambda)\exp\left[\int_a^t \left\{\left(\frac{\lambda}{p(s)}\right)^{1/\alpha} - \frac{q(s)}{\alpha}\right\} |S(\theta(s;\lambda))|^{\alpha-1} S(\theta(s;\lambda)) C(\theta(s;\lambda))ds\right].$$

Thus it is quite important to discuss the initial value problem (3.7)–(3.9). We denote by $f(t,\theta,\lambda)$ the right-hand side of (3.7):

$$f(t,\theta,\lambda) = \left(\frac{\lambda}{p(t)}\right)^{1/\alpha} |C(\theta)|^{\alpha+1} + \frac{q(t)}{\alpha}|S(\theta)|^{\alpha+1}.$$

It is clear that, for each fixed $\lambda > 0$, $f(t,\theta,\lambda)$ is bounded on $a \leq t \leq b$ and $-\infty < \theta < +\infty$. In view of (2.9), $f(t,\theta,\lambda)$ is rewritten as

$$f(t,\theta,\lambda) = \left(\frac{\lambda}{p(t)}\right)^{1/\alpha} + \left\{-\left(\frac{\lambda}{p(t)}\right)^{1/\alpha} + \frac{q(t)}{\alpha}\right\} |S(\theta)|^{\alpha+1}.$$

Since $|S(\theta)|^{\alpha+1}$ has a bounded continuous derivative $(\alpha+1)|S(\theta)|^{\alpha-1}S(\theta)C(\theta)$ on $-\infty < \theta < +\infty$, we see that, for each $\lambda > 0$, $f(t,\theta,\lambda)$ satisfies a Lipschitz condition with respect to θ on $a \leq t \leq b$ and $-\infty < \theta < +\infty$. Consequently we conclude that, for each $\lambda > 0$, the problem (3.7)–(3.9) has a unique solution $\theta = \theta(t;\lambda)$ on $a \leq t \leq b$. By a standard continuous dependence result in the theory of ordinary differential equations, $\theta(t;\lambda)$ is a continuous function of $(t,\lambda) \in [a,b] \times (0,\infty)$.

It is easy to see that $\lambda > 0$ is an eigenvalue of (1.1)–(1.2) if and only if λ satisfies

$$\theta(b;\lambda) = T^{-1}\left(-\left(\frac{\lambda}{p(b)}\right)^{1/\alpha} \frac{B'}{B}\right) + (n+1)\pi_\alpha \tag{3.12}$$

for some $n \in \boldsymbol{Z}$. Here, by virtue of $BB' \geq 0$, we assume without loss of generality that

$$-\frac{\pi_\alpha}{2} < T^{-1}\left(-\left(\frac{\lambda}{p(b)}\right)^{1/\alpha}\frac{B'}{B}\right) \le 0 \qquad \text{for the case } B \neq 0, \text{ and,}$$

$$T^{-1}\left(-\left(\frac{\lambda}{p(b)}\right)^{1/\alpha}\frac{B'}{B}\right) = -\frac{\pi_\alpha}{2} \qquad \text{for the case } B = 0.$$

Lemma 3.1. *The function* $\theta(b;\lambda)$ *is strictly increasing for* $\lambda \in (0,\infty)$.

Proof. As before, let us denote by $f(t,\theta,\lambda)$ the right-hand side of (3.7). Then, $f(t,\theta,\lambda)$ satisfies a Lipshitz condition with respect to θ on $a \le t \le b$ and $-\infty < \theta < +\infty$. Clearly, $f(t,\theta,\lambda)$ is a nondecreasing function of $\lambda \in (0,\infty)$, and, since $AA' \ge 0$, the initial value $\theta(a;\lambda)$ given by (3.9) is also nondecreasing for $\lambda \in (0,\infty)$. Then a standard comparison theorem for the first order scalar differential equations implies that $\theta(t;\lambda)$ is a nondecreasing function of $\lambda \in (0,\infty)$ for each fixed $t \in [a,b]$.

Let $0 < \lambda < \mu$ be fixed. Then, $\theta(t;\lambda) \le \theta(t;\mu)$ for $t \in [a,b]$. Assume that $\theta(t;\lambda) \equiv \theta(t;\mu)$ for all $t \in (a,b)$. Then, $\theta'(t;\lambda) \equiv \theta'(t;\mu)$, and so we have $f(t,\theta(t;\lambda),\lambda) \equiv f(t,\theta(t;\mu),\mu)$, from which it follows that $C(\theta(t;\lambda)) \equiv C(\theta(t;\mu)) \equiv 0$. This implies that $\theta(t;\lambda) \equiv \left(m+\frac{1}{2}\right)\pi_\alpha$ for some integer $m \in \boldsymbol{Z}$, and hence, by equation (3.7), $q(t) \equiv 0$ for all $t \in (a,b)$. This is a contradiction to the assumption that $q(t) > 0$ for some $t \in [a,b]$. Therefore we have

$$\theta(c;\lambda) < \theta(c;\mu) \qquad \text{for some } c \in (a,b).$$

Then, applying a standard comparison theorem again, we conclude that $\theta(b;\lambda) < \theta(b;\mu)$. The proof of Lemma 3.1 is complete.

Now we claim that $x(t;\lambda)$ has no zeros in the interval $(a,b]$ for all sufficiently small $\lambda > 0$. As stated before, $x(t;\lambda) \to x(t;0)$ as $\lambda \to 0$ uniformly on $[a,b]$. We note that $x(t;\lambda)$ satisfies

$$x(t;\lambda) = A' + \int_a^t \left|\frac{p(a)}{p(s)}|A|^{\alpha-1}A - \frac{\lambda}{p(s)}I(s;\lambda)\right|^{(1/\alpha)-1}\left\{\frac{p(a)}{p(s)}|A|^{\alpha-1}A - \frac{\lambda}{p(s)}I(s;\lambda)\right\}ds$$

for $a \le t \le b$, where

$$I(s;\lambda) = \int_a^s q(r)|x(r;\lambda)|^{\alpha-1}x(r;\lambda)dr, \qquad a \le s \le b.$$

Then it is easy to find that if $A = 0$ or $AA' > 0$, then $x(t;\lambda)$ has no zeros in the closed interval $[a,b]$ for all sufficiently small $\lambda > 0$, and that if $A \neq 0$ and $A' = 0$, then $x(t;\lambda)$ has no zeros in the interval $(a,b]$ for all sufficiently small $\lambda > 0$. Further, since

$$p(t)|x'(t;\lambda)|^{\alpha-1}x'(t;\lambda) = p(a)|A|^{\alpha-1}A - \lambda\int_a^t q(s)|x(s;\lambda)|^{\alpha-1}x(s;\lambda)ds$$

for $a \le t \le b$, we see that if $A \neq 0$, then $x'(t;\lambda)$ has no zeros in $[a,b]$ for all sufficiently small $\lambda > 0$.

Next we claim that the number of zeros of $x(t;\lambda)$ in $[a,b]$ can be made as large as possible if $\lambda > 0$ is chosen sufficiently large. To this end, we consider the equation

$$(|x'|^{\alpha-1}x')' + \alpha\mu^{\alpha+1}|x|^{\alpha-1}x = 0,$$

where $\mu > 0$ is a constant. Clearly, $S(\mu t)$ is a solution of the above equation, and has zeros at $t = j\pi_\alpha/\mu$, $j \in \boldsymbol{Z}$, where $S(\tau)$ is the generalized sine function introduced in Section 2. By the assumption that $q(t)$ is positive at some $t \in [a,b]$, there is an interval $[a',b'] \subset [a,b]$ such that $q(t) > 0$ for all $t \in [a',b']$. Let $k \in \boldsymbol{N}$ be any given positive integer and take $\mu > 0$ so that $S(\mu t)$ has at least $k+1$ zeros in $[a',b']$. Let $p^* > 0$ and $\lambda_* > 0$ be numbers such that

$$p^* = \max_{[a',b']} p(t) \quad \text{and} \quad \lambda_* \min_{[a',b']} q(t) = \alpha p^* \mu^{\alpha+1}.$$

Then, comparing the equation (1.1) with $\lambda > \lambda_*$ and the equation

$$(p^*|x'|^{\alpha-1}x')' + \alpha p^*\mu^{\alpha+1}|x|^{\alpha-1}x = 0, \quad a' \le t \le b',$$

we conclude by Lemma 2.2 that all solutions of (1.1) with $\lambda > \lambda_*$ have at least k zeros in $[a',b']$, hence in $[a,b]$. In particular, $x(t;\lambda)$ with $\lambda > \lambda_*$ has at least k zeros in $[a,b]$. Since k is an arbitrary positive integer, this shows that the number of zeros of $x(t;\lambda)$ in $[a,b]$ can be made as large as possible if $\lambda > 0$ is chosen sufficiently large.

Since $\rho(t;\lambda) > 0$ ($a \le t \le b$, $\lambda > 0$), it follows from (3.5) that $x(t;\lambda)$ has a zero at $t = c \in [a,b]$ if and only if there exists $j \in \boldsymbol{Z}$ such that $\theta(c;\lambda) = j\pi_\alpha$. Moreover, if $\theta(c;\lambda) = j\pi_\alpha$ ($c \in [a,b]$, $j \in \boldsymbol{Z}$), then, by (3.7), $\theta'(c;\lambda) = (\lambda/p(c))^{1/\alpha} > 0$. Therefore we easily see that if $\theta(c;\lambda) = j\pi_\alpha$ ($c \in [a,b], j \in \boldsymbol{Z}$), then $\theta(t;\lambda) > j\pi_\alpha$ for $c < t \le b$.

Thus the above results about the number of zeros of $x(t;\lambda)$ may be restated as the following way:

Lemma 3.2. (i) *For all sufficiently small* $\lambda > 0$,

$$\begin{cases} 0 < \theta(b;\lambda) < \dfrac{\pi_\alpha}{2} & \textit{in the case } A \neq 0, \textit{ and} \\ 0 < \theta(b;\lambda) < \pi_\alpha & \textit{in the case } A = 0. \end{cases}$$

(ii) $\lim \theta(b;\lambda) = +\infty$ *as* $\lambda \to +\infty$.

PROOF OF THEOREM 1.1. We are now ready to prove Theorem 1.1. We seek $\lambda > 0$ satisfying (3.12) for some $n \in \boldsymbol{Z}$. The left-hand side $\theta(b;\lambda)$ of (3.12) is a continuous function of $\lambda \in (0, \infty)$, and it is strictly increasing for $\lambda \in (0, \infty)$ by Lemma 3.1, and moreover it has the following properties by Lemma 3.2:

$$\begin{cases} 0 \le \lim_{\lambda \to 0+} \theta(b; \lambda) < \dfrac{\pi_\alpha}{2} & \text{in the case } A \ne 0, \text{ and} \\ 0 \le \lim_{\lambda \to 0+} \theta(b; \lambda) < \pi_\alpha & \text{in the case } A = 0, \end{cases}$$

and

$$\lim_{\lambda \to +\infty} \theta(b; \lambda) = +\infty.$$

On the other hand, by virtue of $BB' \ge 0$, the right-hand side of (3.12) is a nonincreasing function of $\lambda \in (0, \infty)$ for each $n \in \boldsymbol{Z}$. More precisely, in the case $BB' > 0$, it is strictly decreasing and varies from $(n+1)\pi_\alpha$ to $\left(n + \dfrac{1}{2}\right)\pi_\alpha$ as λ varies from 0 to $+\infty$. In the case $B' = 0$, it is the constant function $(n+1)\pi_\alpha$; and in the case $B = 0$, it is the constant function $\left(n + \dfrac{1}{2}\right)\pi_\alpha$.

From what was observed in the above we find that, for each $n = 0, 1, 2, \cdots$, there exists a unique $\lambda_n^+ > 0$ such that

$$\theta(b; \lambda_n^+) = T^{-1}\left(-\left(\frac{\lambda_n^+}{p(b)}\right)^{1/\alpha} \frac{B'}{B}\right) + (n+1)\pi_\alpha. \tag{3.13}$$

Then, each λ_n^+ is an eigenvalue of (1.1)–(1.2), and the associated eigenfunction $x(t; \lambda_n^+)$ has exactly n zeros in the open interval (a, b), where $n = 0, 1, 2, \cdots$. It is clear that

$$\lambda_0^+ < \lambda_1^+ < \cdots < \lambda_n^+ < \cdots, \qquad \lim_{n \to \infty} \lambda_n^+ = +\infty.$$

The proof of Theorem 1.1 is complete.

Theorems 1.2 and 1.3 can be easily derived from Theorem 1.1.

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