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THE COMPLETE DUCKS NOT SATISFYING S^1

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Abstract. In this paper, we will introduce a complete (and an incomplete) duck solution; a singular limit of the duck solution for a parameter. Then, we will give the explicit form of the solution which does not have the property of S^1 under certain conditions.

1. Introduction.

In 1990, Benoit[4] showed how to construct the explicit duck solutions (or simply ducks). Furthermore, he showed, in his paper, that if the difference of each the winding numbers associated with the ducks is more than 3/2, there exists a duck which is not S^1 (S^1 is a smoothness class in nonstandard analysis). However, this solution has not been constructed explicitly yet. In this paper, we would construct the above solution in this framework. In Section2 and Section3, introducing a parameter in the differential equations, an incomplete duck and a complete duck without S^1 are constructed explicitly as a singular limit of the duck for the parameter. The explicit form of the complete duck is the exact solution in the first approximation of the "local model" ([3], [6]).

2. Preliminaries.

Let consider a constrained system(2.1):

$$
\begin{aligned}
dx/dt &= f(x,y,z,u),\\
dy/dt &= g(x,y,z,u),\\
h(x,y,z,u) &= 0,
\end{aligned}
\tag{2.1}
$$

where u is a parameter (any fixed) and f,g,h are defined in $R^3 \times R^1$. Furthermore, let consider the singular perturbation problem of the system (2.1):

$$
\begin{aligned}
dx/dt &= f(x,y,z,u),\\
dy/dt &= g(x,y,z,u),\\
\epsilon dz/dt &= h(x,y,z,u),
\end{aligned}
\tag{2.2}
$$

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where ϵ is infinitesimally small.

We assume that the system (2.1) satisfies the following conditions $(A1)-(A5)$:

$(A1)$ f and g are of class C^1 and h is of class C^2.

$(A2)$ The set $S = \{(x, y, z) \in R^3 : h(x, y, z, u) = 0\}$ is a 2-dimensional differentiable manifold and the set S intersects the set
$T = \{(x, y, z) \in R^3 : \partial h(x, y, z, u)/\partial z = 0\}$ transversely so that the set $PL = \{(x, y, z) \in S \cap T\}$ is a 1-dimensional differentiable manifold.

$(A3)$ Either the value of f or that of g is nonzero at any point $p \in PL$.

Let $(x(t, u), y(t, u), z(t, u))$ be a solution of (2.1). By differentiating $h(x, y, z, u)$ with respect to the time t, the following equation holds:

$$(2.3)\quad h_x(x,y,z,u)f(x,y,z,u) + h_y(x,y,z,u)g(x,y,z,u) + h_z(x,y,z,u)dz/dt = 0,$$

where $h_i(x, y, z, u) = \partial h(x, y, z, u)/\partial i$, $i = x, y, z$. The above system (2.1) becomes the following system:

$$(2.4)\quad \begin{aligned} dx/dt &= f(x,y,z,u), \\ dy/dt &= g(x,y,z,u), \\ dz/dt &= -\{h_x(x,y,z,u)f(x,y,z,u)+ \\ &\quad h_y(x,y,z,u)g(x,y,z,u)\}/h_z(x,y,z,u), \end{aligned}$$

where $(x, y, z) \in S \setminus PL$. The system (2.1) coincides with the system (2.4) at any point $p \in S \setminus PL$. In order to study the system (2.4), let consider the following system:

$$(2.5)\quad \begin{aligned} dx/dt &= -h_z(x,y,z,u)f(x,y,z,u), \\ dy/dt &= -h_z(x,y,z,u)g(x,y,z,u), \\ dz/dt &= h_x(x,y,z,u)f(x,y,z,u) + h_y(x,y,z,u)g(x,y,z,u). \end{aligned}$$

As the system(2.5) is well defined at any point of R^3, it is well defined indeed at any point of PL. The solutions of (2.4) coincide with those of (2.1) on $S \setminus PL$ except the velocity when they start from the same initial points.

$(A4)$ For any $(x, y, z) \in S$, the implicit function theorem holds;

$$(2.6)\quad h_y(x,y,z,u) \neq 0, h_x(x,y,z,u) \neq 0,$$

that is, the surface S can be expressed by using $y = \varphi(x, z, u)$ or $x = \psi(y, z, u)$ in the neighborhood of PL. Let $y = \varphi(x, z, u)$ exist, then the projected system, which restricts the system (2.5) is obtained:

$$(2.7)\quad \begin{aligned} dx/dt &= -h_z(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u), \\ dz/dt &= h_x(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u)+ \\ &h_y(x,\varphi(x,z,u),z,u)g(x,\varphi(x,\varphi(x,z,u),z,u). \end{aligned}$$

$(A5)$ All the singular points of (2.7) are nondegenerate, the matrix induced from the linearized system of (2.6) at a singular point has two nonzero eigenvalues. Note that all the points contained in $PS = \{(x, y, z) \in PL : dz/dt = 0\}$, which is called *pseudo singular points* are the singular points of (2.7).

Definition2.1. Let $p \in PS$ and $\mu_1(u)$, $\mu_2(u)$ be two eigenvalues of the linearized system of (2.7), then the point p is called *pseudo singular saddle* if $\mu_1(u) < 0 < \mu_2(u)$ and called *pseudo singular node* if $\mu_1(u) < \mu_2(u) < 0$ or $\mu_1(u) > \mu_2(u) > 0$.

Definition2.2. A solution $(x(t,u), y(t,u), z(t,u))$ of the system(2.2) is called *a duck*, if there exist standard $t_1 < t_0 < t_2$ such that

(1) $^*(x(t_0,u), y(t_0,u), z(t_0,u)) \in S$, where $^*(X)$ denotes the standard part of X,

(2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t,u), y(t,u), z(t,u))$ is infinitesimally close to the attracting part of the slow curves,

(3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curves, and

(4) the attracting and repelling parts of the trajectory are not infinitesimally small.

Definition2.3. Let E be a set in R^3. We call a point p is a δ- micro-galaxy of E when the distance from p to E is less than $exp(-n/\delta)$, where n is some positive integer and $\delta = \epsilon/\alpha^2$ (α is infinitesimally small).

Definition2.4. Let θ is an angle of the polar coordinate after changing the coordinates in the "local model" such as the orbit passing through the pseudo singular point becomes the z-axis itself as the below. See [3],[5]. Then, the winding number $N(\psi)$ of a duck ψ is defined as follows:

$$N(\psi) = (1/2\psi) \int_\psi d\theta, \tag{2.8}$$

where ψ is contained partially in the δ-micro- galaxy of γ_μ.

Theorem2.1(Benoit). In the system(2.1), if the following two conditions at a pseudo singular saddle or node point;

(1) $f(O,u) \simeq h(O,u) \simeq h_y(O,u) \simeq h_z(O,u) \simeq 0$,

(2) $g(O,u) \not\simeq 0, h_x(O,u) \not\simeq 0, h_{zz}(O,u) \not\simeq 0$, where $O = (0,0,0) \in PS$,

are satisfied, the explicit duck solutions $\gamma_{\mu_i(u)}$ in the first approximation of the local model can be constructed:

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2 t^2 - \delta\mu_i(u), t, \mu_i(u)t)(i = 1, 2), \tag{2.9}$$

wehere δ is an infinitesimally small constant.

The above Definition2.3 is based on the following fact. If ϵ is fixed arbitrarily and $\gamma(t)$ is a duck near $\gamma_{\mu(u)}(t)$ is within $exp(-n/\delta)$ in some neighborhood of the pseudo singular point. See[10].

In the system(2.2), under the conditions (1) and (2) in the Theorem2.1, making the following coordinate transformations (2.10) and (2.11) successively;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha^2 \tilde{x} \\ \alpha \tilde{y} \\ \alpha \tilde{z} \end{pmatrix}, (\alpha \simeq 0, \epsilon/\alpha^2 \simeq 0) \tag{2.10}$$

(2.11)
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} h_x(0,u)h_{zz}(0,u)\tilde{x}/2 + (h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)\tilde{y}^2/4 \\ \tilde{y}/g(0,u) \\ -h_{yz}(0,u)\tilde{y}/2 - h_{zz}(0,u)\tilde{z}/2 \end{pmatrix},$$

the following local model (2.12) is obtained:

$$
\begin{aligned}
dX/dt &= pY + qZ + \xi(X,Y,Z,u),\\
dY/dt &= 1 + \eta(X,Y,Z,u),\\
\delta dZ/dt &= -(Z^2 + X) + \zeta(X,Y,Z,u),
\end{aligned} \tag{2.12}
$$

where

$$
\begin{aligned}
p &= g(0,u)h_x(0,u)(f_y(0,u)h_{zz}(0,u) - f_z(0,u)h_{yz}(0,u))/2\\
&\quad + g(0,u)^2(h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)/2,\\
q &= -h_x(0,u)f_z(0,u),\\
\delta &= \epsilon/\alpha^2.
\end{aligned} \tag{2.13}
$$

Here $\xi(X,Y,Z,u)$,$\eta(X,Y,Z,u)$ and $\zeta(X,Y,Z,u)$ are infinitesimal when X,Y and Z are limited. Note that the solutions (2.9) are in the first approximation system on (2.12).

By applying the following transformations of the coordinates as mentioned above, in Definition2.4, successively;

$$
\begin{aligned}
u &= X + Z^2 + \delta\mu,\\
v &= Y - Z/\mu,\\
z &= Z,
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
u &= rcos\theta,\\
v &= rsin\theta,
\end{aligned} \tag{2.15}
$$

the Hermite equation (2.16) is obtained. This equation associated with $\gamma_{\mu_i(u)}$ $(i = 1,2)$ is the following:

$$
\delta\ddot{z} - \tau\dot{z} + K_i z = 0, \dot{z} = dz/d\tau, t = \tau/\alpha, (i = 1,2), \tag{2.16}
$$

where K_i is a positive integer and $K_1 = 1 + \mu_2(u)/\mu_1(u)$, $K_2 = 1 + \mu_1(u)/\mu_2(u)$. See [3].

It is said that a duck $\psi(t)$ has *a jump* if the shadow of it contains a vertical segment and that $\psi(t)$ is *long* if it is in an infinitesimally small neighborhood at the pseudo singular point. It can be proved that if ψ is not long, the standard part of the winding number $N(\psi_i)$ associated with μ_i is an integer. If the pseudo singular point is node, it is positive. If the point is saddle, it needs some conditions such as K_i is poitive. The relation between $N(\psi_i)$ and K_i $(i = 1,2)$ is as follows.

Theorem2.2(Benoit). If the duck ψ_1, which is not long has 2 jumps, $N(\psi_1) \approx -[K_1/2]$, and if the duck ψ_2 has 2 jumps, $N(\psi_2) \approx 0$.

3. COMPLETE AND INCOMPLETE DUCKS.

In this section, the main theorems of this paper; the sufficicient condition for the existence of the complete duck solution are described explicitly.

Definition3.1. In the system(2.12), if for any parameter u,
it satisfies the conditions $(A1)$-$(A5)$ and has a duck,
then the solution is called *a complete duck*.

Definition3.2. In the system(2.12), if the followings (1) and (2):
(1) for any parameter u except some limited u_0,
it satisfies the conditions $(A1)$-$(A5)$ and has a duck,
(2) for the parameter $(u = u_0)$,
it satisfies the conditions $(A1)$-$(A5)$ but the solution is not a duck,
are established, then it is called *an incomplete duck*.

Definition3.3. A solution $\psi(x,u)$ is called S^1 *at* a,
if there exists a real number b such that

$$\frac{\psi(x,u)-\psi(y,u)}{x-y} \approx b, \tag{3.1}$$

for any $x, y (x \approx a, y \approx a)$.
A duck is called an S^1*duck* if it is S^1 in some neighborhood of the pseudo singular point.

Theorem3.1(Benoit). In the first approximation of the local model (2.12), if $\mu_1(u)/\mu_2(u)$ is positive (> 3) but no an integer, then all the S^1 ducks are exponentially close to one of the two explicit ducks and there exists non S^1 ducks.

In the local model, we assume that

$$f_y(0,u) = g_u(0,u) = h_{yz}(0,u) = h_{yyu}(0,u) = h_{zzu}(0,u) = 0, \tag{3.2}$$

and that the following (1) or (2):
(1) $h_x(0,u) = O(u) and f_z(0,u) = O(1)$,
(2) $f_z(0,u) = O(u) and h_x(0,u) = O(1)$,
where all the coefficients of higher order (more than 2) for u is negligible. In other words, we assume that only the coefficient q can take an unlimited number ($q = \tilde{q}u + o(1)$, a constant $\tilde{q} \not\simeq 0$). Then, blowing up only the variable Z again;

$$Z = (1/u)\zeta, \tag{3.3}$$

the first approximation of the local model becomes the following:

$$\begin{aligned} dX/dt &= pY + \tilde{q}\zeta, \\ dY/dt &= 1, \\ (\delta/u)d\zeta/dt &= -(\zeta^2/u^2 + X), \end{aligned} \tag{3.4}$$

where $\tilde{q}$ is limited (does not contain u) and $(\delta/u) \simeq 0$. The explicit solutions in the system(3.4) are

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2t^2 - u\delta\mu_i(u), u^2t, u^2\mu_i(u)t)(i = 1,2), \tag{3.5}$$

where $\mu_1(u)$, $\mu_2(u)$ ($\mu_1(u) > \mu_2(u)$) are the solutions of the characteristic equation of the system(3.4) in case $\delta/u \simeq 0$.
The above system satifies the conditions $(A1)$-$(A5)$ and the solutions(3.5) satisfy the condition (1) and one of the solutions satisfies the condition (2) in Definition3.2 when $u = u_0 = \omega = 1/\epsilon$. Since $\mu_2(u) \simeq -1/2\epsilon$ as $u = 1/\epsilon$ and for the first component of (3.5), the following

$$\frac{-(1/2\epsilon)^2(2\epsilon)^2 + (1/2\epsilon)^2(\epsilon)^2}{2\epsilon - \epsilon} = -3/4\epsilon, \tag{3.6}$$

is established. In this state, the winding number $N(\psi_2)$ associated with μ_2 is unlimited and the other $N(\psi_1)$ associated with μ_1 is infinitesimal. Then, this duck may be almost tangent to the X-axis when u tends to $1/\epsilon$.

Now, let $v = 1/u$. then $\partial_u = -v^2\partial_v$ holds and then the following conditions are assumed; $f(x,y,z,u) = \tilde{f}(x,y,z,v) \in C^3$, $g(x,y,z,u) = \tilde{g}(x,y,z,v) \in C^1$ and $h(x,y,z,u) = \tilde{h}(x,y,z,v) \in C^3$ at almost everywhere but $v = v_0 = 0$.

Theorem 3.2. *In the first approximation of the local model, under the condition* (3.2) *and* $\tilde{h}_{x,v}(0,v)\tilde{f}_{zv}(0,v) = 0$, *if either the assumption* (1) *or* (2) *;*
(1) $\tilde{f}_z(0,v) = 0$, *and* $\tilde{h}_x(0,v)\tilde{f}_{zvv}(0,v) = 0$,
(2) $\tilde{h}_x(0,v) = 0$, *and* $\tilde{h}_{xvv}(0,v)\tilde{f}_z(0,v) = 0$,
where all the coefficients of higher order (more than 2) for v *is negligible, is satisfied, then this system has an incomplete duck.*

proof. From the above assumptions, the relation $q = -h_x(0,u)f_z(0,u) = \tilde{q}u$ holds. Differentiating the both side of the equation by the parameter v, we can directly lead to the conclusion. □

By blowing up the variable X again, like the same way above, the complete duck which is not S^1 is obtained.

Theorem 3.3. *In the equation* (3.4), *we assume that*

$$\begin{aligned} &\tilde{f}_y(0,v) = \tilde{h}_{yz}(0,v) = 0, \\ &\tilde{g}(0,v) = O(1) \end{aligned} \tag{3.7}$$

and moreover, if either the condition $\tilde{h}_{yy}(0,v) = 0$, *or* $\tilde{h}_{zz}(0,v) = 0$, *is satisfied, that is;* $p = c_1v^2 + o(1)$ ($c_1 \not\simeq 0$ *is limited) and* $\tilde{q} = c_2v^2 + o(1)$ ($c_2 \not\simeq 0$ *is limited), then there exists a complete duck which is not* S^1 *in this system* (3.4).

proof. Blowing up the variable X such as

$$X = v^2\xi, \tag{3.8}$$

the equation (3.4) becomes

$$\begin{aligned} &d\xi/dt = (1/v^2)pY + (1/v^2)\tilde{q}\zeta, \\ &dY/dt = 1, \\ &(\delta/v)d\zeta/dt = -(\zeta^2 + \xi), \end{aligned} \tag{3.9}$$

where $\delta/v \simeq 0$.
Therefore, the explicit ducks $\tilde{\gamma}_{\mu_i(u)}$ of (3.9) are described as follows:

$$\tilde{\gamma}_{\mu_i(v)}(t) = (-\mu_i(v)^2t^2 - \delta\mu_i(v)/v, t, \mu_i(v)t)(i = 1, 2), \tag{3.10}$$

where $\mu_i(v)$ are the eigen values of the linearized system in the constrained system of (3.9). In this situation, the constrained surface $S = \{(\xi, Y, \zeta) : \xi + \zeta^2 = 0\}$ is invariant for the parameter v. Furthermore, one of the ducks tangents to the ξ-axis at the pseudo singular point when v tends to $\epsilon^{1/2}$. So, this duck is not S^1. □

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