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PRECIOUS SIEGEL MODULAR FORMS OF GENUS TWO

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ABSTRACT. We give a review of the recent results concerning Siegel modular forms with respect to the paramodular groups of genus 2 and their applications to Algebraic Geometry and Physics. Some facts mentioned bellow have not been published before.

§0. GENERAL SET UP

Let $L_{2,n}$ be an even integral lattice of signature $(2,n)$. Let us consider the corresponding orthogonal group $O^+(L_{2,n})$ ("plus" denotes the subgroup of elements with the real spin norm 1) and the homogeneous domain of type IV

$$H^+_{L_{2,n}} \cong O^+(L_{2,n}\otimes\mathbb{R})/K_{max} \cong \{\mathbb{C}\omega\subset\mathbb{P}(L_{2,n}\otimes\mathbb{C}) \mid \omega\cdot\omega=0,\ \omega\cdot\overline{\omega}>0\}^+$$

where K_{max} is the maximal compact subgroup of the orthogonal group. The pair $(O^+(L_{2,n}\otimes\mathbb{R}), K_{max})$ is a symmetric pair of BD type. The corresponding bounded symmetric domain having complex dimension n is a complex domain of type IV in the Cartan classification. By $\mathcal{H}^+_{L_{2,n}}$ we denote a standard realization of $O^+(L_{2,n}\otimes\mathbb{R})/K_{max}$ as a tube domain (this is so-called the tube of future) in $\mathbb{C}^n$ (see f.e. [B2], [Od], [G3]).

For an arbitrary subgroup $\Gamma\subset O^+(L_{2,n})$ of finite index we may consider the d-graded ring ($d\in\mathbb{N}$) of modular forms $\mathfrak{M}_{dk}(\Gamma)$ of weight dk with respect to Γ

$$\mathfrak{M}^{(d)}_*(\Gamma)=\bigoplus_{k=0}^{\infty}\mathfrak{M}_{dk}(\Gamma).$$

It is well known that this ring defines the Satake compactification of the modular variety

$$\mathcal{A}_\Gamma=\Gamma\setminus\mathcal{H}^+_{L_{2,n}}.$$

More exactly

$$\operatorname{Proj}\big(\bigoplus_{k=0}^{\infty}\mathfrak{M}_{dk}(\Gamma)\big)=\overline{\mathcal{A}_\Gamma}^{(Satake)}.$$

Classical examples of modular varieties of type $\mathcal{A}_\Gamma$ are the moduli spaces of the polarized $K3$, Enriques, Abelian and Kummer surfaces. In the case of polarized $K3$ surfaces we have the moduli space of dimension 19 related with lattice

$$L_{2,n}=L_{2,19}\cong 2U\oplus 2E_8(-1)\oplus <-2t>$$

where $U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ is the unimodular hyperbolic plane, $E_8(-1)$ is the negative definite unimodular lattice of rank 8 and $< -2t >$ is one dimensional $\mathbb{Z}$-lattice with quadratic form $(-2t)$ (the number $2t$ defines a polarization of $K3$ surface). For Enriques surfaces

$$L_{2,n+2} = L_{2,10} \cong U \oplus U(2) \oplus E_8(-2)$$

(see f.e. [BPV]). The moduli spaces of $(1,t)$-polarized Abelian and corresponding Kummer surfaces have dimension 3. They are related with the lattice

$$L_{2,3} = 2U \oplus < -2t >$$

(see f.e. [vdG], [GH1]). We may formulate the following problem:

Problem A1. *To describe lattices $L_{2,n}$ and corresponding groups $\Gamma \subset O^+(L_{2,n})$ such that the graded ring $\mathfrak{M}_*^{(d)}(\Gamma)$ is a polynomial ring over $\mathbb{C}$ and to find generators of this ring.*

More generally we may put

Problem A2. *For a given lattice $L_{2,n}$ and a group $\Gamma \subset O^+(L_{2,n})$ to write a system of equations of the modular variety A_Γ in terms of generators of the ring $\mathfrak{M}_*^{(d)}(\Gamma)$.*

We would like to mention that a solution of these problems related with description of the graded rings of type $\mathfrak{M}_*^{(d)}(\Gamma)$ gives us not only good models of some classical moduli spaces but defines the so-called *arithmetic mirror symmetry* for $K3$ surfaces (see [GN3]) and provides us with important invariants in the theory of root systems of different types (*of elliptic root systems, of hyperbolic roots systems and of root systems of type IV in the sense of K.Saito*).

Problems A1 and A2 are closely related with the following

Problem B. *To construct automorphic forms with respect to the group $\Gamma \subset O^+(L_{2,n})$ of a small weight with a "simple" divisor.*

One can reformulate this problem as a construction of first generators of the corresponding graded rings. By a simple divisor we mean a union of divisors of *Humbert type.* A Humbert divisor is a projection of a rational quadratic divisor defined by some vector $l \in L_{2,n}^*$ of negative norm (L^* is the lattice dual to L)

$$H_l = \pi_\Gamma(\{\mathfrak{z} \in \mathcal{H}_{L_{2,n}}^+ \mid (\mathfrak{z}, l) = 0\}),$$

where $\pi_\Gamma : \mathcal{H}_{L_{2,n}}^+ \to \Gamma \setminus \mathcal{H}_{L_{2,n}}^+$ is the natural projection. The Humbert divisor H_l is a modular variety of type (2) of dimension $n-1$.

We call a meromorphic automorphic form $F \in \mathfrak{M}_k(\Gamma)$ *Humbert modular form* if its divisor is a sum of Humbert divisors with some multiplicities

$$\mathrm{div}_{A_\Gamma}(F) = \sum_i a_i H_{l_i}.$$

Such form F is called reflective (resp. 2-reflective) if the reflection defined by l_i belongs to the integral orthogonal group $O(L_{2,n})^+$ (resp. $(l_i, l_i) = -2$) for all i. The reflective modular forms are very important in many considerations. For example, they provide us with *automorphic corrections* of hyperbolic Kac–Moody algebras (see [B1]–[B3], [GN1]–[G6]).

A particular answer on Problem B formulated above is given by the lifting construction proposed by the author in [G1]–[G3] (see also [G4] where the case of the paramodular groups, i.e. $n = 3$, was considered). This construction gives us all modular forms of very small weights, more exactly for the weights for which modular forms are defined by the first Fourier–Jacobi coefficient.

The information about divisors of modular forms is given by the Borcherds construction of the exponential lifting. In the present talk we shall concentrate ourselves mainly on the case of the homogeneous domain of dimension 3, i.e. on the lattices on signature $(2,3)$. In our RIMS-preprints ([GN5]–[GN6]) we construct a theory of reflective and 2-reflective forms in the case of the lattices of signature $(2,3)$. As it was mentioned above this case is connected with the theory of the moduli spaces of Abelian and Kummer surfaces (see [G3] and [GH1]–[GH2]). The application of some reflective modular forms constructed in [GN1] and [GN6] to the heterotic string theory were found by T. Kawai (see [Ka1]–[Ka2]), G.L. Cardoso, G. Curio and D. Luest (see [CCL], [Ca]), R. Dijkgraaf, E. Verlinde and H. Verlinde (see [DVV]). See also the papers of J. Harvey and G. Moore [HM1]–[HM2].

The most part of the results presented bellow were obtained by the author together with V. Nikulin and with K. Hulek during his stay at RIMS (09.1996–02.1997) and were published in the RIMS-preprints [GN5]–[GN7] and [GH2]. Using this opportunity we express our gratitude to Research Mathematical Institute of Kyoto University for hospitality. I give my special thanks to all members of Algebraic Geometry Seminars at RIMS for useful and stimulating discussions.

§1. Arithmetic Lifting

In case of the lattice $L_{2,3} = 2U \oplus < -2t >$ of signature $(2,3)$ the corresponding homogeneous domain $\mathcal{H}_{L_{2,3}}$ is isomorphic to the Siegel upper-half plane of genus 2

$$\mathbb{H}_2 = \{Z = {}^tZ \in M_2(\mathbb{C}),\ Z = X + iY,\ Y > 0\}.$$

The integral orthogonal group $PO^+(L_{2,3})$ is isomorphic to a normal extension of the so-called *paramodular* group Γ_t. The last group is conjugated to the integral symplectic group of a skew-symmetric form with elementary divisors $(1,t)$. It can be realized as the following subgroup of $Sp_4(\mathbb{Q})$

$$\Gamma_t := \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & t^{-1}* \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} \in Sp_4(\mathbb{Q}) \mid \text{ all } * \text{ are integral} \right\}.$$

The quotient

$$\mathcal{A}_t = \Gamma_t \setminus \mathbb{H}_2$$

is isomorphic to the coarse moduli space of abelian surfaces with a polarization of type $(1,t)$.

We denote by $|_k$ ($k \in \mathbb{Z}/2$) the standard slash operator on the space of functions on $\mathbb{H}_2$:

$$(F|_k M)(Z) := \det(CZ+D)^{-k} F(M < Z >)$$

where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{R}) \quad \text{and} \quad M < Z >= (AZ+B)(CZ+D)^{-1}.$$

For a half-integral k we choose one of the holomorphic square roots by the condition $\sqrt{\det(Z/i)} > 0$ for $Z = iY \in \mathbb{H}_n$.

Definition. A modular form of weight k ($k \in \mathbb{Z}/2$) with respect to Γ_t with a character (or a multiplier system if k is a half-integer) $\chi : \Gamma_t \to \mathbb{C}^*$ is a holomorphic function on $\mathbb{H}_2$ which for arbitrary $M \in \Gamma_t$ satisfies the functional equation

$$F|_k M = \chi(M) F \qquad \forall M \in \Gamma_t.$$

We denote the space of such modular (resp. cusp) forms by $\mathfrak{M}_k(\Gamma_t, \chi)$ ($\mathfrak{N}_k(\Gamma_t, \chi)$ respectively).

Here we admit a character of the paramodular group Γ_t in order to construct roots of certain orders of modular forms with respect to Γ_t with trivial character. The classical example is the Dedekind eta-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n) = \sum_{n\in\mathbb{N}} \left(\frac{12}{n}\right) q^{n^2/24} \in \mathfrak{N}_{1/2}(SL_2(\mathbb{Z}), v_\eta)$$

where $\tau \in \mathbb{H}_1$, $q = \exp(2\pi i \tau)$,

$$\left(\frac{12}{n}\right) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \bmod 12 \\ -1 & \text{if } n \equiv \pm 5 \bmod 12 \\ 0 & \text{if } (n,12) \neq 1 \end{cases}$$

and $v_\eta : SL_2(\mathbb{Z}) \to \{\sqrt[24]{1}\}$ is η-multiplier system. Then

$$\Delta(\tau) = \eta(\tau)^{24} = q - 24q + 252q^3 - 1472q^4 + \cdots \in \mathfrak{N}_{12}(SL_2(\mathbb{Z}))$$

is the first cusp form with respect to $SL_2(\mathbb{Z})$ with trivial character.

All possible characters of the paramodular group Γ_t are described in [GH2]. In particular we have

Theorem about Commutator. *(see [GH2, Theorem 2.1].) For any integer $t \geq 1$ let $t_1 = (t, 12)$ and $t_2 = (2t, 12)$. If Γ_t' is the commutator subgroup of the paramodular group Γ_t, then*

$$\Gamma_t/\Gamma_t' \cong \mathbb{Z}/t_1 \times \mathbb{Z}/t_2.$$

Remark. The group Γ_t/Γ_t' is isomorphic to the torsion $\operatorname{Tor}\operatorname{Pic}(\underline{\mathcal{A}}_t)$ of the Picard group of the moduli stack $\underline{\mathcal{A}}_t$ of $(1,t)$-polarized abelian surfaces.

A modular form $F \in \mathfrak{M}_k(\Gamma_t, \chi)$ can be considered as a root of some order d of a modular form from $\mathfrak{M}_{dk}(\Gamma_t)$ where d is the order of χ. From Theorem above follows that $d|12$ if k is integer.

The classical example of modular forms with a non-trivial character of $\Gamma_1 = Sp_4(\mathbb{Z})$ is the product of all even Siegel theta-constants

$$\vartheta_{a,b}(Z) = \sum_{l \in \mathbb{Z}^2} \exp\left(\pi i (Z[l + \frac{1}{2}a] + {}^t b l)\right) \qquad (Z[l] = {}^t l Z l), \tag{1.1}$$

i.e.

$$\Delta_5(Z) = 2^{-6} \prod_{{}^t ab = 0(2)} \theta_{a,b}(Z) \in \mathfrak{N}_5(\Gamma_1, \chi_2) \tag{1.2}$$

where $\chi_2 : Sp_4(\mathbb{Z}) \to \{\pm 1\}$ ($[\Gamma_1 : \Gamma_1'] = \mathbb{Z}_2$, thus χ_2 is the unique non-trivial character of Γ_1.) The graded rings $\mathfrak{M}_*^{(2)}(Sp_4(\mathbb{Z}))$ and $\mathfrak{M}_*^{(1)}(Sp_4(\mathbb{Z}))$ where defined by Igusa (see [Ig1]–[Ig2]). In particular

$$\mathfrak{M}_*^{(2)}(Sp_4(\mathbb{Z})) = \mathbb{C}\,[E_4, E_6, F_{10}, F_{12}]$$

is the polynomial ring in four variables, where E_4 and E_6 are the $Sp_4(\mathbb{Z})$-Eisenstein series of weight 4 and 6, $F_{10} = \Delta_5^2$ and F_{12} is the unique, up to a constant, cusp form of weight 12. Moreover

$$\mathfrak{M}_*^{(1)}(Sp_4(\mathbb{Z})) = \mathfrak{M}_*^{(2)}(Sp_4(\mathbb{Z}))[F_{35}]$$

where F_{35} is the unique, up to a constant, Siegel cusp form of weight 35.

The modular forms of very small weights automatically belong to the set of generators of the graded ring $\mathfrak{M}_*^{(1)}(\Gamma_t)$. In the above example the Eisenstein series E_4 and E_6 are the first two $Sp_4(\mathbb{Z})$-modular forms. Similar Eisenstein series can be defined for arbitrary Γ_t. The modular forms $F_{10} = \Delta_5^2$ and F_{12} are the first two $Sp_4(\mathbb{Z})$-cusp forms.

The cusp forms of the smallest possible weight can be constructed using a special procedure, *Arithmetic Lifting*. The datum for this lifting is a Jacobi form $\phi_{k,R}(\tau, z)$ of integral weight k and index R ($R \in \mathbb{N}/2$) together with a character of the full Jacobi group. We define *Jacobi forms* as modular forms with respect to the maximal parabolic subgroup Γ_∞ of $Sp_4(\mathbb{Z}) = \Gamma_1$ which fixes a line, i.e.

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}.$$

Let k and R be integral or half-integral.

Definition. We call a holomorphic function $\phi(\tau, z)$ on $\mathbb{H}_1 \times \mathbb{C}$ *a Jacobi form of weight* k *and index* R *with a multiplier system (or a character)* $v : \Gamma_\infty \to \mathbb{C}^*$ if the function

$$\widetilde{\phi}(Z) := \phi(\tau, z) \exp(2\pi i R\omega), \qquad Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2,$$

satisfies the functional equation

$$\widetilde{\phi}(M < Z >) = v(M) \det(CZ + D)^k \widetilde{\phi}(Z) \qquad \text{for any } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty$$

and ϕ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{\substack{n,\, l \\ 4Rn - l^2 \geq 0}} f(n, l) \exp(2\pi i(n\tau + lz))$$

where the summation is taken over n and l from some free $\mathbb{Z}$-modules depending on v. The condition $f(n, l) = 0$ unless $4Rn - l^2 \geq 0$ is equivalent to the holomorphicity of ϕ at infinity. The form $\phi(\tau, z)$ is called a Jacobi *cusp* form if $f(n, l) = 0$ unless $4Rn - l^2 > 0$. We call the number $4Rn - l^2$ *the norm* of the index of the Fourier coefficient $f(n, l)$. We denote the finite dimensional space of such Jacobi forms (resp. cusp forms) by $J_{k,R}(v)$ (resp.$J^c_{k,R}(v)$). If the function ϕ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n \geq 0,\, l} f(n, l) \exp(2\pi i(n\tau + lz)),$$

then we call it a *weak Jacobi form*. The space $J^w_{k,R}(v)$ of all such forms is again finite dimensional.

Here we admit Jacobi forms of *half-integral* indices. This is the only difference between the definition of the Jacobi forms given above and the definition of [EZ].

Many examples of Jacobi forms of half-integral index see in [GN6]. The oldest example of such forms is *the Jacobi triple product*. The Jacobi theta-series is defined as

$$\vartheta(\tau, z) = \sum_{n \equiv 1 \, mod\, 2} (-1)^{\frac{n-1}{2}} \exp\left(\frac{\pi i\, n^2}{4}\tau + \pi i\, nz\right) = \sum_{m \in \mathbb{Z}} \left(\frac{-4}{m}\right) q^{m^2/8} r^{m/2},$$

where $q = \exp(2\pi i\tau)$, $r = \exp(2\pi iz)$ and

$$\left(\frac{-4}{m}\right) = \begin{cases} \pm 1 & \text{if } m \equiv \pm 1 \bmod 4 \\ 0 & \text{if } m \equiv \ \ 0 \bmod 2. \end{cases}$$

This is a Jacobi form of weight 1/2 and index 1/2, i.e. an element of $J_{\frac{1}{2},\frac{1}{2}}(v^3_\eta \times v_H)$. The multiplier system $v^3_\eta \times v_H$ is induced by the $SL_2(\mathbb{Z})$-multiplier system v_η of the Dedekind η-function and the following character of the integral Heisenberg group

$$v_H([\lambda, \mu; \kappa]) := (-1)^{\lambda + \mu + \lambda\mu + \kappa}. \tag{1.3}$$

We recall the famous Jacobi triple product formula

$$\vartheta(\tau,\, z) = -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1} r)(1 - q^n r^{-1})(1 - q^n). \tag{1.4}$$

The next theorem is proved in [GN6, Theorem 1.12] (compare with the main theorem in [G4])

Arithmetic Lifting Theorem. *Let*

$$\phi(\tau, z) = \sum_{\substack{N\equiv 1 \, mod\, Q\\ l\equiv \varepsilon\, mod\, 2\\ N>0,\, 4NDt>l^2}} f(ND, l) \exp\left(2\pi i(\frac{N}{Q}\tau + \frac{l}{2}z)\right) \in J^{cusp}_{k,R}(v_\eta^{24/Q} \times v_H^\varepsilon),$$

where $Q = 1, 2, 3, 4, 6$ *or* 12, $\varepsilon \equiv 2R$ *mod* 2, *the weight* k *is an integer. We assume for simplicity that* $t = QR \in \mathbb{N}$ *and fix* $\mu \in (\mathbb{Z}/Q\mathbb{Z})^*$. *Then the function*

$$\begin{aligned}&\mathrm{Lift}_\mu(\phi)(Z)\\ &= \sum_{\substack{N,\, M>0\\ N,M\equiv\mu\, mod\, Q\\ L\equiv\varepsilon\, mod\, 2}} \Bigl(\sum_{a|(N,L,M)} a^{k-1} v_\eta^D(\sigma_a) f(\frac{NMD}{a^2}, \frac{L}{a})\Bigr) \exp\left(2\pi i(\frac{N}{Q}\tau + \frac{L}{2}z + Mt\omega)\right),\end{aligned}$$

where $\sigma_a \in SL_2(\mathbb{Z})$ *such that* $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0\\ 0 & a\end{pmatrix}$ *mod* Q, *is a cusp form of weight* k *with respect to the paramodular group* Γ_t *with a character* $\chi_Q : \Gamma_t \to \{\sqrt[Q]{1}\}$ *of order* Q *induced by* $v_\eta^{24/Q} \times v_H^\varepsilon$.

If $\mu = 1$, *then* $\mathrm{Lift}(\phi)(Z) = \mathrm{Lift}_1(\phi)(Z) \not\equiv 0$ *for* $\phi \not\equiv 0$, *i.e. we have an embedding of the space* $J^{cusp}_{k,t}(v_\eta^D \times v_H^\varepsilon)$ *into the space of Siegel modular forms.*

Remarks. 1. If QR is half-integral see Theorem 1.12 in [GN6].

2. For $\mu \neq 1$ the lifting could be zero for non-vanishing Jacobi forms. An examples of non-zero μ-lifting see (1.11) bellow.

3. The construction of the arithmetic lifting in the case of orthogonal groups of signature $(2, n)$ see in [G3]. Its variant with a "commutator" character is similar to the theorem above. See also [G3] for the case of non-cusp Jacobi forms.

Δ-series of modular forms. We define below the Siegel cusp forms Δ_1, Δ_2, Δ_5, Δ_{11} (the index denotes the weight). Let

$$\phi_{5,1/2}(\tau, z) = \eta(\tau)^9\vartheta(\tau, z), \quad \phi_{2,1/2}(\tau, z) = \eta(\tau)^3\vartheta(\tau, z), \quad \phi_{1,1/2}(\tau, z) = \eta(\tau)\vartheta(\tau, z).$$

Then using the arithmetic lifting with $\mu = 1$ we can define the following modular forms

$$\begin{aligned}\Delta_5 &= \mathrm{Lift}(\phi_{5,1/2}) \in \mathfrak{M}_5(\Gamma_1, \chi_2),\\ \Delta_2 &= \mathrm{Lift}(\phi_{2,1/2}) \in \mathfrak{M}_2(\Gamma_2, \chi_4),\\ \Delta_1 &= \mathrm{Lift}(\phi_{1,1/2}) \in \mathfrak{M}_1(\Gamma_1, \chi_6).\end{aligned}$$

Moreover the lifting construction gives us the following Fourier expansions

$$\Delta_1(Z) = \sum_{M\geq 1} \sum_{\substack{n,\, m>0,\, l\in\mathbb{Z}\\ n,\, m\equiv 1\, mod\, 6\\ 4nm-3l^2=M^2}} \left(\frac{-4}{l}\right)\left(\frac{12}{M}\right) \sum_{a|(n,l,m)} \left(\frac{6}{a}\right) q^{n/6} r^{l/2} s^{m/2} \in \mathfrak{M}_1(\Gamma_3, \chi_6) \tag{1.5}$$

and

$$\Delta_2(Z) = \sum_{N\geq 1} \sum_{\substack{n,\, m>0,\, l\in\mathbb{Z} \\ n,\, m\equiv 1 \bmod 4 \\ 2nm-l^2=N^2}} N\left(\frac{-4}{Nl}\right) \sum_{a\,|\,(n,l,m)} \left(\frac{-4}{a}\right) q^{n/4} r^{l/2} s^{m/2} \in \mathfrak{M}_2(\Gamma_2, \chi_4) \qquad (1.6)$$

where

$$Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2,\ e(z) = \exp(2\pi i z),\ q = e(\tau),\ r = e(z),\ s = e(\omega). \qquad (1.7)$$

Hence all primitive (i.e. with primitive matrices N) Fourier coefficients $a(N)$ of the cusp form $\Delta_1(Z)$ of weight one are equal to ± 1 or 0 (compare with the Fourier expansion of the Dedekind η-function)!

According Igusa Δ_5^2 is the first cusp form with respect to $Sp_4(\mathbb{Z})$. One can prove (see [G5] and [GH2]) that Δ_2^4 (resp. Δ_1^6) is the first cusp form with respect to Γ_2 (resp. Γ_3) with trivial character. The first Γ_2-cusp form of an odd weight (with trivial character) is the modular form of weight 11

$$\Delta_{11}(Z) = \mathrm{Lift}(\eta(\tau)^{21}\vartheta(\tau, 2z)) \in \mathfrak{M}_{11}(\Gamma_2). \qquad (1.8)$$

D-**series.** Let us consider the Jacobi form of weight 1/2 and index 3/2 with the multiplier system $v_\eta \times v_H$ (see (1.3))

$$\vartheta_{3/2}(\tau, z) = \sum_{n\in\mathbb{Z}} \left(\frac{12}{n}\right) q^{n^2/24} r^{n/2} \in J_{\frac{1}{2},\frac{3}{2}}(v_\eta \times v_H).$$

This is the so-called quintiple product

$$\vartheta_{3/2}(\tau, z) = q^{\frac{1}{24}} r^{-\frac{1}{2}} \prod_{n\geq 1} (1+q^{n-1}r)(1+q^n r^{-1})(1-q^{2n-1}r^2)(1-q^{2n-1}r^{-2})(1-q^n). \qquad (1.9)$$

Then we can define two cusp forms of weight one and two with elementary Fourier coefficients

$$D_1(Z) = \mathrm{Lift}(\eta\vartheta_{3/2}) =$$

$$\sum_{M\geq 1} \sum_{\substack{n,\, m>0,\, l\in\mathbb{Z} \\ n,\, m\equiv 1 \bmod 12 \\ 2nm-l^2=M^2}} \left(\frac{12}{Ml}\right) \sum_{a|(n,l,m)} \left(\frac{-4}{a}\right) q^{n/12} r^{l/2} s^{3m/2} \in \mathfrak{M}_1(\Gamma_{18}, v_\eta^2 \times v_H),$$

(compare with the quintiple product) and

$$D_2(Z) = \mathrm{Lift}(\eta^3\vartheta_{3/2}) =$$

$$\sum_{N\geq 1} \sum_{\substack{n,\, m>0,\, l\in\mathbb{Z} \\ n,\, m\equiv 1 \bmod 6 \\ 4nm-l^2=N^2}} N\left(\frac{-4}{N}\right)\left(\frac{12}{l}\right) \sum_{a|(n,l,m)} \left(\frac{6}{a}\right) q^{n/6} r^{l/2} s^{3m/2} \in \mathfrak{M}_2(\Gamma_9, v_\eta^4 \times v_H).$$

The first cusp forms for the paramodular groups Γ_5, Γ_6, Γ_7, Γ_{10}. One can prove that

$$\vartheta(\tau, az)\vartheta(\tau, bz) \in J^{cusp}_{1,\frac{1}{2}(a^2+b^2)}(v_\eta^6 \times v_H)$$

is a cusp Jacobi form of weight one. It gives us an infinite series of Siegel cusp forms of weight one

$$\operatorname{Lift}(\vartheta(\tau, az)\vartheta(\tau, bz)) \in \mathfrak{M}_1(\Gamma_{2(a^2+b^2)}, v_\eta^6 \times v_H)$$

with a character of order 4. One can prove that

$$F_1^{(10)}(Z)^4 = \operatorname{Lift}(\vartheta(\tau, z)\vartheta(\tau, 2z))^4 \in \mathfrak{M}_4(\Gamma_{10})$$

is the Γ_{10}-cusp form of the minimal weight with trivial character.

For the paramodular group Γ_5 the first cusp form with trivial character has weight 5. This is the lifting

$$F_5^{(5)}(Z) = \operatorname{Lift}\,(\eta(\tau)^3\vartheta(\tau, z)^6\vartheta(\tau, 2z)) \in \mathfrak{M}_5(\Gamma_5). \tag{1.10}$$

For Γ_6 the first cusp form is the product of two liftings

$$F_6^{(6)}(Z) = \operatorname{Lift}\big(\eta(\tau)^5\vartheta(\tau, 2z)\big) \cdot \operatorname{Lift}_2\big(\eta(\tau)^5\vartheta(\tau, 2z)\big) \in \mathfrak{M}_6(\Gamma_6) \tag{1.11}$$

where Lift_2 is 2-lifting of the theorem above.

For the group Γ_7 the first cusp form is the square of the lifting

$$F_2^{(7)}(Z)^2 = \operatorname{Lift}\,(\vartheta(\tau, z)^3\vartheta(\tau, 2z))^2 \in \mathfrak{M}_4(\Gamma_7). \tag{1.12}$$

In the next section we show that all these modular forms have an infinite product expansion.

Some other interesting examples of the arithmetic liftings with non-trivial commutator character see in [GN6] and [GH2].

Applications to Algebraic Geometry. At this point it is natural to discuss the geometric implications of our results. Weight 3 cusp forms are closely related to canonical differential forms on smooth models of the corresponding modular variety. If F is a cusp form of weight 3 with respect to a group Γ, then $\omega_F = F(Z)dZ$ is a holomorphic 3-form on the space $\mathcal{A}_\Gamma^o = (\Gamma \setminus \mathbb{H}_2)^o$, where ^{o} means that we consider the threefold outside the branch locus of the natural projection from $\mathbb{H}_2$ to $\mathcal{A}_\Gamma$. A very useful extension theorem due to E. Freitag implies that such a form can be extended to any smooth model of $\mathcal{A}_\Gamma$. To be more precise, let Γ be an arbitrary subgroup of $Sp_4(\mathbb{R})$, which contains a principal congruence subgroup $\Gamma_1(q) \subset Sp_4(\mathbb{Z})$ of some level q. We then have the following

Criterion. *(Freitag) An element $\omega_F = F(Z)dZ \in H^0(\mathcal{A}_\Gamma^o, \Omega_3(\mathcal{A}_\Gamma^o))$ can be extended to a canonical differential form on a non-singular model $\widetilde{\mathcal{A}}_\Gamma$ of a compactification of $\mathcal{A}_\Gamma$ if and only if the differential form ω_F is square integrable.*

Proof. See [F], Hilfsatz 3.2.1.

It is well known that a Γ-invariant differential form $\omega_F = F(Z)dZ$ is square-integrable if and only if F is a cusp form of weight 3 with respect to the group Γ. Thus we have the following identity for the geometric genus of the variety $\widetilde{\mathcal{A}}_\Gamma$:

$$p_g(\widetilde{\mathcal{A}}_\Gamma) = h^{3,0}(\widetilde{\mathcal{A}}_\Gamma) = \dim_{\mathbb{C}} \mathfrak{N}_3(\Gamma). \tag{1.14}$$

In [G1] cusp forms of weight 3 with respect to the paramodular group Γ_t with trivial character were constructed for all t except

$$t = 1,\ 2,\ \ldots,\ 12,\ 14,\ 15,\ 16,\ 18,\ 20,\ 24,\ 30,\ 36. \tag{1.15}$$

We call these polarizations exceptional. Thus we have

Non-Unirationality Theorem. *$\mathcal{A}_t$ is not unirational for all non-exceptional polarizations.*

Remark. A similar (but a little weaker) result is true in the case of moduli space of polarized $K3$ surfaces (see [G3]).

We also remark at this point that Γ_t-cusp forms of weight 2 can be very useful when one wants to prove that some modular threefolds are of general type (see [GS]). All these facts explain our interest in Siegel cusp forms of small weight k ($k \le 3$).

One can formulate a standard conjecture:

$\mathcal{A}_t$ is unirational for all exceptional polarizations.

In [GH2] it was proved that the geometrical genus of $\mathcal{A}_t$ is zero for $t \le 8$ (see also [GP]). Nevertheless as it was found in [GH2] the threefolds $\mathcal{A}_t$ have very natural modular coverings of degree 2 or 3 with positive geometrical genus.

Definition. Let Γ be an arithmetic subgroup of $Sp_4(\mathbb{R})$. For any character $\chi_\Gamma : \Gamma \to \mathbb{C}^*$ we define the threefold

$$\mathcal{A}(\chi_\Gamma) = \ker(\chi_\Gamma) \setminus \mathbb{H}_2.$$

The covering $\mathcal{A}(\chi_\Gamma) \to \mathcal{A}_\Gamma = \Gamma \setminus \mathbb{H}_2$ is Galois with a finite abelian Galois group.

For example, $A_t^{com} = \Gamma_t' \setminus \mathbb{H}_2$ is *the maximal abelian covering* of the moduli space $\mathcal{A}_t$ (Γ_t' is the commutator subgroup of Γ_t).

The next theorem is a direct corollary of the Arithmetic Lifting Theorem.

Theorem on Abelian Modular Coverings. *(see* [GH2, Corollary 1.6]*.) Let t be one of the exceptional polarizations.*

(1) *If $t \ne 1,\ 2,\ 4,\ 5,\ 8,\ 16$, then the modular double covering*

$$\mathcal{A}(\chi_2) \xrightarrow{2:1} \mathcal{A}_t$$

of the moduli space $\mathcal{A}_t$ of abelian surfaces with polarization of type $(1,t)$ has positive geometric genus, and in particular the Kodaira dimension of $\mathcal{A}(\chi_2)$ is not negative. Moreover, for $t = 12,\ 15,\ 18,\ 20,\ 30$ and 36 the Kodaira dimension of the corresponding $\mathcal{A}_t(\chi)$ is positive.

(2) *If $t = 6,\ 12,\ 15,\ 18,\ 24,\ 30,\ 36$, then the threefold $\mathcal{A}_t(\chi_3) \xrightarrow{3:1} \mathcal{A}_t$ has positive geometric genus.*

(3) *If $t = 8$ or 16, then the covering $\mathcal{A}_t(\chi_4) \xrightarrow{4:1} \mathcal{A}_t$ has positive geometric genus.*

In particular all these modular varieties are not unirational.

Moreover we have the following corollary about maximal abelian covering of $\mathcal{A}_t$ (compare with Theorem on Non-Unirationality of $\mathcal{A}_t$ above).

Theorem on Maximal Abelian Coverings. *Let $\widetilde{\mathcal{A}}_t^{com}$ be a smooth projective model of the maximal abelian covering $\mathcal{A}_t^{com} = \Gamma_t' \setminus \mathbb{H}_2$ of $\mathcal{A}_t$. Then $\widetilde{\mathcal{A}}_t^{com}$ has geometric genus* 0 *if and only if* $t = 1$, 2, 4 *or* 5.

Let us consider the particular case of polarization of type $(1,3)$. The cube of the cusp form Δ_1 (see example above) defines a canonical differential form on the double cover $\mathcal{A}_3(\chi_6^3)$ of $\mathcal{A}_3$ where $\Delta_1^3 \in \mathfrak{N}_3(\Gamma_3, \chi_6^2)$ thus $h^{3,0}(\mathcal{A}_3(\chi_6^3)) \geq 1$. In fact we have equality! In [GH2] we found *six coverings of $\mathcal{A}_3$ with geometrical genus one.* One can prove the following result

Threefolds with the Geometrical Genus One. *Let $\mathcal{A}_3^{com} \xrightarrow{18:1} \mathcal{A}_3$ and $\mathcal{A}_7^{com} \xrightarrow{2:1} \mathcal{A}_7$ be two maximal abelian coverings. Then*

$$h^{3,0}(\mathcal{A}_3^{com}) = 1, \qquad h^{3,0}(\mathcal{A}_7^{com}) = 1.$$

Remark. One can prove that the threefolds $\widetilde{\mathcal{A}_3}$ and $\widetilde{\mathcal{A}_7}$ are rational (see [G6])

We would like to put the following question

Question. *How far are the threefolds $\mathcal{A}_3^{com}$ and $\mathcal{A}_7^{com}$ from Calabi–Yau?*

§2. Exponential or Borcherds Lifting

In this section we give another construction of modular forms, *exponential or Borcherds lifting*, which gives us modular forms with Humbert divisors. In the case of Sp_4 the Humbert divisors are classical Humbert surfaces. For our purpose it is more convenient to consider Humbert surfaces as divisors of a double cover of the moduli space of $(1,t)$-polarized Abelian surfaces $\mathcal{A}_t$. To define this covering we consider a double normal extension of the paramodular group Γ_t

$$\Gamma_t^+ = \Gamma_t \cup \Gamma_t V_t, \qquad V_t = \frac{1}{\sqrt{t}} \begin{pmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \end{pmatrix}.$$

The double quotient

$$\mathcal{A}_t \xrightarrow{2:1} \mathcal{A}_t^+ = \Gamma_t^+ \setminus \mathbb{H}_2$$

of $\mathcal{A}_t$ can be interpreted as a moduli space of lattice-polarized $K3$ surfaces for arbitrary t or as the moduli space of Kummer surfaces of $(1,p)$-polarized abelian surfaces for a prime $t = p$ (see [GH1, Theorem 1.5]).

Any Humbert surface in $\mathcal{A}_t^+$ of discriminant D can be represented in the form

$$H_D^+(b) = \pi_t^+ \Big(\bigcup_{g \in \Gamma_t^+} g^*(\{Z \in \mathbb{H}_2 \,|\, a\tau + bz + t\omega = 0\})\Big)$$

where $a, b \in \mathbb{Z}$, $D = b^2 - 4ta$, $b \bmod 2t$ and $\pi_t^+ : \mathbb{H}_2 \to \mathcal{A}_t^+$ (see [vdG], [GH1]). Remark that $H_D^+(b)$ depends only on $\pm b \mod 2t$.

The datum for the exponential lifting is a nearly-holomorphic Jacobi form

$$\phi_{0,t}(\tau, z) = \sum_{n,l \in \mathbb{Z}} f(n,l) q^n r^l \in J_{0,t}^{nh} \tag{2.1}$$

of weight 0 and index t. Nearly holomorphic means that there exist a number m such that $\Delta^m(\tau)\phi(\tau, z)$ is a weak Jacobi form. If we chose the minimal non-negative m with this property then $n \geq -m$ in the Fourier expansion (2.1). Bellow we use the notation (1.7).

Exponential Lifting Theorem. *(see [GN6]) Assume that the Fourier coefficients of Jacobi form $\phi_{0,t}$ from (2.1) are integral. Then the product*

$$\text{Exp-Lift}(\phi_{0,t})(Z) = B_\phi(Z) = q^A r^B s^C \prod_{\substack{n,l,m \in \mathbb{Z} \\ (n,l,m)>0}} (1 - q^n r^l s^{tm})^{f(nm,l)}, \tag{2.2}$$

where

$$A = \frac{1}{24} \sum_l f(0,l), \quad B = \frac{1}{2} \sum_{l>0} l f(0,l), \quad C = \frac{1}{4} \sum_l l^2 f(0,l),$$

and $(n,l,m) > 0$ means that if $m > 0$, then l and n are arbitrary integers, if $m = 0$, then $n > 0$ and $l \in \mathbb{Z}$ or $l < 0$ if $n = m = 0$, defines a meromorphic modular form of weight $\frac{f(0,0)}{2}$ with respect to Γ_t^+ with a character (or a multiplier system if the weight is half-integral) induced by $v_\eta^{24A} \times v_H^{2B}$. All divisors of $\text{Exp-Lift}(\phi_{0,t})(Z)$ *on $\mathcal{A}_t^+$ are the Humbert modular surfaces $H_D(b)$ of discriminant $D = b^2 - 4ta$ with multiplicities*

$$m_{D,b} = \sum_{n>0} f(n^2 a, nb).$$

Moreover

$$B_\phi(V_t(Z)) = (-1)^D B_\phi(Z) \qquad \text{with} \quad D = \sum_{\substack{n<0 \\ l \in \mathbb{Z}}} \sigma_1(-n) f(n,l)$$

where $\sigma_1(n) = \sum_{d|n} d$.

The infinite product expansion of the modular form $\Delta_5(Z)$. We recall that there exists the unique, up to a constant, weak Jacobi form of weight zero and index one

$$\phi_{0,1}(\tau, z) = \frac{\phi_{12,1}(\tau, z)}{\eta(\tau)^{24}} = (r + 10 + r^{-1}) + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2) + q^2(\dots)$$

where $\phi_{12,1}$ is the unique Jacobi cusp form of weight 12 and index 1. There are several formulae for Fourier coefficients of this Jacobi form. In [EZ] one can find a formula for Fourier coefficients of $\phi_{12,1}$ it in terms of Cohen's numbers (values of special L-functions at integral points). For a very convenient formula in terms of Hecke operators see [GN6, (3.34)]). Using the theorem above for the function $\phi_{0,1}$ we get the following result from [GN1]:

$$\Delta_5(Z) = (qrs)^{1/2} \prod_{\substack{n,\, l,\, m \in \mathbb{Z} \\ (n,l,m)>0}} \left(1 - q^n r^l s^m\right)^{f_1(nm,l)} \in \mathfrak{M}_5(\Gamma_1, \chi_2)$$

where $f_1(n,l)$ are the Fourier coefficients of $\phi_{0,1}(\tau, z)$.

The products expression for Δ_1 and Δ_2. Let us define a weak Jacobi form

$$\phi_{0,3}(\tau,z)=\left(\frac{\vartheta_{3/2}(\tau,z)}{\eta(\tau)}\right)^2=\left(\frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)}\right)^2=\sum_{n\ge 0,\,l} f_3(n,l)q^n r^l$$

$$=r^{-1}\Bigl(\prod_{n\ge 1}(1+q^{n-1}r)(1+q^n r^{-1})(1-q^{2n-1}r^2)(1-q^{2n-1}r^{-2})\Bigr)^2$$

$$=(r+2+r^{-1})+q(-4r^{\pm 3}-4r^{\pm 2}+2r^{\pm 1}+4)+q^2(\dots)$$

Then we have

$$\Delta_1(Z)=\sum_{M\ge 1}\sum_{\substack{m>0,\,l\in\mathbb{Z}\\ n,\,m\equiv 1\,mod\,6\\ 4nm-3l^2=M^2}}\left(\frac{-4}{l}\right)\left(\frac{12}{M}\right)\sum_{a|(n,l,m)}\left(\frac{6}{a}\right)q^{n/6}r^{l/2}s^{m/2}$$

$$=q^{\frac{1}{6}}r^{\frac{1}{2}}s^{\frac{1}{2}}\prod_{\substack{n,\,l,\,m\in\mathbb{Z}\\ (n,l,m)>0}}\left(1-q^n r^l s^{3m}\right)^{f_3(nm,l)}\in\mathfrak{M}_1(\Gamma_3^+,\chi_6).$$

To get a similar formula for Δ_2 we consider the following holomorphic Jacobi form of weight 2

$$\phi_{2,2}(\tau,z)=\frac{1}{2\pi i}\Bigl(3\frac{\partial\vartheta(\tau,z)}{\partial z}\vartheta_{3/2}(\tau,z)-\frac{\partial\vartheta_{3/2}(\tau,z)}{\partial z}\vartheta(\tau,z)\Bigr)$$

$$=\frac{1}{2}\sum_{m,\,n\in\mathbb{Z}}(3m-n)\left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)q^{\frac{3m^2+n^2}{24}}r^{\frac{m+n}{2}}\in J_{2,2}(v_\eta^4\times \mathrm{id}_H).$$

Moreover the divisor of these modular forms is the irreducible Humbert surface H_1 in the corresponding moduli space

$$\mathrm{Div}_{\mathcal{A}_1}(\Delta_5(Z))=H_1,\qquad \mathrm{Div}_{\mathcal{A}_2^+}(\Delta_2(Z))=H_1,\qquad \mathrm{Div}_{\mathcal{A}_3^+}(\Delta_1(Z))=H_1.$$

One can show that these Δ-functions are *discriminant of some moduli space of $K3$ surfaces with a condition on the Picard lattice* (see [GN3]).

The singular modular forms $\Delta_{1/2}$ and $D_{1/2}$. Now we define two special singular modular forms with respect to the full paramodular group Γ_4 and Γ_{36}. The first form is one of the classical Siegel theta-constants. The "most odd" even theta-constant

$$\Theta_{1,1}(Z)=\frac{1}{2}\sum_{l_1,l_2\in\mathbb{Z}}\exp\left(\pi i\,(Z\left[\begin{smallmatrix}l_1+\frac{1}{2}\\ l_2+\frac{1}{2}\end{smallmatrix}\right]+l_1+l_2)\right)=\frac{1}{2}\sum_{n,\,m\in\mathbb{Z}}\left(\frac{-4}{n}\right)\left(\frac{-4}{m}\right)q^{n^2/8}r^{nm/4}s^{m^2/8},$$

where $Z[M]={}^tMZM$, is a modular form with respect to the subgroup $\Gamma_{2,1}\subset Sp_4(\mathbb{Z})$ conjugated to the paramodular group Γ_4 (see [GN6]). Our second example is a modular form of weight 1/2 with respect to the full paramodular group Γ_{36}. Firstly we define a Jacobi form

$$\phi_{0,4}(\tau,z)=\frac{\vartheta(\tau,3z)}{\vartheta(\tau,z)}=\sum_{n\ge 0,\,l\in\mathbb{Z}} f_4(n,l)q^nr^l$$

$$=r^{-1}\prod_{m\ge 1}(1+q^{m-1}r+q^{2m-2}r^2)(1+q^mr^{-1}+q^{2m}r^{-2})\prod_{\substack{n\equiv 1,2\, mod\, 3\\ n\ge 1}}(1-q^nr^3)(1-q^nr^{-3})$$

$$=(r+1+r^{-1})-q(r^4+r^3-r+2-r^{-1}+r^{-3}+r^{-4})+q^2(\dots) \tag{2.10}$$

where all Fourier coefficients $f_4(n,l)$ of the weak Jacobi form are integral (in fact they are Fourier coefficients of automorphic forms of weight $-1/2$). Moreover we get the following identity for the exponential lifting of $\phi_{0,4}$

$$\Delta_{1/2}(Z)=\frac{1}{2}\sum_{n,m\in\mathbb{Z}}\left(\frac{-4}{n}\right)\left(\frac{-4}{m}\right)q^{n^2/8}r^{nm/2}s^{m^2/2}=$$

$$q^{1/8}r^{1/2}s^{1/2}\prod_{(n,l,m)>0}(1-q^nr^ls^{4m})^{f_4(nm,l)}\in\mathfrak{M})1/2(\Gamma_4,\chi_8)$$

where χ_8 is a multiplier system of Γ_4 of order 8 induced by the multiplier system $v_\eta^3\times v_H$ of the Jacobi group. The divisor of this theta-series is again H_1

$$\mathrm{Div}_{\mathcal{A}_4}(\Delta_{1/2}(Z))=H_1.$$

To define the form $D_{1/2}$ we determine a Jacobi form

$$\phi_{0,36}(\tau,z)=\frac{\vartheta_{3/2}(\tau,5z)}{\vartheta_{3/2}(\tau,z)}=\frac{\vartheta(\tau,10z)\vartheta(\tau,z)}{\vartheta(\tau,5z)\vartheta(\tau,2z)}=\sum_{n\ge 0,\ l\in\mathbb{Z}} f_{36}(n,l)\,q^nr^l$$

$$=r^{-2}\prod_{n\ge 1}\frac{(1+q^{n-1}r^5)(1+q^nr^{-5})(1-q^{2n-1}r^{10})(1-q^{2n-1}r^{-10})}{(1+q^{n-1}r)(1+q^nr^{-1})(1-q^{2n-1}r^2)(1-q^{2n-1}r^{-2})}$$

$$=(r^2-r^1+1-r^{-1}+r^{-2})+q^2(-r^{17}+\dots)+q^5(r^{27}+\dots)$$

$$+q^7(r^{32}+\dots)+q^8(r^{34}+\dots)+\dots,$$

where we include in the last formula only summands q^nr^l with the negative norm $144n-l^2$: $144\cdot 2-17^2=-1$, $144\cdot 5-27^2=-9$, $144\cdot 7-32^2=-16$, $144\cdot 8-34^2=-4$. Then one can prove the following identity

$$\frac{1}{2}\sum_{m,n\in\mathbb{Z}}\left(\frac{12}{n}\right)\left(\frac{12}{m}\right)q^{n^2/24}r^{nm/2}s^{3m^2/2}=q^{1/24}r^{1/2}s^{3/2}\prod_{(n,l,m)>0}(1-q^nr^ls^{36m})^{f_{36}(nm,l)}.$$

The divisor of this modular form is more complicated, but this is still a reflective modular form:

$$\mathrm{Div}_{\mathcal{A}_{36}^+}(D_{1/2}))=H_4(2)+H_4(34)+H_9(27)+H_{16}(32).$$

The example of modular forms given above give us some evidence for the following conjecture

Conjecture C. *Let $t > 1$ be square free. Then the space $\mathfrak{M}_{min}(\Gamma_t')$ of modular form of the minimal weight with respect to the commutator subgroup Γ_t' of the paramodular group Γ_t contains a Humbert modular form. In other words, we can choose a Humbert modular form as a generator of the minimal weight of the graded ring $\mathfrak{M}_*^{(1)}(\Gamma_t')$.*

We can formulate the same conjecture for arbitrary t, but then we should admit modular forms of half-integral weight.

Using considerations from the last section of [GH2] we can prove rather important property of modular forms of type Δ.

Uniqueness Theorem. *The modular forms Δ_5, Δ_2, Δ_1, $\Delta_{1/2}$ are the only Siegel modular forms with respect to Γ_t^+ with the divisor equal exactly to H_1^+ (with multiplicity one).*

The result about the divisor of modular forms of Δ-type gives us a particular answer on Problems A and B formulated in §1. As corollary we get

Rationality Theorem K. *The moduli space of Kummer surfaces associated with* $(1,2)$-, $(1,3)$- *or* $(1,4)$-*polarized Abelian surfaces is rational.*

Remark. For $t = 2$ this result was obtained by Freitag (see [F1]) and Ibukiyama (see [Ib]) by other methods.

Using Theorem on Exponential Lifting one can find divisor of the first cusp forms for Γ_5, Γ_6, Γ_7 and Γ_{10} from the example above. It gives us rationality of the corresponding moduli spaces (see [G6]).

All modular forms constructed above are *symmetric*, i.e. invariant with respect to the exterior involution V_t. To prove rationality of the moduli space of Abelian surfaces we should construct modular forms anti-invariant with respect to V_t.

Anti-symmetric modular forms. Using the exponential lifting, one can construct anti-invariant modular forms, i.e. forms satisfying $F(V_t(Z)) = -F(Z)$. For example for $t = 1$ we have the anti-invariant form $F(Z) = \Delta_{35}(Z)$. This is so called *Igusa modular forms.* In [GN4] we proposed a new construction of this form as a Hecke product of $\Delta_5(Z)$.

Let us take the modular form $\Delta_5(Z)$ which has the divisor H_1 in $\mathcal{A}_1$. Using the system of representatives $T(p)$, we then get

$$[\Delta_5(Z)]_{T(2)} = \prod_{a,b,c \bmod 2} \Delta_5(\tfrac{z_1+a}{2}, \tfrac{z_2+b}{2}, \tfrac{z_3+c}{2}) \prod_{a \bmod 2} \Delta_5(\tfrac{z_1+a}{2}, z_2, 2z_3)\, \Delta_5(2z_1, z_2, \tfrac{z_3+a}{2})$$

$$\times\, \Delta_5(2z_1, 2z_2, 2z_3) \prod_{b \bmod 2} \Delta_5(2z_1, -z_1+z_2, \tfrac{z_1-2z_2+z_3+b}{2}).$$

One can check that $\operatorname{div}_{\mathcal{A}_1}\big([\Delta_5(Z)]_{T(2)}\big) = 9H_1 + H_4$. Thus

$$\Delta_{35}(Z) = \frac{[\Delta_5(Z)]_{T(2)}}{\Delta_5(Z)^8} = \text{Exp-Lift}\big(\phi_{0,1}|(T_0(2) - 2)\big) \in \mathfrak{N}_{35}(\Gamma_1)$$

and

$$\Delta_{35}(Z) = q^2 r s^2\,(q-s) \prod_{\substack{n,\,l,\,m \in \mathbb{Z}\\ (n,l,m)>0}} \big(1 - q^n r^l s^m\big)^{f_1^{(2)}(4nm-l^2)}$$

where $f_1(4n-l^2)=f_1(n,l)$ are the Fourier coefficients of $\phi_{0,1}(\tau,z)$ and

$$f_1^{(2)}(N)=8f_1(4N)+2(\left(\frac{-N}{2}\right)-1)f_1(N)+f_1(\frac{N}{4}).$$

Remark that we cannot construct $\Delta_{35}(Z)$ as an arithmetic lifting of a holomorphic Jacobi form. Nevertheless (3.29) gives us $\Delta_{35}(Z)$ as a finite Hecke product of the lifted form $\Delta_5(Z)$.

A similar construction gives us the unique anti-symmetric modular forms for Γ_2, Γ_3 and Γ_4 of weight 12. The uniqueness will follow from the fact that these modular forms have only the Humbert surface $H_{4t}(0)=\pi_t\{\tau-t\omega=0\}$ as their divisor.

Let us consider the function $\psi_{0,t}(\tau,z)=\Delta(\tau)^{-1}E_{12,t}(\tau,z)$ where

$$\Delta(\tau)=q\prod_{n\ge1}(1-q^n)^{24}=q-24q^2+253q^3+\dots$$

and $E_{12,t}(\tau,z)$ is a Jacobi–Eisenstein series of weight 12 and index t.

There exists a formula for Fourier coefficients of $E_{k,1}$ in terms of H. Cohen's numbers (see [EZ, §2]). One can find the table of the values of Fourier coefficients of $E_{4,1}(\tau,z)$ and $E_{6,1}(\tau,z)$ in [EZ, §1]. Using the basic Jacobi forms $\phi_{0,1}$, $\phi_{0,2}$ and $\phi_{0,3}(\tau,z)$ we define

$$\begin{aligned}\psi_{0,2}(\tau,z)&=\Delta(\tau)^{-1}E_{6,1}(\tau,z)^2-2\phi_{0,1}^2+216\phi_{0,2}(\tau,z)\\&=\sum_{n\ge0,\,l\in\mathbb{Z}}c_2(n,l)q^nr^l=q^{-1}+24+q(\dots),\\ \psi_{0,3}(\tau,z)&=\Delta(\tau)^{-1}E_{4,1}(\tau,z)^3-3\phi_{0,3}|T_0(2)(\tau,z)-180\phi_{0,3}(\tau,z)\\&=\sum_{n\ge0,\,l\in\mathbb{Z}}c_3(n,l)q^nr^l=q^{-1}+24+q(\dots).\end{aligned}$$

The Jacobi forms $\psi_{0,p}$ ($p=2$, 3) contain the only type of Fourier coefficients with indices of negative norm. This is q^{-1} of norm $-4p$. Thus we can use both functions to produce the exponential liftings

$$\Psi_{12}^{(2)}(Z)=\text{Exp-Lift}(\psi_{0,2})=q\prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}}(1-q^nr^ls^{2m})^{c_2(nm,l)}\in\mathfrak{M}_{12}(\Gamma_2),$$

$$\Psi_{12}^{(3)}(Z)=\text{Exp-Lift}(\psi_{0,3})=q\prod_{\substack{n,l,m\in\mathbb{Z}\\(n,l,m)>0}}(1-q^nr^ls^{3m})^{c_3(nm,l)}\in\mathfrak{M}_{12}(\Gamma_3).$$

According to Theorem on Exponential lifting

$$\Psi_{12}^{(p)}(V_p<Z>)=-\Psi_{12}^{(p)}(Z)\quad(p=2,3)\quad\text{and}\quad\operatorname{Div}_{\mathcal{A}_p}(\Psi_{12}^{(p)})=\begin{cases}H_8&\text{for }p=2\\H_{12}&\text{for }p=3.\end{cases}$$

The Fourier-Jacobi expansion of $\Psi_{12}^{(p)}$ starts with coefficients

$$\Psi_{12}^{(p)}(Z)=\Delta_{12}(\tau)-\Delta_{12}(\tau)\psi_{0,p}(\tau,z)\exp(2\pi ip\omega)+\dots.$$

Therefore the constructed modular forms $\Psi_{12}^{(p)}(Z)$ ($p = 2, 3$) are not cusp forms.

Remark. An expression for $\Phi_{12}^{(2)}$ in terms of theta-constants was found recently in [IO].

If we do the same for $t = 4$, we get a Jacobi form we used to construct $\Delta_{35}(Z)$. Let us take the Jacobi form

$$\phi_{0,1}|(T_0(2) - 2)(\tau, 2z) = q^{-1} + (r^4 + 70 + r^{-4}) + q(\dots).$$

Its exponential lifting is zero along two Humbert surfaces with discriminant 16. To delete the second component, we consider the additional Jacobi–Eisenstein series which has the constant term equals zero (such a series exists if the index contains a perfect square). For $t = 4$ this Jacobi–Eisenstein series is the eight power of the Jacobi theta-series $\vartheta(\tau, z)$.

Using $\vartheta(\tau, z)^8$, we define

$$\begin{aligned}\psi_{0,4}(\tau, z) &= \big(\phi_{0,1}|(T_0(2) + 26)\big)(\tau, 2z) - \Delta(\tau)^{-1}E_4(\tau)\vartheta(\tau, z)^8 - 8\big(\phi_{0,4}|(T_0(3) + 4)\big)(\tau, z)\\ &= \sum_{n\ge 0,\, l\in\mathbb{Z}} c_4(n, l)q^n r^l = q^{-1} + 24 + q(\dots).\end{aligned}$$

Similarly to $\psi_{0,2}$ and $\psi_{0,3}$ the Jacobi form $\psi_{0,4}$ contains only the Fourier coefficients of type q^{-1} with index of negative norm. Taking its exponential lifting we obtain the Γ_4-modular form of weight 12

$$\Psi_{12}^{(4)}(Z) = \text{Exp-Lift}(\psi_{0,4})(Z) = q \prod_{\substack{n,l,m\in\mathbb{Z}\\ (n,l,m)>0}} (1 - q^n r^l s^{4m})^{c_4(nm,l)} \in \mathfrak{M}_{12}(\Gamma_4).$$

$\Psi_{12}^{(4)}(Z)$ is anti-invariant and $\mathrm{Div}_{\mathcal{A}_4}(\Psi_{12}^{(4)}) = H_8(0)$.

The construction of the modular forms $\Psi_{12}^{(t)}$ together with information about their divisor gives us

Rationality Theorem A. *The moduli space of $(1,2)$-, $(1,3)$- or $(1,4)$-polarized Abelian surfaces is rational.*

Remark. In March, 1997, a new preprint of Ibukiyama and Onodera [IO] where the rationality of $\mathcal{A}_2$ was also proved by another method.

The pair of modular forms (Δ_5, Δ_{35}) and $(\Delta_2, \Psi_{12}^{(2)})$ were used in [Ka2] and in [CCL], [C] to define mirror symmetry for heterotic and IIA strings.

Applications to Lorentzian Kac–Moody algebras. Almost all identities between arithmetic and exponential liftings mentioned above give denominator formulae for Lorentzian Kac–Moody algebras (see [GN1] and specially [GN5]–[GN6] for more details). These algebras are generalized Kac–Moody superalgebras. Bellow we give a correspondence between constructed modular forms and generalized Cartan matrices of hyperbolic type which describe the set of real simple roots of the algebras under construction.

We denote by $M_t = U(4t) \oplus \langle 2\rangle$ ($t \in \mathbb{N}/4$) a hyperbolic lattice with the bases $f_2, \widehat{f_3}, f_{-2}$ and with the Gram matrix

$$\begin{pmatrix} 0 & 0 & -4t \\ 0 & 2 & 0 \\ -4t & 0 & 0 \end{pmatrix}.$$

We denote an element $nf_2 + l\widehat{f_3} + mf_{-2}$ of this lattice (or of $M_t \otimes \mathbb{Q}$) by its coordinates (n, l, m).

All our formulae have the form:

$$\Phi(Z) = \sum_{(n,l,m)\in M_t} N(n,l,m) q^{\rho_1+n} r^{\rho_2+l} s^{t(\rho_3+m)} =$$

$$= \sum_{w\in W} \epsilon(w) \sum_{a\in\mathbb{R}_{++}\mathcal{M}\cap M_t} (e^{w(\rho)} + N(a)e^{w(\rho+a)}) = q^{\rho_1} r^{\rho_2} s^{t\rho_3} \prod_{(n,l,m)>0} (1 - q^n r^l s^{tm})^{f(n,l,m)}$$

for some W, $\mathcal{M}$, $\rho = (\rho_1, \rho_2, \rho_3)$ which we define below. The "exponent" $e^{(n,l,m)} := q^n r^l s^{tm}$. The *Weyl group* W is a reflection subgroup $W \subset W(M_t) \subset O(M_t)$, and $\mathcal{M}_t$ is a fundamental polyhedron of W in the hyperbolic space $\mathcal{L}(M_t)$ defined by the hyperbolic lattice M_t. ¿From the automorphic point of view the Weyl group W is defined by the modular property of the modular form Φ.

The elements a together with coefficients $N(a)$ at the left hand side of the denominator formula form a multi-set of imaginary simple roots ${}_s\Delta^{im}$ of the constructed generalized Kac–Moody algebra (one should take every a exactly $|N(a)|$-times in this multi-set). The $\mathbb{Z}_2$-graduation of the superalgebra is defined by the sign of the Fourier coefficients $N(a)$.

Then the Siegel modular forms are related with the following generalized Cartan matrices, equivalently, hyperbolic root systems

1. *The product of all even theta-constant Δ_5 — the right triangle with vertices at infinity* (i.e the fundamental polygon of the Weyl group is a right triangle).

$$A_{2,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

2. *The Igusa modular form Δ_{35} — a right triangle with two vertices at infinity.*

$$A_{1,0} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

This is the "simplest" hyperbolic Cartan matrix (see [FF]).

3. *The form Δ_2 — the right square with all vertices at infinity.*

$$A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}.$$

4. *The form Δ_1 — the right hexagon with all vertices at infinity.*

$$A_{2,III} = \begin{pmatrix} 2 & -2 & -8 & -16 & -18 & -14 & -8 & 0 \\ -2 & 2 & 0 & -8 & -14 & -18 & -16 & -8 \\ -8 & 0 & 2 & -2 & -8 & -16 & -18 & -14 \\ -16 & -8 & -2 & 2 & 0 & -8 & -14 & -18 \\ -18 & -14 & -8 & 0 & 2 & -2 & -8 & -16 \\ -14 & -18 & -16 & -8 & -2 & 2 & 0 & -8 \\ -8 & -16 & -18 & -14 & -8 & 0 & 2 & -2 \\ 0 & -8 & -14 & -18 & -16 & -8 & -2 & 2 \end{pmatrix}.$$

See [GN5]–[GN6] for more information.

References

[BVP] W. Barth, C. Peters, A. van de Ven, *Compact complex surfaces*, Erg. Math. Grenzgeb., 3.Folge, **4**, Springer Verlag, 1984.

[B1] R. Borcherds, *Sporadic groups and string theory*, Proc. European Congress of Mathem. 1992, pp. 411–421.

[B2] R. Borcherds, *Automorphic forms on $O_{s+2,2}$ and infinite products*, Invent. Math. **120** (1995), 161–213.

[B3] R. Borcherds, *The moduli space of Enriques surfaces and the fake monster Lie superalgebra*, Topology **35** (1996), no. 3, 699–710.

[B4] R. Borcherds, *Automorphic forms with singularities on Grassmanians*, Preprint (1996); alg-geom/9609022.

[C] G.L. Cardoso, *Perturbative gravitational coupling and Siegel modular forms in $D = 4$, $N = 2$ string models*, Preprint (1996); hep-th/9612220.

[CCL] G.L. Cardoso, G. Curio and D. Lüst, *Perturbative coupling and modular forms in $N = 2$ string models with a Wilson line*, Preprint (1996); hep-th/9608154.

[DMVV] R. Dijkgraaf, G. Moore, E. Verlinde, H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Preprint (1996); hep-th/9608096.

[DVV] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Counting dyons in $N = 4$ string theory*, Preprint (1996); hep-th/9607026.

[FF] A.J. Feingold and I.B. Frenkel, *A hyperbolic Kac–Moody algebra and the theory of Siegel modular forms of genus 2*, Math. Ann. **263** (1983), no. 1, 87–144.

[F1] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der math. Wissensch., 254, Springer Verlag, Berlin, Heidelberg, New York, 1983.

[F2] E. Freitag, *Singular modular forms and theta relations*, Lecture Notes in Math., 1487, Springer Verlag, Berlin, Heidelberg, New York, 1991.

[G1] V.A. Gritsenko, *Jacobi functions of n-variables*, Zap. Nauch. Sem. LOMI **168** (1988), 32–45; English transl. in J. Soviet Math. **53** (1991), 243–252.

[G2] V.A. Gritsenko, *Jacobi functions and Euler products for Hermitian modular forms*, Zap. Nauch. Sem. LOMI **183** (1990), 77–123; English transl. in J. Soviet Math. **62** (1992), 2883–2914.

[G3] V. Gritsenko, *Modular forms and moduli spaces of Abelian and K3 surfaces*, Algebra i Analyz **6:6** (1994), 65–102; English transl. in St.Petersburg Math. Jour. **6:6** (1995), 1179–1208.

[G4] V.A. Gritsenko, *Irrationality of the moduli spaces of polarized Abelian surfaces*, Int. Math. Res. Notices **6** (1994), 235–243; in full form in Abelian varieties (Barth, Hulek, Lange, eds.), de Gruyter, Berlin, 1995, pp. 63–81.

[G5] V.A. Gritsenko, *Modulformen zur Paramodulgruppe und Modulräume der Abelschen Varietäten*, Mathematica Gottingensis Schrift. des SFB "Geometrie und Analysis", Heft 12 (1995), 1–89.

[G6] V. Gritsenko, *Rationality of some modular threefolds* (in preparation).

[GH1] V. Gritsenko, K. Hulek, *Minimal Siegel modular threefolds*, Proceedings of the Cambridge Philosophical Society (1997) to appear; alg-geom/9506017.

[GH2] V. Gritsenko, K. Hulek, *Commutator coverings of Siegel threefolds*, Preprint-1128 RIMS Kyoto Univ. (1997); alg-geom/9702007.

[GN1] V.A. Gritsenko, V.V. Nikulin, *Siegel automorphic form correction of some Lorentzian Kac–Moody Lie algebras*, Amer. J. Math. **119** (1997), 181–224; alg-geom/9504006.

[GN2] V.A. Gritsenko, V.V. Nikulin, *Siegel automorphic form correction of a Lorentzian Kac–Moody algebra*, C. R. Acad. Sci. Paris Sér. A–B **321** (1995), 1151–1156.

[GN3] V.A. Gritsenko, V.V. Nikulin, *K3 surfaces, Lorentzian Kac–Moody algebras and mirror symmetry*, Math. Res. Let. **3** (1996), no. 2, 211–229; alg-geom/9510008.

[GN4] V.A. Gritsenko, V.V. Nikulin, *The Igusa modular forms and "the simplest" Lorentzian Kac–Moody algebras*, Matem. Sbornik **187** (1996), 1601–1643; alg-geom/9603010.

[GN5] V.A. Gritsenko, V.V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. Part I*, Preprint-1116 RIMS Kyoto Univ. (1996); alg-geom/9610022.

[GN6] V.A. Gritsenko, V.V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. Part II*, Preprint-1122 RIMS Kyoto Univ. (1996); alg-geom/9611028.

[GN7] V.A. Gritsenko, V.V. Nikulin, *The arithmetic mirror symmetry and Calabi–Yau manifolds*, Preprint-1129 RIMS Kyoto Univ. (1997); alg-geom/9612002.

[GS] V. Gritsenko, G.K. Sankaran, *Moduli of abelian surfaces with a* $(1,p^2)$ *polarization*, Izv. Akad. Nauk of Russia, Ser. Mathem. **60** (1996), no. 5.

[GP] M. Gross, S. Popescu, *Equations of* $(1,d)$*-polarized abelian surfaces*, alg-geom/9609001.

[Ib] T. Ibukiyama, *On Siegel modular variety of level 3*, Inter. J. of Math. **2** (1991), 17–35.

[IO] T. Ibukiyama, F. Onodear, *On the graded ring of modular forms of the Siegel paramodular group of level 2*, Preprint MPI für Mathematik MPI 97-30 (1997).

[Ig1] J. Igusa, *On Siegel modular forms of genus two (II)*, Amer. J. Math. **84** (1964), no. 2, 392–412.

[Ig2] J. Igusa, *Ring of modular forms of degree two over* $\mathbb{Z}$, Am. J. Math. **101** (1979), 132–148.

[HM1] J. Harvey, G. Moore, *Algebras, BPS-states, and strings*, Nucl. Physics. **B463** (1996), 315; hep-th/9510182.

[HM2] J. Harvey, G. Moore, *On the algebras of BPS-states*, Preprint (1996); hep-th/9609017.

[K] V. Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, 1990.

[Ka1] T. Kawai, $N=2$ *Heterotic string threshold correction, K3 surfaces and generalized Kac–Moody superalgebra*, Phys. Lett. **B371** (1996), 59; hep-th/9512046.

[Ka2] T. Kawai, *String duality and modular forms*, Preprint (1996); hep-th/9607078.

[M1] H. Maass, *Die Multiplikatorsysteme zur Siegelschen Modulgruppe*, Nachrichten der Akad. der Wissen. Göttingen (II. Math.-Phys.Klasse) **Nr 11** (1964), 125–135.

[M2] H. Maass, *Über ein Analogon zur Vermutung von Saito-Kurokawa*, Invent. math. **60** (1980), 85–104.

[N] V.V. Nikulin, *Reflection groups in Lobachevsky spaces and the denominator identity for Lorentzian Kac–Moody algebras*, Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 2, 73–106; English transl. in Izvestiya Math. **60** (1996), no. 2, 305–334; alg-geom/9503003.

[Od] T. Oda, *On modular forms associated with indefinite quadratic forms of signature* $(2, n-2)$, Math. Ann. **231** (1977), 97–144.

[vdG] G. van der Geer, *Hilbert modular surfaces*, Erg. Math. Grenzgeb., 3.Folge, **16**, Springer Verlag, 1988.

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