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Dynamics of rational semigroups and skew products*

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1 Introduction

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of maps. A *rational semigroup* is a subsemigroup of $\mathrm{End}(\overline{\mathbb{C}})$ without any constant elements. We define hyperbolic, sub-hyperbolic, and semi-hyperbolic semigroups. We show that if G is a finitely generated rational semigroup and satisfies some semi-hyperbolicity, then there exists an *attractor* in the Fatou set of G for G(Theorem 3.12). In Section 4, we will consider the skew products of rational functions or $\overline{\mathbb{C}}$-fibrations. The "Julia set" of any skew product is defined to be the closure of the union of the fiberwise Julia sets. We will define hyperbolicity and semi-hyperbolicity. We will show that if a skew product is semi-hyperbolic, then the Julia set is equal to the union of the fiberwise Julia sets and the skew product has the contraction property with respect to the backward dynamics along fibers. The results in section 4 are generalized to those of version of $\overline{\mathbb{C}}$-fibrations.

In Section 5, we will consider necessary and sufficient conditions to be semi-hyperbolic. We will show that any sub-hyperbolic semigroup without any superattracting fixed point of any element of the semigroup in the Julia set is semi-hyperbolic.

We consider the Hausdorff dimension of the Julia sets of rational semigroups. To investigate that we construct the *subconformal measures*(Section 6). If a rational semigroup has at most countably many elements and the δ-Poincaré series converges, then we can construct δ-subconformal measures. We will see that if G is a finitely generated semi-hyperbolic rational semigroup, then the Hausdorff dimension of the Julia set is less than the expo-

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nent δ(Theorem 6.7). To show those results, the contracting property of backward dynamics will be used.

In section 7, for any finitely generated rational semigroup $G = \langle f_1, \dots, f_m \rangle$ we consider the skew product costructed by the generator system. we construct the *(backward) self-similar measure.* That is, a kind of invariant measures whose projection to the base space(space of one-sided infinite words) are some Bernoulli measures. We will show the uniqueness for any weight without any assumption about hyperbolicity. Furthermore we caliculate the metric entropy of those measures. We show that the topological entropy of the skew product constructed by the generator system $\{f_1, \dots, f_m\}$ is equal to

$$\log(\Sigma_{j=1}^{m} \deg(f_j))$$

and there exists a unique maximal entropy measure, which coincides with the backward self-similar measure corresponding to the weight

$$a_0 := \left(\frac{\deg(f_1)}{\sum\limits_{j=1}^{m} \deg(f_j)}, \dots, \frac{\deg(f_m)}{\sum\limits_{j=1}^{m} \deg(f_j)}\right).$$

Hence the projection of the maximal entropy measure of the skew product to the base space is equal to the Bernoulli measure corresponding to the above weight a_0. Applying this result if $\{f_j^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint, then we get a lower estimate of Hausdorff dimension of the Julia set of G.

This paper is a summary of a part of the author's thesis([S5]).

2 preliminaries

For a Riemann surface S, let End(S) denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of maps. A *rational semigroup* is a subsemigroup of End($\overline{\mathbb{C}}$) without any constant elements. We say that a rational semigroup G is a *polynomial semigroup* if each element of G is a polynomial.

Definition 2.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \quad J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G.

Definition 2.2. Let G be a rational semigroup and z be a point of $\overline{\mathbb{C}}$. The backward orbit $G^{-1}(z)$ of z and the set of exceptional points $E(G)$ are defined by:

$$G^{-1}(z) \overset{\text{def}}{=} \{w \in \overline{\mathbb{C}} \mid \text{there is some } g \in G \text{ such that } g(w) = z\},$$

$$E(G) \overset{\text{def}}{=} \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) \leq 2\}.$$

Lemma 2.3. *Let G be a rational semigroup.*

1. *For any $f \in G$,*

$$f(F(G)) \subset F(G), \ f^{-1}(J(G)) \subset J(G),$$

$$F(G) \subset F(\langle f \rangle), \ J(\langle f \rangle) \subset J(G)$$

2. *Assume G is generated by a compact subset Λ of End$(\overline{\mathbb{C}})$. Then*

$$J(G) = \bigcup_{f \in \Lambda} f^{-1}(J(G)).$$

We call this property the backward self-similarity of the Julia set.

The Julia set of any rational semigroup is a perfect set, backward orbit of any point of the Julia set is dense in the Julia set and the set of repelling fixed points of the semigroup is dense in the Julia set. In general, the Julia set of a rational semigroup G is not forward invariant under G and the Julia set of a rational semigroup may have non-empty interior points. For example, $J(\langle z^2, 2z \rangle) = \{|z| \leq 1\}$. In fact, in [HM2] it was shown that if G is a finitely generated rational semigroup, then any super attracting fixed point of any element of G does not belong to $\partial J(G)$. Hence we can easily get many examples that the Julia sets have non-empty interior points. For more details about these properties, see [HM1], [HM2], [S1] and [S2]. In this paper we use the notations in [HM1] , [S1] and [S2].

Since the Julia set of a rational semigroup may have non-empty interior points, it is significant for us to get sufficient conditions such that the Julia set has no interior points, to know when the area of the Julia set is equal to 0 or to get an upper estimate of the Hausdorff dimension of the Julia set. We will try that using various information about forward dynamics of the semigroup in the Fatou set or backward dynamics of the semigroup in the Julia set.

3 Hyperbolicity

Definition 3.1. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}.$$

We call $P(G)$ the post critical set of G. We say that G is *hyperbolic* if $P(G) \subset F(G)$. Also we say that G is *sub-hyperbolic* if $\sharp\{P(G) \cap J(G)\} < \infty$ and $P(G) \cap F(G)$ is a compact set.

We denote by $B(x, \epsilon)$ a ball of center x and radius ϵ in the spherical metric. We denote by $D(x, \epsilon)$ a ball of center $x \in \mathbb{C}$ and radius ϵ in the Euclidian metric. Also for any hyperboplic manifold M we denote by $H(x, \epsilon)$ a ball of center $x \in M$ and radius ϵ in the hyperbolic metric. For any rational map g, we denote by $B_g(x, \epsilon)$ a connected component of $g^{-1}(B(x, \epsilon))$. For each open set U in $\overline{\mathbb{C}}$ and each rational map g, we denote by $c(U, g)$ the set of all connected components of $g^{-1}(U)$. Note that if g is a polynomial and $U = D(x, r)$ then any element of $c(U, g)$ is simply connected by the maximal principle.

For each set A in $\overline{\mathbb{C}}$, we denote by A^i the set of all interior points of A.

Definition 3.2. Let G be a rational semigroup and A a set in $\overline{\mathbb{C}}$. We set $G(A) = \cup_{g \in G} g(A)$ and $G^{-1}(A) = \cup_{g \in G} g^{-1}(A)$.

We can show the following Lemma immediately.

Lemma 3.3. *Let G be a rational semigroup. Assume that $\{f_\lambda\}_{\lambda \in \Lambda}$ is a generator system of G. Then we have*

$$\bigcup_{g \in G} \{\textit{critical values of } g\} = \bigcup_{\lambda \in \Lambda} (G \cup \{Id\})(\{\textit{critical values of } f_\lambda\}).$$

Definition 3.4. Let G be a rational semigroup and N a positive integer. We set

$$\begin{aligned} & SH_N(G) \\ = \ & \{x \in \overline{\mathbb{C}} \mid \exists \delta(x) > 0, \ \forall g \in G, \ \forall B_g(x, \delta(x)), \ \deg(g : B_g(x, \delta) \to B(x, \delta)) \leq N\} \end{aligned}$$

and $UH(G) = \overline{\mathbb{C}} \setminus (\cup_{N \in \mathbf{N}} SH_N(G))$.

Remark 1. By definition, $SH_N(G)$ is an open set in $\overline{\mathbb{C}}$ and $g^{-1}(SH_N(G)) \subset SH_N(G)$ for each $g \in G$. Also $UH(G)$ is a compact set and $g(UH(G)) \subset UH(G)$ for each $g \in G$. For each rational map g with $\deg(g) \leq 2$, any parabolic or attracting periodic point of g belongs to $UH(G)$.

Definition 3.5. Let G be a rational semigroup. We say that G is *semi-hyperbolic* (resp. *weakly semi-hyperbolic*) if there exists a positive integer N such that $J(G) \subset SH_N(G)$(resp.$\partial J(G) \subset SH_N(G)$).

Remark 2.

1. If G is semi-hyperbolic and $N = 1$, then G is hyperbolic.
2. If G is hyperbolic, then G is semi-hyperbolic.
3. For a rational map f with the degree at least two, $\langle f \rangle$ is semi-hyperbolic if and only if f has no parabolic orbits and each critical point in the Julia set is non-recurrent([CJY], [Y]). If $\langle f \rangle$ is semi-hyperbolic, then there are neither indifferent cycles, Siegel disks nor Hermann rings. In

[CJY], it is shown that for a polynomial P of degree at least two, P is semi-hyperbolic if and only if the basin of infinity of P is a *John domain.*

Lemma 3.6 ([CJY]). *For any positive integer N and real number r with $0 < r < 1$, there exists a constant $C = C(N, r)$ such that if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with $\deg(f) = N$, then*

$$H(f(z_0), C) \subset f(H(z_0, r)) \subset H(f(z_0), r)$$

for any $z_0 \in D(0,1)$. Here we can take $C = C(N, r)$ independent of f.

Corollary 3.7 ([Y]). *Let V be a simply connected domain in $\mathbb{C}$, $0 \in V$, $f : V \to D(0,1)$ be a proper holomorphic map of degree N and $f(0) = 0$, W be the component of $f^{-1}(D(0,r))$ containing 0, $0 < r < 1$. Then there exists a constant K depending only on r and N, not depending on V and f, so that*

$$|\frac{x}{y}| \leq K$$

for all $x, y \in \partial W$.

Lemma 3.8. *Let V be a domain in $\overline{\mathbb{C}}$, K a continuum in $\overline{\mathbb{C}}$ with $diam_S K = a$. Assume $V \subset \overline{\mathbb{C}} \setminus K$. Let $f : V \to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N, a)$ depending only on N and a such that for each r with $0 < r \leq r(N, a)$, there exists a constant $C = C(N, r)$ depending only on N and r satisfying that for each connected component U of $f^{-1}(D(0,r))$,*

$$diam_S\ U \leq C,$$

where we denote by $diam_S$ the spherical diameter. Also we have $C(N, r) \to 0$ as $r \to 0$.

Definition 3.9. Let G be a rational semigroup. We set

$$A_0(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}_0(G) = \overline{G(\{z \in F(G) \mid \exists g \in G \text{ with } \deg(g) \geq 2,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$A(G) = \overline{G(\{z \in \overline{\mathbb{C}} \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

$$\tilde{A}(G) = \overline{G(\{z \in F(G) \mid \exists g \in G,\ g(x) = x \text{ and } |g'(x)| < 1.\})},$$

where the closure in the definition of $\tilde{A}_0(G)$ and $\tilde{A}(G)$ is considered in $\mathbb{C}$.

Definition 3.10. Let G be a rational semigroup and U a open set in $\overline{\mathbb{C}}$. We say that a non-empty compact subset K of U is an *attractor* in U for G if $g(K) \subset K$ for each $g \in G$ and for any open neighborhood V of K in U and each $z \in U$, $g(z) \in U$ for all but finitely many $g \in G$.

Remark 3. By definition, $A_0(G) \subset A(G) \cap P(G)$. For each $g \in G$, $g(A_0(G)) \subset A_0(G)$ and $g(A(G)) \subset A(G)$. We have also similar statements for $\tilde{A}_0(G)$ and $\tilde{A}(G)$.

Lemma 3.11. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup and E a finite subset of $\overline{\mathbb{C}}$. Assume that each $x \in E$ is not any non-repelling fixed point of any element of G. Then there exists an open neighborhood V of E in $\overline{\mathbb{C}}$ such that for each $z \in V$, if there exists a word $w = (w_1, w_2, \ldots) \in \{1, \ldots, m\}^{\mathbb{N}}$ satifying that:*

1. *for each n, $(f_{w_n} \cdots f_{w_1})(z) \in V$,*
2. *$(f_{w_n} \cdots f_{w_1}(z))$ accumulates only in E and*
3. *for each n, $(f_{w_n} \cdots f_{w_1})(\zeta) \in E$ and $(f_{w_n} \cdots f_{w_1})'(\zeta) \neq 0$ where ζ is the closest point to z in E,*

then z is equal to the point $\zeta \in E$.

By Lemma 3.8 and Lemma 3.11, we get the following result.

Theorem 3.12. *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(G) \neq \emptyset$, there is an element $g \in G$ such that $\deg(g) \geq 2$ and each element of $\mathrm{Aut}\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic. Also we assume all of the following conditions;*

1. *$\tilde{A}_0(G)$ is a compact subset of $F(G)$,*
2. *any element of G with the degree at least two has neither Siegel disks nor Hermann rings.*
3. *$\sharp(UH(G) \cap \partial J(G)) < \infty$ and each point of $UH(G) \cap \partial J(G)$ is not a non-repelling fixed point of any element of G.*

Then $\tilde{A}_0(G) = \tilde{A}(G) \neq \emptyset$ and for each compact subset L of $F(G)$,

$$\sup\{d(f_{i_n} \cdots f_{i_1}(z),\ \tilde{A}(G)) \mid z \in L,\ (i_n, \ldots, i_1) \in \{1, \ldots, m\}^n\} \to 0,$$

as $n \to \infty$, where we denote by d the spherical metric. Also $\tilde{A}(G)$ is the smallest attractor in $F(G)$ for G. Moreover we have that if (h_n) is a sequence in G consisting of mutually disjoint elements and converges to a map ϕ in a subdomain V of $F(G)$, then ϕ is constant taking its value in $\tilde{A}(G)$.

Remark 4. Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is sub- or semi-hyperbolic. Assume that there is an element $g \in G$ such that $\deg(g) \geq 2$ and each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic. If $F(G) \neq \emptyset$, then all of the conditions in the assumption in Theorem 3.12 are satisfied. Note that by [HM2] if z is an attracting fixed point of some element of G, then z does not belong to $\partial J(G)$.

4 Rational Skew Product

Definition 4.1 (rational skew product). Let X be a topological space. If a continuous map $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is represented by the following form:

$$\tilde{f}((x, y)) = (p(x),\ q_x(y)),$$

where $p : X \to X$ is a continuous map and $q_x : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map with the degree at least 1 for each $x \in X$, then we say that $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is a rational skew product. In this paper we always assume that X is a compact metric space.

O.Sester investigated polynomial skew products(in particular, quadratic case) in [Se]. M.Jonsson investigated $\overline{\mathbb{C}}$-fibration in [J2].

Definition 4.2. For each $n \in \mathbb{N}$ and $x \in X$, we set $q_x^{(n)} := q_{p^{n-1}(x)} \circ \cdots \circ q_x$ and $\tilde{f}_x^n := \tilde{f}^n|_{\pi_X^{-1}(\{x\})}$. We define the following sets. For each $x \in X$,

$$d(x) = \deg(q_x),$$

$$F_x = \{y \in \overline{\mathbb{C}} \mid \{q_x^{(n)}\}_n \text{ is normal in a neigborhood of } y\},$$

$$J_x = \overline{\mathbb{C}} \setminus F_x,\ \tilde{J}_x = \{x\} \times J_x.$$

Further we set

$$\tilde{J}(\tilde{f}) = \overline{\bigcup_{x \in X} \tilde{J}_x},\ \tilde{F}(\tilde{f}) = (X \times \overline{\mathbb{C}}) \setminus \tilde{J}(\tilde{f}).$$

$$C(\tilde{f}) = \{(x, y) \in X \times \overline{\mathbb{C}} \mid q_x'(y) = 0\},\ P(\tilde{f}) = \overline{\bigcup_{n \in \mathbb{N}} \tilde{f}^n(C(\tilde{f}))}.$$

$C(\tilde{f})$ is called the critical set for $\tilde{f}$ and $P(\tilde{f})$ is called the post critical set for $\tilde{f}$. Moreover we set

$$(\tilde{f}^n)'((x, y)) = (q_x^{(n)})'(y).$$

If (x, y) is a period point of $\tilde{f}$ with the period n, then we say that (x, y) is repelling(resp. indifferent, attracting, etc.) if $|(\tilde{f}^n)'((x, y))| > 1$(resp.$=$ 1, < 1, etc.).

Lemma 4.3. *Let $\tilde{f}: X\times\overline{\mathbb{C}} \to X\times\overline{\mathbb{C}}$ be a rational skew product represented by $\tilde{f}((x,y)) = (p(x),\ q_x(y))$. Then the following hold.*

1. *if $x \in X$, then $q_x^{-1}(F_{p(x)}) = F_x$, $q_x^{-1}(J_{p(x)}) = J_x$, $\tilde{f}(\tilde{J}(\tilde{f})) \subset \tilde{J}(\tilde{f})$.*

2. *if $p: X \to X$ is surjective, then $\tilde{f}: X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is surjective.*

3. *if $p : X \to X$ is a surjective and open map, then $\tilde{f}^{-1}(\tilde{J}(\tilde{f})) = \tilde{f}(\tilde{J}(\tilde{f})) = \tilde{J}(\tilde{f})$.*

Now we need some notations from [J2], concerning potential theoritic aspects. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product represented by $\tilde{f}((x,y)) = (p(x),\ q_x(y))$. Let ω be the spherical probability measure on $\overline{\mathbb{C}}$. Let $\omega_x = (i_x)_*\omega$ for each $x \in X$ where we denote by $i_x : \overline{\mathbb{C}} \to \pi_X^{-1}(\{x\})$ the natural isomorphism. For each continuous function φ on $\pi_X^{-1}(\{x\})$ let $(\tilde{f}_x^n)^*\varphi$ be the continuous function on $\pi_X^{-1}(\{p^n(x)\})$ defined by $((\tilde{f}_x^n)^*\varphi)(z) = \sum_{\tilde{f}_x^n(w)=z} \varphi(w)$ for each $n \in \mathbb{N}$. Let $\mu_{x,n}$ be the probability measure on $\pi_X^{-1}(\{x\})$ defined by $\langle \mu_{x,n},\ \varphi\rangle = \frac{1}{\prod_{j=0}^{n-1} d(p^j(x))}\langle \omega_{p^n(x)},\ (\tilde{f}_x^n)^*\varphi\rangle$. For each $x \in X$, we denote by $R_x : \mathbb{C}^2\setminus\{0\} \to \mathbb{C}^2\setminus\{0\}$ the homogenious polynomial mapping of degree $d(x)$ such that $q_x \circ \pi' = \pi' \circ R_x$ where $\pi' : \mathbb{C}^2\setminus\{0\} \to \overline{\mathbb{C}}$ is the natural projection and $\sup\{|R_x(z,w)| \mid |(z,w)| = 1\} = 1$. R_x is determined uniquely up to multiplication by a complex number of units. We can assume $x \mapsto R_x$ is continuous. For each $x \in X$ and $n \in \mathbb{N}$ let $G_{x,n} := \frac{1}{\prod_{j=0}^{n-1} d(p^j(x))}\log|R_x^n|$ where $R_x^n := R_{p^{n-1}(x)} \circ \cdots \circ R_x$. Then the following results hold.

Proposition 4.4. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product represented by $\tilde{f}((x,y)) = (p(x),\ q_x(y))$ and assume $d(x) \geq 2$ for each $x \in X$. Then we have the following.*

1. *$\mu_{x,n}$ converges to a probability measure μ_x on $\pi_X^{-1}(\{x\})$ weakly as $n \to \infty$ for each $x \in X$.*

2. *$G_{x,n}$ converges to a continuous plurisubharmonic function G_x locally uniformly on $\mathbb{C}^2 \setminus \{0\}$ as $n \to \infty$ for each $x \in X$.*

3. *$\mu_x = (i_x^{-1})_*(dd^c(G_x \circ s))$ where s is a local section of $\pi' : \mathbb{C}^2\setminus\{0\} \to \overline{\mathbb{C}}$. Further $G_x(z,w) \leq \log|(z,w)| + O(1)$ as $|(z,w)| \to \infty$ and $G_x(\lambda z, \lambda w) = G_x(z,w) + \log\lambda$ for each $\lambda \in \mathbb{C}$, for each $x \in X$.*

4. *$G_{p(x)} \circ R_x = d(x) \cdot G_x$ for each $x \in X$.*

5. *if $x \to x'$ then $G_x \to G_{x'}$ uniformly on $\mathbb{C}^2 \setminus \{0\}$.*

6. *$(\tilde{f}_x)_*\mu_x = \mu_{p(x)}$, $(\tilde{f}_x)^*\mu_{p(x)} = d(p(x)) \cdot \mu_x$ for each $x \in X$.*

7. *μ_x puts no mass on polar subsets of $\pi_X^{-1}(\{x\})$ for each $x \in X$.*

8. *$x \mapsto \mu_x$ is continuous with respct to the weak topology of measures in $X \times \overline{\mathbb{C}}$.*

9. *$supp(\mu_x) = \tilde{J}_x$ for each $x \in X$.*

10. *$\tilde{J}_x$ has no isolated points for each $x \in X$.*

11. *$x \mapsto \tilde{J}_x$ is lower semicontinuous with respect to the Hausdorff metric in the space of compact subsets of $X \times \overline{\mathbb{C}}$.*

Proof. Since $d(x) \geq 2$ for each $x \in X$, we can show the statements in the same way as that in section 3 in [J2]. □

Definition 4.5. Let G be a rational semigroup generated by $\{f_\lambda\}_{\lambda \in \Lambda}$. Let $X = \Lambda^{\mathbf{N}}$. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be the map defined by:

$$\tilde{f}((x, y)) = (p(x),\ f_{x_1}(y)),$$

where $p : X \to X$ is the shift map and $x \in X$ is represented by: $x = (x_1, x_2, \dots)$. Then we say that $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ is the rational skew product constructed by the generator system $\{f_\lambda\}_{\lambda \in \Lambda}$.

Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the rational skew product constructed by the generator system $\{f_1, \dots, f_m\}$, where $\Sigma_m = \{1, \dots, m\}^{\mathbf{N}}$. Then $\tilde{f}$ is a finite-to-one and open map. We have that a point $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ satisfies $f'_{w_1}(x) \neq 0$ if and only if $\tilde{f}$ is a homeomorphism in a small neighborhood of (w, x). Moreover the following proposition holds.

Proposition 4.6. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the rational skew product defined by $\tilde{f}((w, x)) = (\sigma(w), f_{w_1}(x))$. Then the following hold.*

1. *$\tilde{F}$ and $\tilde{J}$ are completely invariant under $\tilde{f}$. $\tilde{F}$ is open and $\tilde{J}$ is compact. $\tilde{f}(\tilde{J}_w) = \tilde{J}_{\sigma w}$. $\tilde{F}(\tilde{f})$ is equal to the set of all the points $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$ which satisfies that there exists an open neighborhood U of x and an open neighborhood V of w such that for each $a \in V$ the family of maps $\{f_{a_n} \circ \cdots \circ f_{a_1}\}$ is normal in U.*

2. *$\tilde{J} = \cap_{n=0}^{\infty} \tilde{f}^{-n}(\Sigma_m \times J(G))$. $\pi_2(\tilde{J}) = J(G)$, where we denote by $\pi_2 : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the second projection.*

3. *$\tilde{J}$ has no interior points or is equal to $\Sigma_m \times \overline{\mathbb{C}}$.*

4. *If $\sharp(J(G)) \geq 3$, then $\tilde{J}$ is a perfect set.*

5. *If $\sharp(J(G)) \geq 3$, then $\tilde{J}$ is equal to the closure of the set of all repelling period points of $\tilde{f}$.*

6. *Assume* $\sharp(J(G)) \geq 3$ *and* $E(G) \subset F(G)$. *Let* K *be a compact subset of* $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$. *If* U *is an open set in* $\Sigma_m \times \overline{\mathbb{C}}$ *satisfying* $U \cap \tilde{J} \neq \emptyset$, *then there exists a positive integer* N *such that for each integer* n *with* $n \geq N$, *we have* $\tilde{f}^n(U) \supset K$.

Definition 4.7 (hyperbolicity). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ is hyperbolic along fibers if $P(\tilde{f}) \subset \tilde{F}(\tilde{f})$.

Definition 4.8. Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ is expanding along fibers if there exists a positive constant C and a constant λ with $\lambda > 1$ such that for each $n \in \mathbb{N}$,

$$\inf_{z \in \tilde{J}(\tilde{f})} \|(\tilde{f}^n)'(z)\| \geq C\lambda^n,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric.

Definition 4.9 (semi-hyperbolicity). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. Let N be a positive integer. We say that a point $(x_0, y_0) \in X \times \overline{\mathbb{C}}$ belongs to $SH_N(\tilde{f})$ if there exists a neighborhood U of x_0 and a positive number δ satisfying that for any $x \in U$, any $n \in \mathbb{N}$, any element $x_n \in p^{-n}(x)$ and any element V of $c(B(y_0,\ \delta),\ q_{x_n}^{(n)})$,

$$\deg(q_{x_n}^{(n)} : V \to B(y_0,\ \delta)) \leq N.$$

We set

$$UH(\tilde{f}) = (X \times \overline{\mathbb{C}}) \setminus \cup_{N \in \mathbb{N}} SH_N(\tilde{f}).$$

We say that $\tilde{f}$ is semi-hyperbolic along fibers if for any $(x_0, y_0) \in \tilde{J}(\tilde{f})$ there exists a positive integer N such that $(x_0, y_0) \in SH_N(\tilde{f})$.

Lemma 4.10. *Let* $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ *be a rational skew product. If* $\tilde{f}$ *is hyperbolic along fibers, then it is semi-hyperbolic along fibers.*

Lemma 4.11. *Let* $G = \langle f_1, f_2, \ldots, f_m \rangle$ *be a finitely generated rational semigroup. Then* G *is semi-hyperbolic if and only if the rational skew product* $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ *constructed by the generator system* $\{f_1, f_2, \ldots, f_m\}$ *is semi-hyperbolic along fibers.* G *is hyperbolic if and only if* $\tilde{f}$ *is hyperbolic along fibers.*

Definition 4.12 (Condition(C1)). Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. We say that $\tilde{f}$ satisfies the condition (C1) if there exists a family $\{D_x\}_{x \in X}$ of discs in $\overline{\mathbb{C}}$ such that the following three conditions are satisfied:

1. $\overline{\bigcup_{n \geq 0} \tilde{f}^n(\{x\} \times D_x)} \subset \tilde{F}(\tilde{f})$.

2. for any $x \in X$, we have that $\mathrm{diam}(q_x^{(n)}(D_x)) \to 0$, as $n \to \infty$.

3. $\inf_{x\in X}$ diam $(D_x) > 0$.

Now we will show the following lemma and theorem.

Lemma 4.13. *Let $\tilde{f}: X\times\overline{\mathbb{C}} \to X\times\overline{\mathbb{C}}$ be a rational skew product satisfying the condition (C1). Assume that there exists a point $(x_0, y_0) \in X\times\overline{\mathbb{C}}$ with $y_0 \in F_{x_0}$, a connected open neighborhood U of y_0 in $\overline{\mathbb{C}}$ and a sequence (n_j) of positive integers such that $R_j := q_{x_0}^{(n_j)}$ converges to a non-constant map ϕ uniformly on U as $j\to\infty$. Let $(x_j, y_j) = \tilde{f}^{n_j}(x_0, y_0)$ and $(x_\infty, y_\infty) = \lim_{j\to\infty}(x_j, y_j)$. Let $S_{i,j} = q_{x_i}^{(n_j - n_i)}$ for $1\le i<j$. Let*

$$V = \{y\in\overline{\mathbb{C}} \mid \exists\epsilon>0,\ \lim_{i\to\infty}\sup_{j>i}\sup_{d(\xi,y)\le\epsilon} d(S_{i,j}(\xi),\ \xi) = 0\}.$$

Then V is a non-empty open set and for any $y\in\partial V$, we have that

$$(x_\infty,\ y)\in\tilde{J}(\tilde{f})\cap UH(\tilde{f}). \qquad (1)$$

Theorem 4.14. *Let $\tilde{f}: X\times\overline{\mathbb{C}}\to X\times\overline{\mathbb{C}}$ be a rational skew product. Assume $\tilde{f}$ is semi-hyperbolic along fibers and satisfies the condition (C1). Then the following hold.*

1. *Let $(x_0, y_0)\in X\times\overline{\mathbb{C}}$ be any point with $y_0\in F_{x_0}$. Then for any open connected neighborhood U of y_0 in $\overline{\mathbb{C}}$, there exists no subsequence of $(q_{x_0}^{(n)})_n$ converging to a non-constant map locally uniformly on U.*

2. $$\tilde{J}(\tilde{f}) = \bigcup_{x\in X}\tilde{J}_x.$$

3. *If there exists a disc D in $\overline{\mathbb{C}}$ such that $D_x = D$ for all $x\in X$ in the condition (C1), then there exist positive constants δ, L and $\lambda (0<\lambda<1)$ such that for any $n\in\mathbb{N}$,*

   $$\sup\{\textit{diam}\ U \mid U\in c(B(y,\delta),\ q_{x_n}^{(n)}),\ (x,y)\in\tilde{J}(\tilde{f}),\ x_n\in p^{-n}(x)\}\le L\lambda^n.$$

4. *Assume $d(x)\ge 2$ for each $x\in X$. Then we have that $x\mapsto\tilde{J}_x$ is continuous with respect to the Hausdorff metric in the space of compact subsets of $X\times\overline{\mathbb{C}}$.*

5. *Assume $d(x)\ge 2$ for each $x\in X$. Then for any compact subset K of $\tilde{F}(\tilde{f})$, we have that $\overline{\cup_{n\ge0}\tilde{f}^n(K)}\subset\tilde{F}(\tilde{f})$ and there exist constants $C>0$ and $\tau<1$ such that for each n, $\sup_{z\in K}\|(\tilde{f}^n)'(z)\|\le C\tau^n$.*

To show Lemma 4.13 and Theorem 4.14, we need the following lemma.

Lemma 4.15. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product satisfying the condition (C1). Assume $(x_0, y_0) \in SH_N(\tilde{f})$ for some $N \in \mathbb{N}$. Then there exists a positive number δ_0 such that for each δ with $0 < \delta < \delta_0$ there exists a neighborhood U of x_0 in $\overline{\mathbb{C}}$ satisfying that for each $n \in \mathbb{N}$, each $x \in U$ and each $x_n \in p^{-n}(x)$, we have that each element of $c(B(y_0, \delta),\ q_{x_n}^{(n)})$ is simply connected.*

Proof. By Lemma 3.8 and condition (C1). □

Proof. (outline) of Lemma 4.13 and Theorem 4.14. We will show Lemma 4.13. We will develop a method in [J1]. Since ϕ is non-constant, we have that V is non-empty. Take any point $y \in \partial V$. By condition (C1), we can show that $(x_\infty, y) \in \tilde{J}(\tilde{f})$. Suppose $(x_\infty, y) \in SH_N(\tilde{f})$ for some N. Then by Corollary 3.7 and Lemma 4.15 we can show that $\{S_{i,j}\}_{j>i}$ is normal in a neighborhood of y. But it implies that $y \in V$ and this is a contradiction. Hence Lemma 4.13 holds. By this lemma the statement 1 of Theorem 4.14 holds. The statement 2 of Theorem 4.14 follows from Lemma 4.13 and the condition (C1). The statement 3 of Theorem 4.14 follows from the statement 2 and some arguments on moduli of annuli. The statement 4 follows from the statement 2 of Theorem 4.14 and 11 in Proposition 4.4. The statement 5 follows from Lemma 3.8, 11 in Proposition 4.4 and the statement 1 of Theorem 4.14. □

Corollary 4.16. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume G contains an element with the degree at least two and each element of $Aut\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic. Also assume $F(G) \neq \emptyset$. Then there exists a $\delta > 0$, a constant L with $L > 0$ and a constant λ with $0 < \lambda < 1$ such that*

$$\sup\{\mathit{diam}\ U \mid U \in c(B(x,\delta),\ f_{i_n} \cdots f_{i_1}),\ x \in J(G),\ (i_1, \dots, i_n) \in \{1, \dots, m\}^n\}$$

$$\leq L\lambda^n, \quad \textit{for each } n.$$

Theorem 4.17. *Let $\tilde{f} : X \times \overline{\mathbb{C}} \to X \times \overline{\mathbb{C}}$ be a rational skew product. Assume $\tilde{f}$ is hyperbolic along fibers and satisfies the condition (C1) with a family of discs $(D_x)_{x \in X}$ such that there exists a disc D satisfying $D_x = D$ for all $x \in X$. Then $\tilde{f}$ is expanding along fibers.*

Remark 5. We can show that the results in this section are generalized to the version of $\overline{\mathbb{C}}$-fibration. For the definition of $\overline{\mathbb{C}}$-fibration, see [J2].

5 Conditions to be semi-hyperbolic

Theorem 5.1. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $z_0 \in J(G)$ be a point. Assume all of the following conditions:*

1. *there exists a neighborhood U_1 of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U_1$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V.*

2. *there exists a neighborhood U_2 of z_0 in $\overline{\mathbb{C}}$ and a positive real number $\tilde{\epsilon}$ such that if we set*

$$T = \{c \in \overline{\mathbb{C}} \mid \exists j,\ f_j'(c) = 0,\ (G \cup \{id\})(f_j(c)) \cap U_2 \neq \emptyset\}$$

then for each $c \in T \cap C(f_j)$, we have $d(c,\ (G \cup \{id\})(f_j(c))) > \tilde{\epsilon}$.

3. *$F(G) \neq \emptyset$.*

Then $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$.

Notation: For any family $\{g_\lambda\}_{\lambda \in \Lambda}$ of rational functions, we denote by $F(\{g_\lambda\})$ the set of all points $z \in \overline{\mathbb{C}}$ such that z has a neighborhood where the family $\{g_\lambda\}$ is normal. We set $J(\{g_\lambda\}) = \overline{\mathbb{C}} \setminus F(\{g_\lambda\})$. $F(\{g_\lambda\})$ is called the Fatou set and $J(\{g_\lambda\})$ is called the Julia set for the family.

Corollary 5.2. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Let $z_0 \in J(G)$ be a point. Assume all of the following conditions:*

1. *there exists a neighborhood U_1 of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$ consisting of mutually distinct elements and any domain V in $F((g_n))$, there exists a point $x \in V$ such that the sequence $\overline{\cup_n \{g_n(x)\}} \cap \overline{\mathbb{C}} \setminus U_1 \neq \emptyset$.*

2. *there exists a neighborhood U_2 of z_0 in $\overline{\mathbb{C}}$ and a positive real number $\tilde{\epsilon}$ such that if we set*

$$T = \{c \in \overline{\mathbb{C}} \mid \exists j,\ f_j'(c) = 0,\ (G \cup \{id\})(f_j(c)) \cap U_2 \neq \emptyset\}$$

then for each $c \in T \cap C(f_j)$, we have $d(c,\ (G \cup \{id\})(f_j(c))) > \tilde{\epsilon}$.

3. *$F(G) \neq \emptyset$.*

Then $z_0 \in SH_N(G)$ for some $N \in \mathbb{N}$ and there exists a neighborhood W of z_0 in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$ consisting of mutually distinct elements, we have

$$\sup\{\ diam\ S \mid S \in c(W, g_n)\} \to 0,\ \ as\ n \to \infty.$$

We will consider the proof of Theorem 5.1. We may assume $U_1 = U_2 = U$ for some small disc U . By condition 1 and 3, we may assume $\infty \in F(G)$ and $g^{-1}(U) \subset \mathbb{C}$ for each $g \in G$. Now we will show the above theorem by developing a lemma in [Ma4] and using the methods in [KS]. The stories are almost same as those in [KS], except some modifications.

First we need some new notations. An "square" is a set S of the form

$$S = \{z \in \mathbb{C} \mid |\Re(z-p)| < \delta, \ |\Im(z-p)| < \delta\}.$$

The point p is called the center of S and δ its radius . For each $k > 0$, given a square S with center p and radius δ, we denote by S^k the square with the center p and radius $k\delta$. Take a $\sigma > 0$ such that U contains a closed square Q' with the center a point in U and its radius 2σ. Let $Q'' = (Q')^{1/2}$. Q'' is called "admissable square at level 1." We will define asmissable squares at level n for each $n \in \mathbb{N}$. Let Q be an admissable square at level n with the radius a. Then Q is covered by 16 squares with the radius $a/8$. We have 20 squares with the radius $a/8$ adjacent to Q. We call all these 36 squares admissable at level $n+1$. These squares are denoted by $\{Q_{\mu,n+1}\}$. The union of these 36 squares is denoted by $\tilde{Q}$, which is called the "square attached to Q." Each admissable and each attached square is a relative compact subset of U.

Notation: For any open set V_1 and for any rational map g, if $V_2 \in c(V_1, \ g)$ then we set $\Delta(V_1, \ g) = \sharp\{x \in V_1 \mid g'(x) = 0\}$, counting the multiplicity.

We need some lemmas to show Theorem 5.1.

Lemma 5.3. *For given $\epsilon > 0$ and $N \in \mathbb{N}$, there exists some $n_0 \in \mathbb{N}$ such that the following holds: If Q is an admissable square at some level $n \geq n_0$, $\tilde{Q}$ the corresponding attached square, V an element of $c(\tilde{Q}, \ f)$ for some $f \in G$, and $\Delta(V, f) \leq N$, then diam $(K) \leq \epsilon$ for each element $K \in c(Q, f)$ contained in V.*

Now, let $t = \sharp T$, $N = (\max_{j=1,\dots,m} \deg(f_j))^t$ and $\epsilon < \frac{\tilde{\epsilon}}{36N}$. We can assume that $\tilde{\epsilon}$ is sufficiently small and diam $U \leq \tilde{\epsilon}$. Let $n_0 \in \mathbb{N}$ be an integer in Lemma 5.3 for these ϵ and N.

Lemma 5.4. *Let G be an element of the form $f = f_{w_1} \circ \cdots \circ f_{w_k}$. Let B be a simply connected subdomain of U, $B' \in c(B, f)$ an element such that $\Delta(B', f) > N$. Then there exists some $\nu \in \{0, \dots, k-1\}$ such that if we set $B_\nu = f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(B')$, then B_ν is simply connected, diam $(B_\nu) \geq \tilde{\epsilon}$, and*

$$\deg(f_{w_1} \circ \cdots f_{w_{k-\nu-1}} | B_\nu : B_\nu \to B) \leq N.$$

Now we will show the Theorem 5.1.

Proof. Take $\tilde{\epsilon}$, ϵ and N as before. Take n_0 in Lemma 5.3 for ϵ and N. Let k be the smallest integer such that there exists some admissable square $Q = Q_{\mu,n}$ at level $n \geq n_0$ with diam $(K) > \epsilon$ for some element K of $c(Q, \ f_{w_1} \circ \cdots \circ f_{w_k})$ where $(w_1, \dots, w_k)$ is some word of length k. We have $k \geq 1$. Let $\tilde{Q}$ be the square attached to Q. By lemma 5.3, there exists an element $S \in c(\tilde{Q},\ f_{w_1} \circ \cdots \circ f_{w_k})$ such that $\Delta(S,\ f_{w_1} \circ \cdots \circ f_{w_k}) > N$. Take a integer ν with $1 \leq \nu < k$ in Lemma 5.4. Then we have

$$\text{diam } (f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(S)) > \tilde{\epsilon}.$$

If we set $\tilde{S} = f_{w_{k-\nu}} \circ \cdots \circ f_{w_k}(S)$ then

$$\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{\tilde{S}}) \leq N$$

and

$$\tilde{S} \subset \bigcup_{\mu} (f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}})^{-1}(Q_{\mu,\ n+1}).$$

By the minimality of k, we have that the diameter of each element of $c(Q_{\mu,\ n+1},\ f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}})$ is less than ϵ. Since $\deg(f_{w_1} \circ \cdots \circ f_{w_{k-\nu-1}}|_{\tilde{S}}) \leq N$, we have that

$$\tilde{\epsilon} < \text{ diam } \tilde{S} \leq 36N\epsilon.$$

This contradicts to $\epsilon < \frac{\tilde{\epsilon}}{36N}$. Hence we have proved that for each admissable square $Q_{\mu,n}$ with $n \geq n_0$ and each $g \in G$, each element $K \in c(Q_{\mu,n},\ g)$ satisfies that diam $(K) < \epsilon$. Since ϵ is sufficiently small, K is simply connected. By Lemma 5.4, we have that

$$\deg(f|_K : K \to Q_{\mu,n}) \leq N + 1.$$

Hence $z_0 \in SH_{N+1}$. □

By Theorem 4.14 and Theorem 5.1, we get the following result.

Theorem 5.5 (criterion to be semi-hyperbolic). *Let $G = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic and that $F(G) \neq \emptyset$. Then G is semi-hyperbolic if and only if all of the following conditions are satisfied.*

1. *for each $z \in J(G)$ there exists a neighborhood U of z in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V*

2. *for each $j = 1, \ldots, m$ each $c \in C(f_j) \cap J(G)$ satisfies*

$$d(c,\ (G \cup \{id\})(f_j(c))) > 0$$

Theorem 5.6 (expandingness of sub-hyperbolicity). *Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated sub-hyperbolic rational semigroup. Assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic and that there is no super* *attracting fixed point of any element of* G *in* $J(G)$. *Then there exists a Riemannian metric* ρ *on a neighborhood* V *of* $J(G)\setminus P(G)$ *such that for each* $z_0 \in J(G)\setminus G^{-1}(P(G)\cap J(G))$, *if there exists a word* $w=(w_1,w_2,\dots,)\in \{1,\dots,m\}^{\mathbb{N}}$ *satisfying* $(f_{w_n}\cdots f_{w_1})(z_0)\in J(G)$ *for each* n, *then*

$$\|(f_{w_n}\cdots f_{w_1})'(z_0)\|\to\infty,\ \text{as } n\to\infty,$$

where $\|\cdot\|$ *is the norm of the derivative measured from* ρ *on* V *to it.*

Now we will give an application of Theorem 5.5.

Theorem 5.7. *Let* $G=\langle f_1,f_2,\dots,f_m\rangle$ *be a finitely generated sub-hyperbolic rational semigroup. Assume that there exists an element of* G *with the degree at least two, that each element of* $Aut\,\overline{\mathbb{C}}\cap G$*(if this is not empty) is loxodromic and that there is no super attracting fixed point of any element of* G *in* $J(G)$. *Then* G *is semi-hyperbolic.*

Proof. We will appeal to Theorem 5.5. Since there is no super attracting fixed point of any element of G in $J(G)$, the condition 2. in Theorem 5.5 is satisfied. By Theorem 3.12, there exists an attractor K in $F(G)$ for G. Let z_0 be any point and U a neighborhood of z_0 such that $\overline{U}\cap K=\emptyset$. Suppose that there exists a sequence $(g_n)\subset G$, a domain V in $\overline{\mathbb{C}}$ and a point $\zeta\in U$ such that $g_n\to\zeta$ as $n\to\infty$ locally uniformly on V. We will deduce a contradiction. We can assume that there exists a word $w\in\{1,\dots,m\}^{\mathbb{N}}$ such that for each n,

$$g_n=\alpha_n f_{w_n}\circ\cdots\circ f_{w_1},$$

where $\alpha_n\in G$ is an element. Then from Theorem 3.12 and that $\overline{U}\cap K=\emptyset$, we have that

$$f_{w_n}\circ\cdots\circ f_{w_1}(V)\subset J(G), \tag{2}$$

for each n. Hence $(f_{w_n}\circ\cdots\circ f_{w_1})_n$ is normal in V. Let $z_1\in V\cap G^{-1}(P(G)\cap J(G))$ be a point. By the backward self-similarity of $J(G)$ and Lemma 3.11, there exists a sequence (n_j) of positive integers and a neighborhood W of $P(G)\cap J(G)$ in $\overline{\mathbb{C}}$ such that for each j,

$$f_{w_{n_j}}\circ\cdots\circ f_{w_1}(z_1)\in\overline{\mathbb{C}}\setminus W.$$

By Theorem 5.6, we have that

$$\|(f_{w_{n_j}}\circ\cdots\circ f_{w_1})'(z_1)\|\to\infty,\quad \text{as } j\to\infty, \tag{3}$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric. Since $(f_{w_n}\circ\cdots\circ f_{w_1})_n$ is normal in V, this is a contradiction. Hence the condition 1 in Theorem 5.5 is satisfied. By Theorem 5.5, we get that G is semi-hyperbolic. □

6 δ-subconformal measures and Hausdorff dimension of the Julia sets

Definition 6.1. Let G be a rational semigroup and δ a non-negative number. We say that a Borel probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each Borel measurable set A

$$\mu(g(A)) \leq \int_A \|g'(z)\|^{\delta} d\mu,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric. For each $x \in \overline{\mathbb{C}}$ and each real number s we set

$$S(s,\ x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,\ x) < \infty\}.$$

If there is not s such that $S(s,\ x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ s(G) = \inf\{\delta \mid \exists \mu : \delta\text{-subconformal measure}\}$$

Theorem 6.2 (S2). *Let G be a rational semigroup which has at most countably many elements. If there exists a point $x \in \overline{\mathbb{C}}$ such that $S(x) < \infty$ then there is a $S(x)$-subconformal measure. In particular, we have $s(G) \leq s_0(G)$.*

Definition 6.3. Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. We say that G satisfies the *open set condition* with respect to the generators $f_1, f_2, \dots, f_m$ if there exists an open set O such that for each $j = 1, \dots, m$, $f_j^{-1}(O) \subset O$ and $\{f_j^{-1}(O)\}_{j=1,\dots,m}$ are mutually disjoint.

Proposition 6.4. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \dots, f_m$ and $O \setminus J(G) \neq \emptyset$ where O is an open set in the definition of the open set condition. If there exists an attractor in $F(G)$ for G, then*

$$s_0(G) \leq 2.$$

Lemma 6.5. *Let G be a rational semigroup. Assume that $\infty \in F(G)$, $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. We also assume that there exist a countable set E in $\overline{\mathbb{C}}$, positive numbers a_1 and a_2 and a constant c with $0 < c < 1$ such that for each $x \in J(G) \setminus E$, there exist two sequences (r_n) and (R_n) of positive real numbers and a sequence (g_n) of elements of G satisfying all of the following conditions:*

1. *$r_n \to 0$ and for each n, $0 < \frac{r_n}{R_n} < c$ and $g_n(x) \in J(G)$.*
2. *for each n, $g_n(D(x, R_n)) \subset D(g_n(x), a_1)$.*
3. *for each n $g_n(D(x, r_n)) \supset D(g_n(x), a_2)$.*

Let δ be a real number with $\delta \geq s(G)$ and μ a δ- subconformal measure. Then δ-Hausdorff measure on $J(G)$ is absolutely continuous with respect to μ such that the Radon-Nikodim derivative is bounded from above. In particular, we have

$$dim_H(J(G)) \leq s(G).$$

By Theorem 4.14 and Lemma 6.5, we get the following result.

Theorem 6.6. *Let G be a rational semigroup generated by a generator system $\{f_\lambda\}_{\lambda \in \Lambda}$ such that $\cup_{\lambda \in \Lambda}\{f_\lambda\}$ is a compact subset of $End(\overline{\mathbb{C}})$. Let $\tilde{f}$ be a rational skew product constructed by the generator system. Assume $\tilde{f}$ is semi-hyperbolic along fibers and satisfies the condition C1 with a family of discs $\{D_x\}_{x \in X}$ such that $D_x = D$, $\forall x \in X$ with some D. Then we have*

$$\dim_H(J(G)) \leq s(G).$$

Theorem 6.7. *Let $G = \langle f_1, f_2, \cdots, f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic. Assume that G contains an element with the degree at least two, each element of $Aut\,\overline{\mathbb{C}} \cap G$ (if this is not empty) is loxodromic and $F(G) \neq \emptyset$. Then we have*

$$\dim_H(J(G)) \leq s(G) \leq s_0(G).$$

Proof. By Theorem 6.6 and Theorem 6.2. □

Remark 6. Let $G = \langle f_1, f_2, \cdots, f_m \rangle$ be a finitely generated hyperbolic rational semigroup which satisfies that $\{f_j^{-1}(J(G))\}_{j=1,\ldots m}$ are mutually disjoint . We assume that the degree of f_1 is at least two. By the results in Theorem 3.2 and the proof, Theorem 3.4 and Corollary 3.5 in [S2], we have

$$0 < \dim_H J(G) = s(G) = s_0(G) = \delta(G) < 2,$$

where we denote by $\delta(G)$ the infimum of δ which allows us the δ-conformal measure on $J(G)$.

Example 6.8. Let n be a positive integer such that $n \geq 4$. We set $G = \langle z^n,\ n(z-4)+4 \rangle$. Then G is a finitely generated hyperbolic rational semigroup. By Theorem 6.7 and some arguments, we get

$$1 \leq \dim_H J(G) \leq \frac{\log(n+1)}{\log(n)}.$$

Example 6.9. Let $G = \langle f_1, f_2 \rangle$ where $f_1(z) = z^2 + 2$, $f_2(z) = z^2 - 2$. Since $P(G) \cap J(G) = \{2, -2\}$ and $P(G) \cap F(G)$ is compact, we have G is sub-hyperbolic. By Theorem 5.7, G is also semi-hyperbolic. Since $f_j^{-1}(D(0,2)) \subset D(0,2)$ for $j = 1, 2$ and$f_1^{-1}(D(0,2)) \cap f_2^{-1}(D(0,2)) = \emptyset$, G satisfies the open set condition. Also $J(G)$ is included in $B = \cup_{j=1}^{2} f_j^{-1}(\overline{D(0,2)})$. Since $B \cap \partial D(0,2) = \{2, -2, 2i, -2i\}$, we get $\sharp(J(G) \cap \partial D(0,2)) < \infty$. By Theorem 1.15 in [S6], we have $m_2(J(G)) = 0$, where we denote by m_2 the 2-dimensional Lebesgue measure. By Theorem 6.7 and Proposition 6.4, we have also

$$dim_H(J(G)) \leq s(G) \leq s_0(G) \leq 2.$$

7 backward self-similar measure

In the following sections we assume the following situation. Let m be a positive integer and $\Sigma_m = \{1, \dots, m\}^{\mathbf{N}}$. We denote by $\sigma : \Sigma_m \to \Sigma_m$ the shift map, that is , $(w_1, \dots) \mapsto (w_2, \dots)$. Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the rational skew product constructed by the generator system $\{f_1, \dots, f_m\}$. Hence for each $(w, x) \in \Sigma_m \times \overline{\mathbb{C}}$, $\tilde{f}((w, x)) = (\sigma w, f_{w_1} x)$. We now consider about invariant measures and self-similar measures on Julia sets. In the cases of iterations of rational functions, Brolin's and Lyubich's studies are well known([Br], [L]). Recently, D.Boyd investigated "invariant measure" (that is, the measure $(\pi_2)_* \tilde{\mu}$ in the notation in Theorem 7.1) in the case that each f_j is of degree at least two and have shown the uniqueness in [Bo].

Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. We set $d_j = \deg(f_j)$ for each $j = 1, \dots, m$ and $d = \sum_{j=1}^{m} d_j$. For each compact set K of $\overline{\mathbb{C}}$ we denote by $C(K)$ all continuous complex valued functions on K. It is a Banach space with supremum norm on K. Assume that K is backward invariant under G. For each j and for each element φ we set

$$(A_j \varphi)(z) = \frac{1}{d_j} \sum_{\zeta \in f_j^{-1}(z)} \varphi(\zeta),$$

where z is any point of K. Then $A_j \varphi$ is an element of $C(K)$ and A_j is a bounded operator on $C(K)$. We set

$$\mathcal{W} = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_j a_j = 1,\ a_j \geq 0\}.$$

And for each $a \in \mathcal{W}$ we set

$$(B_a \varphi)(z) = \sum_{j=1}^{n} a_j (A_j \varphi)(z).$$

Then B_a is a bounded operator on $C(K)$.

Similarly, let $\tilde{K}$ be a compact subset of $\Sigma_m \times \overline{\mathbb{C}}$ which is backward invariant under $\tilde{f}$. We define an operator $\tilde{B}_a$ on $C(\tilde{K})$ as follows. For each element $\tilde{\varphi} \in C(K)$ we set

$$(\tilde{B}_a\tilde{\varphi})(z) = \Sigma_{\zeta \in \tilde{f}^{-1}(z)} \tilde{\varphi}(\zeta)\tilde{\psi}_a(\zeta)$$

where $\tilde{\psi}_a(\zeta) = \frac{a_{w_1}}{d_{w_1}}$ if $\pi_1(\zeta) = (w_1, w_2, \dots)$.

$\tilde{B}_a$ is a bounded operator on $C(\tilde{K})$. Furtheremore, if $\pi_2(\tilde{K}) = K$, then we get

$$\pi_2^* B_a = \tilde{B}_a \pi_2^*$$

and $\pi_2^*; C(K) \to C(\tilde{K})$ is an isometry.

Theorem 7.1. *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup. Assume that there exists an element $g_0 \in G$ of degree at least two, the exceptional set $E(G)$ for G is included in $F(G)$ and $F(H) \supset J(G)$ where H is a rational semigroup defined by $H = \{h^{-1} \mid h \in Aut\ (\overline{\mathbb{C}}) \cap G\}$.(if H is empty, put $F(H) = \overline{\mathbb{C}}$.) Then all of the following hold.*

1. *For each $a \in \mathcal{W}$ with $a \neq 0$ there exists a unique regular Borel probability measure $\tilde{\mu}_a$ on $\Sigma_m \times \overline{\mathbb{C}}$ such that for each compact set $\tilde{K}$ which is included in $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ and backward invariant under $\tilde{f}$,*

   $$\|\tilde{B}_a^n(\tilde{\varphi}) - \tilde{\mu}_a(\varphi)\mathbf{1}\|_{\tilde{K}} \to 0,$$

   as $n \to \infty$, for each $\tilde{\varphi} \in C(\tilde{K})$, where we denote by $\mathbf{1}$ the constant function taking its value 1. Similarly, there exists a unique regular Borel probability measure μ_a on $\overline{\mathbb{C}}$ such that for each compact set K which is included in $\overline{\mathbb{C}} \setminus E(G)$ and backward invariant under G,

   $$\|B_a^n(\varphi) - \mu_a(\varphi)\mathbf{1}\|_K \to 0,$$

   as $n \to \infty$, for each $\varphi \in C(K)$.

   Moreover, $(\pi_2)_*(\tilde{\mu}_a) = \mu_a$. The support of $\tilde{\mu}_a$ is equal to $\tilde{J}$ and the support of μ_a is equal to $J(G)$.

2. *For each $a \in \mathcal{W}$ with $a \neq 0$, we have that $\tilde{\mu}_a$ is $\tilde{f}$-invariant and $(\tilde{f}, \tilde{\mu}_a)$ is exact.*

3. *For each $a \in \mathcal{W}$ with $a \neq 0$, we have that $(\pi_1)_*\tilde{\mu}_a$ is the Bernoilli measure on Σ_m corresponding to the weight a.*

4. *For each* $a \in \mathcal{W}$ *with* $a \neq 0$, *we have that*

$$\begin{aligned} h_{\tilde{\mu}_a}(\tilde{f}) &= H(\epsilon|(\tilde{f})^{-1}\epsilon) \\ &= -\sum_{j=1}^{m} a_j \log a_j + \sum_{j=1}^{m} a_j \log d_j \\ &= h_{(\pi_1)_*\tilde{\mu}_a}(\sigma) + \int_{\Sigma_m} \log d_{w_1}\ (\pi_1)_*\tilde{\mu}_a(dw), \end{aligned}$$

where we denote by ϵ *the partition of* $\Sigma_m \times \overline{\mathbb{C}}$ *into one point subsets.*

5. *Let* $\tilde{\mu} = \tilde{\mu}_{a_0}$ *where* $a_0 = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$. *Then* $\tilde{\mu}$ *is the unique maximizing measure for* $\tilde{f}$ *and we have*

$$h(\tilde{f}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^{m} \deg(f_j)).$$

Definition 7.2. We call $\tilde{\mu}_a$ or μ_a the *self-similar measure* with respect to the weight a.

To prove Theorem 7.1, we introduce some notations and results from [L]. Let A be a bounded operator in the complex Banach space $\mathcal{B}$. The operator A is called *almost periodic* if the orbit $\{A^m\varphi\}_{m=1}^{\infty}$ of any vector $\varphi \in \mathcal{B}$ is strongly conditionally compact. The eigenvalue λ and related eigenvector are called unitary if $|\lambda| = 1$. The set of unitary eigenvectors of the operator A will be denoted by $\text{spec}_u A$. We denote by $\mathcal{B}_u$ the closure of the linear span of the unitary eigenvectors of the operator A. And we set

$$\mathcal{B}_0 = \{\varphi \mid A^m\varphi \to 0\ (m \to \infty)\},$$

here the convergence is assumed to be strong.

Theorem 7.3. *([L]) If* $A : \mathcal{B} \to \mathcal{B}$ *is an almost periodic operator in the complex Banach space* $\mathcal{B}$, *then* $\mathcal{B} = \mathcal{B}_u \oplus \mathcal{B}_0$.

Corollary 7.4. *([L]) Let* $A : \mathcal{B} \to \mathcal{B}$ *be an almost periodic operator in the complex Banach space* $\mathcal{B}$. *Assume that* $\text{spec}_u A = \{1\}$ *and the point* $\lambda = 1$ *is a simple eigenvalue. Let* $h \neq 0$ *be an invariant vector of the operator* A. *Then there exists an* A^* *invariant functional* $\mu \in \mathcal{B}^*$, $\mu(h) = 1$, *such that* $A^m\varphi \to \mu(\varphi)h \quad m \to \infty$.

We need some lemmas to prove Theorem 7.1.

Lemma 7.5. *Let* $G = \langle f_1, \dots, f_m\rangle$ *be a finitely generated rational semigroup. Assume that there exists an element* $g_0 \in G$ *of degree at least two and* $F(H) \supset J(G)$ *where* H *is a rational semigroup defined by* $H = \{h^{-1} \mid$

$h \in Aut(\overline{\mathbb{C}}) \cap G\}$.(if H is empty, put $F(H) = \overline{\mathbb{C}}$.) Then there exists a $\delta > 0$ such that for each $x \in J(G)$, if we denote by $\mathcal{F}_{x,\delta}$ the family of maps satisfying that each element of it is a well-defined inverse branch of some element of G on $B(x,\delta)$ where $B(x,\delta)$ is a ball about x with the radius δ with respect to the spherical metric, then $\mathcal{F}_{x,\delta}$ is a normal family on $B(x,\delta)$.

Proof. (outline) This lemma is shown by using a theorem in [HM3] and the assumption $F(H) \supset J(G)$.

□

Lemma 7.6. *Under the same assumption as Theorem 7.1, let $\tilde{K}$ be a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ which is backward invariant under $\tilde{f}$. If $\tilde{B}_a\varphi = \lambda\varphi$, $|\lambda| = 1$, then $\lambda = 1$ and φ is constant. That is, $(C(\tilde{K}))_u = \mathbb{C} \cdot \mathbf{1}$.*

Lemma 7.7. *Under the same assumption as Theorem 7.1, if K is a compact subset of $\pi_2^{-1}(\overline{\mathbb{C}} \setminus E(G))$ which is backward invariant under $\tilde{f}$, then $\tilde{B}_a$ is an almost periodic operator on $C(K)$.*

Proof. (outline) We will develop the methods of key lemma about equicontinuity of $\{B_{a_0}^n\phi\}_n$ where $a_0 = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$, $\phi \in C(K)$ in [Bo]. Let $a \in \mathcal{W}$ with $a \neq 0$. Let $\varphi \in C(K)$ be any element. We have $\|\tilde{B}_a{}^n\varphi\|_K \leq \|\varphi\|_K$ for each positive integer n. By the Ascoli-Arzela Theorem, we have only to show that the family $\{\tilde{B}_a{}^n\varphi\}_n$ is equicontinuous on $\tilde{K}$.

For each $t = (t_1, \dots, t_m) \in \mathbb{N}^m$, we set

$$a_{r,t} = \frac{t_r d_r}{\sum_{k=1}^n t_k d_k}, \ r = 1, \dots, m,$$

and $a(t) = (a_{1,t}, \dots, a_{m,t}) \in \mathcal{W}$. Then there exists a sequence $(t^l)_l$ of elements of $\mathbb{N}^m$ such that $a(t^l) \to a$, as $l \to \infty$. For each $i = 1, \dots, m$ and $l \in \mathbb{N}$, we set $g_{i,j}^l = f_i$, $j = 1, \dots, t_i^l$. For each $l \in \mathbb{N}$ we consider $\{g_{i,j}^l\}_{i,j}$ as a generator system and let $\tilde{f}_l : \Sigma_{m(l)} \times \overline{\mathbb{C}} \to \Sigma_{m(l)} \times \overline{\mathbb{C}}$ be the skew product map constructed by that generator system in the same way as the beginning of this section where $m(l) = \sum_{i=1}^m t_i^l$. For each n, we investigate the cardinality of "good" inverse branches of $\tilde{f}_l{}^n$ devided by the cardinality of all inverse branches of $\tilde{f}_l{}^n$. It turns out that it is sufficiently small. By using Lemma 7.5, we see that $\{\tilde{B}_{a(t^l)}^n\phi\}_{n,l}$ is equicontinuous on K. Letting $l \to \infty$, we get that $\{\tilde{B}_a^n\phi\}_n$ is equicontinuous.

We have to consider some long arguments and we will omit the detail of the proof. □

Proof. of the statements 1, 2 and 3 of Theorem 7.1. By Corollary 7.4, Lemma 7.6 and Lemma 7.7 we can show the statement about convergence of the operator and that the support of $\tilde{\mu}_a$ is included in $\tilde{J}$ in the same way as that in [L]. Since $\tilde{\mu}_a$ is $\tilde{B}_a^*$-invariant and $\inf_{z \in \tilde{J}} \tilde{\psi}_a(z) > 0$, by Proposition 4.6.6, we can show that the support of $\tilde{\mu}_a$ is equal to $\tilde{J}$ immediately.

It implies that the support of μ_a is equal to $J(G)$. Hence the statement 1 holds. From the statement 1, the statements 2 and 3 follow.

$\square$

To show the statements 4 and 5 in Theorem 7.1, we need some lemmas.

Lemma 7.8. *Under the same assumption as Theorem 7.1, for any $a \in \mathcal{W}$ with $a \neq 0$, we have μ_a is non-atomic.*

Lemma 7.9 (Ruelle's inequality). *Let $G = \langle f_1, \dots, f_m \rangle$ be a finitely generated rational semigroup and $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ the skew product map constructed by the generator system $\{f_1, \dots, f_m.\}$ Let ρ be an $\tilde{f}$-invariant probability measure on $\Sigma_m \times \overline{\mathbb{C}}$. Then we have*

$$h_\rho(\tilde{f}) \leq 2\max\{0, \int_{\Sigma_m \times \overline{\mathbb{C}}} \lim_{n\to\infty} \frac{1}{n} \log \|(\tilde{f}^n)'(z)\| d\rho(z)\} + h_{(\pi_1)_*\rho}(\sigma).$$

Let ρ be an $\tilde{f}$-invariant probability measure on $\Sigma_m \times \overline{\mathbb{C}}$. As in p108 in [Par], there exists a ρ-integrable function $J_\rho : \Sigma_m \times \overline{\mathbb{C}} \to [1, \infty)$ such that

$$\rho(\tilde{f}(A)) = \int_A J_\rho(z) d\rho(z),$$

for any Borel set A in $\Sigma_m \times \overline{\mathbb{C}}$ such that $\tilde{f}_{|A}$ is injective. Now we will generalize some Mañé's results([Ma1]), using the methods in [Ma1] and Lemma 7.9.

Lemma 7.10. *Let ρ be an $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. Then there exists a measurable partition $\mathcal{P}$ of $\Sigma_m \times \overline{\mathbb{C}}$ such that $h_\rho(\tilde{f}, \mathcal{P}) < \infty$ and $\mathcal{P}$ is a generator for $(\tilde{f}, \rho)$ i.e. $\vee_{i=1}^\infty \tilde{f}^{-n}(\mathcal{P}) = \epsilon$ (mod 0) where ϵ denotes the partition of $\Sigma_m \times \overline{\mathbb{C}}$ into one point subsets.*

Lemma 7.11. *Let ρ be an $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. Then*

$$h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z) = \int_{\Sigma_m \times \overline{\mathbb{C}}} I(\epsilon | \tilde{f}^{-1}(\epsilon))(z) d\rho(z).$$

Proof. By Lemma 7.10, there exists a generator $\mathcal{P}$ with $h_\rho(\tilde{f},\ \mathcal{P}) < \infty$. By Remark 8.10 and Lemma 10.5 in [Par], we get $h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z)$.

$\square$

Theorem 7.12. *Let $G = \langle f_1, \dots, f_m \rangle$ be a rational semigroup and $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ the skew product map constructed by the generator system $\{f_1, \dots, f_m\}$. Then the toplogical entropy $h(\tilde{f})$ on $\Sigma_m \times \overline{\mathbb{C}}$ satifies that*

$$h(\tilde{f}) \leq \log(\sum_{j=1}^m \deg f_j).$$

Proof. of Theorem 7.12 Suppose $h(\tilde{f}) \leq \log m$. Then we have nothing to do. Suppose $h(\tilde{f}) > \log m$. Let ρ be any $\tilde{f}$-invariant ergodic probability measure on $\Sigma_m \times \overline{\mathbb{C}}$ with $h_\rho(\tilde{f}) > \log m$. Then since $h(\sigma) = \log m$, by variational principle we get $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. By Lemma 10.5 in [Par] and Lemma 7.11, we have $I(\epsilon|\tilde{f}^{-1}\epsilon)(z) = \log J_\rho(z)$ and $h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z)$. Since $\tilde{f}$ is a $d:1$ map where $d = \sum_{j=1}^m \deg(f_j)$, we have $I(\epsilon|\tilde{f}^{-1}\epsilon)(z) \leq \log(\sum_{j=1}^m \deg(f_j))$. Hence we get $h_\rho(\tilde{f}) \leq \log(\sum_{j=1}^m \deg(f_j))$. By the variational principle, we get $h(\tilde{f}) \leq \log(\sum_{j=1}^m \deg(f_j))$. □

Proof. of statements 4 and 5 in Theorem 7.1. By Lemma 7.8 and the statement 1 of Theorem 7.1 , we get $H(\epsilon|(\tilde{f})^{-1}\epsilon) = -\sum_{j=1}^m a_j \log a_j + \sum_{j=1}^m a_j \log d_j$. Since $H(\epsilon|(\tilde{f})^{-1}\epsilon) = \int_{\Sigma_m \times \overline{\mathbb{C}}} I(\epsilon|\tilde{f}^{-1}(\epsilon))(z) d\rho(z)$, by Lemma 7.11 we get the statement 4 of Theorem 7.1.

Now we will show the statement 5 in Theorem 7.1. By the previous paragraph and Theorem 7.12, we get $h(\tilde{f}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^m \deg(f_j))$. Now assume there exists an $\tilde{f}$-invariant probability measure ρ on $\Sigma_m \times \overline{\mathbb{C}}$ with $\tilde{\mu} \neq \rho$ and $h_\rho(\tilde{f}) = \log d$ where $d = \sum_{j=1}^m \deg(f_j)$. We will show it causes a contradiction. We can assume ρ is ergodic. Since there exists an element $g \in G$ with the degree at least two, we have $\log d > \log m$. Hence $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$. By Lemma 7.11, we have

$$h_\rho(\tilde{f}) = \int_{\Sigma_m \times \overline{\mathbb{C}}} \log J_\rho(z) d\rho(z).$$

By Lemma 10.5 in [Par], we have $I(\epsilon|\tilde{f}^{-1}\epsilon)(z) = \log J_\rho(z)$. Since $\tilde{f}$ is a $d:1$ map, we have $\log J_\rho(z) \leq \log d$ for ρ almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Hence we get $\log J_\rho(z) = \log d$ for ρ almost all $z \in \Sigma_m \times \overline{\mathbb{C}}$. By Proposition 2.2 in [DU], we get that $\tilde{B}_a^*(\rho) = \rho$ where $a = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$ and $\tilde{B}_a$ denotes the operator on $C(\Sigma_m \times \overline{\mathbb{C}})$ defined in Section 7. If $E(G) = \emptyset$, then by the statement 1 in Theorem 7.1, we get $\rho = \tilde{\mu}$ and this is a contradiction. Assume $E(G) \neq \emptyset$. Let V be the union of connected components of $F(G)$ having non-empty intersection with $E(G)$. Let $\varphi \in C(\Sigma_m \times \overline{\mathbb{C}})$ be any element with $\varphi(z) \geq 0$ for all $z \in \Sigma_m \times \overline{\mathbb{C}}$. Let $\epsilon > 0$ be any number. Let A_ϵ be the ϵ-open hyperbolic neighborhood in V. Then $K_\epsilon = \pi_2^{-1}(\overline{\mathbb{C}} \setminus A_\epsilon)$ is compact and backward invariant under $\tilde{f}$. Then by the statement 1 in Theorem 7.1,

$$\begin{aligned} \int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) &= \int_{\Sigma_m \times \overline{\mathbb{C}}} (\tilde{B}_a^n \varphi)(z) d\rho(z) \\ &\geq \int_{K_\epsilon} (\tilde{B}_a^n \varphi)(z) d\rho(z) \\ &\to \rho(K_\epsilon) \cdot \int_{K_\epsilon} \varphi(z) d\tilde{\mu}(z), \end{aligned}$$

as $n \to \infty$. Hence we have for each $\epsilon > 0$,

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) \geq \rho(K_\epsilon) \cdot \int_{K_\epsilon} \varphi(z) d\tilde{\mu}(z).$$

Since $h_\rho(\tilde{f}) > h_{(\pi_1)_*\rho}(\sigma)$ and ρ is ergodic, we have $\rho(\pi_2^{-1}(E(G))) = 0$. Letting $\epsilon \to 0$, we get

$$\int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\rho(z) \geq \int_{\Sigma_m \times \overline{\mathbb{C}}} \varphi(z) d\tilde{\mu}(z).$$

It implies that $\rho \geq \tilde{\mu}$. Since ρ and $\tilde{\mu}$ are probability measures, it follows that $\rho = \tilde{\mu}$ but it is a contradiction. $\square$

The following result is shown by Theorem 7.1 and using Mañé 's methods([Ma3]).

Theorem 7.13. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(H) \supset J(G)$ where $H = \{h^{-1} \mid h \in Aut(\overline{\mathbb{C}}) \cap G\}$ (if $H = \emptyset$, put $F(H) = \overline{\mathbb{C}}$.) Also assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. Then*

$$\dim_H(J(G)) \geq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{J(G)} \log(\|f'\|)\, d\mu},$$

where $\mu = (\pi_2)_* \tilde{\mu}_a$, $a = (\frac{d_1}{d}, \dots, \frac{d_m}{d})$ and $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$.

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