数理解析研究所講究録
1087 巻 1999 年 53-56

On ergodic properties of expanding piecewise smooth maps.

Masato TSUJII (Hokkaido University)

February 10, 1999

In the talk, the author presented three recent results of him on ergodic properties of expanding piecewise smooth maps, which is given in the preprints [8, 9, 10].

Lasota and Yorke showed, in their famous work[3], the existence of absolutely continuous invariant measures for piecewise C^2 expanding maps on intervals. They made use of the Perron-Frobenius operator and functions of bounded variation, and their idea has been used extensively in the study of one dimensional dynamical systems. After their work, efforts have been paid for the generalization of their result to higher dimensional case. Though it is natural to expect similar results in higher dimension, it has been turned out that things are not simple. The main difficulty in higher dimension exists in the fact that the partition of the domain into the regions where an iteration of the map is smooth can be very complicated. As we show below, some examples of expanding piecewise C^r-maps on bounded regions in higher dimensional Euclidean space have quite singular ergodic properties and these examples seems to suggest that expanding piecewise C^r-maps do not necessarily admits absolutely continuous invariant measures.

Towards the positive direction, Gerhard Keller treated piecewise C^2 expanding maps on bounded regions on the plane in his thesis[4, 5] and gave some criterion for the existence of absolutely continuous invariant measure. Góra and Boyarski[6] gives a lower bound for the expansion rate that assures the existence of absolutely continuous invariant measures. Their result is valid for arbitrary dimension. But their lower bound depends on the minimal angle on the boundaries of the regions in the partition associated to the map. See [7] for a modification of their result. See also [1].

Before stating our results, let us give some definitions, to be precise. We call a map $c : [a, b] \to \mathbf{R}^2$ a C^r-curve if it is a restriction of a C^r-map defined on a neighborhood of $[a, b]$ and satisfies $c'(t) \neq 0$ for $t \in [a, b]$. A continuous map $c : [a, b] \to \mathbf{R}^2$ is called a piecewise C^r-curve if there is a sequence $a = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = b$ such that the restrictions $c|_{[\xi_i, \xi_{i+1}]}$, $0 \leq i < n$, are C^r-curves. Let D be a region on the plane $\mathbf{R}^2$ whose boundary consists of finitely many simple closed piecewise C^r-curves. We consider a finite (quasi-)partition $\xi = \{D_i\}_{i=1}^k$ of the domain D such that

- $D_i \subset D$ is a region whose boundary is a finite union of simple closed piecewise C^r-curves,
- $D_i \cap D_j = \phi$ if $i \neq j$, and
- $\cup_{i=1}^k \overline{D}_i = \overline{D}$ where $\overline{D}$ and $\overline{D}_i$ denote the closures of D and D_i respectively.

We call such partition a piecewise C^r-partition of D. We denote $E = \cup_{i=1}^k \partial D_i = \overline{D} - \cup_{i=1}^k D_i$.

A map $f : D \to D$ is called a piecewise C^r-map on D if there is a C^r-partition $\xi = \{D_i\}_{i=1}^k$ of D as above such that each restriction $f|_{D_i}$ of f to D_i, $1 \leq i \leq k$, can be extended to a neighborhood of $\overline{D}_i$ as a C^r-map.

For a tangent vector v at $x \in D - E$, we define its expansion rate $\rho(v, f)$ by

$$\rho(v, f) = \frac{\|Df(v)\|}{\|v\|}.$$

The expansion rate $\rho(f)$ of the map f is the infinimum of the expansion rate over all non-zero vectors at all points in $D - E$. If $\rho(f) > 1$ for a piecewise C^r-map, we call f a expanding piecewise C^r-map.

In the talk, the author first considered piecewise real-analytic maps (the case $r = \omega$) on bounded regions in the plane. The real-analytic property somewhat relax the difficulty we mentioned above. In fact, we can prove the following theorem.

Theorem 1 *Absolutely continuous invariant finite measures exist for arbitrary piecewise real-analytic expanding maps on bounded regions on the plane.*

This result improves a theorem of Keller in his thesis[4, 5], which give the same conclusion under one additional assumption that the map is piecewise conformal.
Remark The author learned from Gerhard Keller that Jerome Buzzi at Marseille obtained a similar result when he was preparing the manuscript of this paper.[2]

Next the author gave an example of expanding piecewise C^r-maps ($r < \infty$) with singular ergodic properties.

Theorem 2 *For $1 \leq r < \infty$, there exists an example of expanding piecewise C^r maps F on an open rectangle $D = (0,1) \times (-1,1)$ such that there exists an open subset $B \subset D$ with the following properties:*

(A) *the diameter of the set $F^n(B)$ converges to 0 as $n \to \infty$, and*

(B) *the empirical measures $n^{-1} \sum_{i=0}^{n-1} \delta_{F^i(x)}$ for $x \in B$ converges to the point measure δ_p at $p = (0,0)$ as $n \to \infty$.*

These examples say, at least, that the approach using the spectral properties of Perron-Frobenius operator is not valid for general expanding piecewise C^r maps when $r < \infty$. At present we do not know whether this kind of example exists for C^∞ case.

In dimension higher than 2, we only have a result for piecewise linear map. Let U be a bounded polyhedron in $\mathbf{R}^d$ with non-empty interior. An expanding piecewise linear map on U is a map $T : U \to U$ with a family $\mathcal{U} = \{U_k\}_{k=1}^{\ell}$ of polyhedra $U_k \subset U$, $k = 1, 2, \ldots, \ell$, satisfying the conditions

1. the interiors of polyhedra U_k are mutually disjoint,
2. $\cup_{k=1}^{\ell} U_k = U$, and
3. the restriction of the map T to the interior of each U_k is an affine map.

Then we have

Theorem 3 *An arbitrary expanding piecewise linear map admits an absolutely continuous invariant finite measure.*

We expect the same conclusion for piecewise real-analytic case. But, at present, we have not get such result because of complexity of intersections of real analytic hypersurfaces.

References

[1] Blank, M., Discreteness and continuity in problems of chaotic dynamics, Translations of Mathematical Monographs, 161. AMS, (1997).

[2] Buzzi, J.,*A.C.I.M.'S for arbitrary expanding piecewise* $\mathbf{R}$*-analytic mappings of the plane*, (preprint, IML)(1998)

[3] Lasota, A.& Yorke, J.,*On the existence of invariant measure for piecewise monotonic transformations*, Trans. A.M.S., Vol 186, 481–488,(1973)

[4] Keller, G., *Propriété ergodique des endomorphismes dilatants,* C^2 *par morceaux, des rǵions bornées du plan*, thesis, Universite de Rennes, (1979)

[5] Keller, G., *Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan*, C.R.Acad. Sci. Paris **289** Serie A, 625–627,(1979)

[6] Góra, P., &Boyarski, A., *Absolutely continuous invariant measures for piecewise expanding transformations in* $\mathbf{R}^N$, Israel J. Math. Vol 67, 272–276, (1989)

[7] Adl-Zarabi, K., *Absolutely continuous invariant measure for piecewise expanding C^2 transformations in $\mathbf{R}^n$ on domains with cusps on the boundaries*, Ergod. Th.& Dynam. Sys. **16**, 1–18, (1996)

[8] Tsujii, M.,*Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane.*, (preprint, Hokkaido University) (1998)

[9] Tsujii, M.,*Piecewise expanding maps on the plane with singular ergodic properties*, (preprint, Hokkaido University) (1998) *To appear in Ergod. Th. & Dynam. Sys.*

[10] Tsujii, M.,*Absolutely continuous invariant measures for expanding piecewise linear maps*, (preprint, Hokkaido University) (1998)
These three papers are available at the author's web page:

`http:\\www.math.sci.hokudai.ac.jp\~tsujii\`