数理解析研究所講究録
1087 巻 1999 年 108-119

Complex Ruelle Operator in a Parabolic Basin

Shigehiro Ushiki
Graduate School of Human and Environmental Studies
Kyoto University
京都大学大学院 人間・環境学研究科 宇敷重広

1. Parabolic basin and holomorphic quadratic differentials

In this note, we investigate the behavior of partial Ruelle operator associated to a parabolic basin of a complex dynamical system. Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping of the Riemann sphere to itself. We assume that the infinity is a parabolic fixed point of R of the form :

$$R(z) = z + 1 + \frac{P(z)}{Q(z)}, \quad \deg P \leq \deg Q - 2,$$

where $P(z)$ and $Q(z)$ are polynomials without common factor. Let A_∞ denote the immediate parabolic basin of the infinity, and let $K = \mathbb{C} \setminus A_\infty$ and $\overline{K} = K \cup \{\infty\}$. We call K the filled Julia set of R. Further, we assume that all the critical points in A_∞ are non-degenerate, and the forward orbit of each critical point does not contain other critical points. For the sake of simplicity, we assume $\overline{K}$ is connected.

Let $\mathcal{O}_0(\overline{K})$ denote the space of functions $g : \overline{K} \to \mathbb{C}$ holomorphic in a neighborhood of $\overline{K}$ and $g(\infty) = 0$. The topology is defined as follows : sequence of functions $\{g_n\}$ in $\mathcal{O}_0(\overline{K})$ converges to some function g_∞ in $\mathcal{O}_0(\overline{K})$ if there exists a neighborhood of $\overline{K}$ such that $\{g_n\}$ are extendable to this neighborhood and the sequence converges to g_∞ uniformly in this neighborhood.

Let $\mathcal{O}(A_\infty)$ denote the space of holomorphic functions $f : A_\infty \to \mathbb{C}$ with the topology of local uniform convergence. We denote by $\mathcal{O}_0(A_\infty)$ the set of holomorphic functions $f \in \mathcal{O}(A_\infty)$ satisfying $\lim_{z\to\infty} f(z) = 0$. We define the pairing of functions in these spaces.

DEFINITION 1.2 (pairing) For $g \in \mathcal{O}_0(\overline{K})$ and $f \in \mathcal{O}(A_\infty)$, Let

$$\langle f, g\rangle = \frac{1}{2\pi i}\int_\gamma f(\tau)g(\tau)d\tau,$$

where γ is a closed curve surrounding and passing near $\overline{K}$ with an orientation looking $\overline{K}$ on the left hand side. The contour courve γ should be chosen so that there is no critical point of R between $\partial\overline{K}$ and γ. The choice of γ depends on g, but the value of $\langle f, g\rangle$ does not depend on the choice, provided that the curve γ passes sufficiently near the filled Julia set $\overline{K}$.

PROPOSITION 1.3 Each $f \in \mathcal{O}(A_\infty)$ defines a continuous, holomorphic, and complex linear functional $\hat{f} : \mathcal{O}_0(\overline{K}) \to \mathbb{C}$ by $\hat{f}[g] = \langle f, g\rangle$ for $g \in \mathcal{O}_0(\overline{K})$.

Here, functional $\hat{f}$ is said to be holomorphic if $\hat{f}[g_\nu]$ is holomorphic with respect to ν for all holomorphic family $\{g_\nu\}$ in $\mathcal{O}_0(\overline{K})$.

PROOF Let $\{g_n\}$ be a sequence of functions in $\mathcal{O}_0(\overline{K})$ and assume g_n converges to 0 in $\mathcal{O}_0(\overline{K})$. Then by the definition of the topology of $\mathcal{O}_0(\overline{K})$, there exists a neighborhood U of $\overline{K}$ such that g_n are extendable to U and $\sup_{z\in U}|g_n(z)| \to 0$. Take a curve $\gamma \subset U$ and set $M = \sup_{\tau\in\gamma}|f(\tau)|$, and let $|\gamma|$ denote the length of γ. Then

$$|\langle f, g_n\rangle| = |\frac{1}{2\pi i}\int_\gamma f(\tau)g_n(\tau)d\tau| \leq \frac{1}{2\pi}|\gamma|M\sup_{z\in U}|g_n(z)| \to 0.$$

Clearly by definition, the functional is complex linear and holomorphic in the sense above.

DEFINITION 1.4 The dual space $\mathcal{O}_0^*(\overline{K})$ is the space of continuous, holomorphic and complex linear functionals $F : \mathcal{O}_0(\overline{K}) \to \mathbb{C}$.

PROPOSITION 1.5 For a functional $F \in \mathcal{O}_0^*(\overline{K})$,

$$f(\zeta) = F[\frac{1}{\zeta - z}], \quad \zeta \in A_\infty$$

defines a holomorphic function $f \in \mathcal{O}(A_\infty)$ and for $g \in \mathcal{O}_0(\overline{K})$,

$$F[g] = \langle f, g\rangle$$

holds.

PROOF For each $\zeta \in A_\infty$, $\frac{1}{\zeta - z} \in \mathcal{O}_0(\overline{K})$. It is a holomorphic family of holomorphic functions. Hence we have $f \in \mathcal{O}(A_\infty)$. Next, for $g \in \mathcal{O}_0(\overline{K})$, by applying the residue theorem, we have

$$g(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(\tau)}{\tau - z} d\tau, \quad z \in \overline{K}$$

since $g(\infty) = 0$, the resudue at the infinity vanishes. Therefore,

$$F[g] = F[\frac{1}{2\pi i}\int_\gamma \frac{g(\tau)}{\tau - z} d\tau] = \frac{1}{2\pi i}\int_\gamma F[\frac{1}{\tau - z}]g(\tau)d\tau$$
$$= \frac{1}{2\pi i}\int_\gamma f(\tau)g(\tau)d\tau = \langle f, g\rangle.$$

Propositions 1.3 and 1.5 yield the following.

PROPOSITION 1.6 $\mathcal{O}_0^*(\overline{K})$ is isomorphic to $\mathcal{O}(A_\infty)$.

The isomorphism defined in proposition 1.5 is called the Cauchy transformation.

2. Complex Ruelle operator and its adjoint operator

We define a linear operator $L : \mathcal{O}_0(\overline{K}) \to \mathcal{O}_0(\overline{K})$ by

$$(Lg)(x) = \frac{1}{2\pi i}\int_\gamma \frac{g(\tau)d\tau}{R'(\tau)(R(\tau) - x)}, \quad g \in \mathcal{O}_0(\overline{K}), x \in \overline{K}.$$

We call this operator a *complex Ruelle operator.* More precisely, it is a component of a Ruelle operator for a perticular weight $(R'(z))^{-2}$ in the decompostion of the operator described in [4]. The coutour curve γ depends upon g. Observe that Lg is holomorphic in a neighborhood of $\overline{K}$ and $g(\infty) = 0$. Note that Lg can be expressed as

$$(Lg)(x) = \sum_{y \in R^{-1}(x)} \frac{g(y)}{(R'(y))^2} + \sum_{c \in C(R)\cap\overline{K}} \frac{g(c)}{R''(c)(R(c) - x)}$$

in a neighborhood of $\overline{K}$.

The dual operator $L^* : \mathcal{O}_0^*(\overline{K}) \to \mathcal{O}_0^*(\overline{K})$ defines the *adjoint Ruelle operator* $\mathcal{L}^* : \mathcal{O}(A_\infty) \to \mathcal{O}(A_\infty)$ through the Cauchy transformation described in the previous section.

PROPOSITION 2.1 The adjoint operator $\mathcal{L}^* : \mathcal{O}_0(A_\infty) \to \mathcal{O}_0(A_\infty)$ is given by

$$(\mathcal{L}^* f)(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(R(\tau))d\tau}{R'(\tau)(z-\tau)}, \quad f \in \mathcal{O}_0(A_\infty), z \in A_\infty.$$

Moreover,

$$(\mathcal{L}^* f)(z) = \frac{f(R(z))}{R'(z)} - \sum_{c \in C(R)\cap A_\infty} \frac{f(R(c))}{R''(c)(z-c)}.$$

PROOF This is verified by a direct calculation. Let $\hat{f} \in \mathcal{O}_0^*(\overline{K})$ be a functional and $f \in \mathcal{O}(A_\infty)$ be the corresponding holomorphic function. Then we have

$$\begin{aligned}
(\mathcal{L}^* f)(z) &= (L^*\hat{f})\left[\frac{1}{z-\zeta}\right] = \hat{f}\left[L\left[\frac{1}{z-\zeta}\right]\right] \\
&= \hat{f}\left[\frac{1}{2\pi i}\int_\gamma \frac{d\tau}{R'(\tau)(R(\tau)-\zeta)(z-\tau)}\right] \\
&= \frac{1}{2\pi i}\int_\gamma f(\zeta)d\zeta\left(\frac{1}{2\pi i}\int_\gamma \frac{d\tau}{R'(\tau)(R(\tau)-\zeta)(z-\tau)}\right) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{d\tau}{R'(\tau)(z-\tau)}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{R(\tau)-\zeta}\right) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(R(\tau))d\tau}{R'(\tau)(z-\tau)} \\
&= \frac{f(R(z))}{R'(z)} - \sum_{c \in C(R)\cap A_\infty} \mathrm{Res}_{\tau=c}\frac{f(R(\tau))}{R'(\tau)(z-\tau)} \\
&= \frac{f(R(z))}{R'(z)} - \sum_{c \in C(R)\cap A_\infty} \frac{f(R(c))}{R''(c)(z-c)}.
\end{aligned}$$

This proposition shows that the adjoint Ruelle operator decomposes into two parts. This decomposition is similar to that introduced in [1], and the analysis of spectrum below is almost same as described there. Let $A_R = \{z \in A_\infty \mid (R^{\circ n})'(z) \neq 0 \text{ for } n \geq 0\}$, and let $\mathcal{O}(A_R)$ denote the space of holomorphic functions on A_R with the topology of local uniform convergence. Note that $\mathcal{O}(A_\infty) \subset \mathcal{O}(A_R)$. Define a linear operator $\mathcal{K} : \mathcal{O}(A_R) \to \mathcal{O}(A_R)$ by

$$(\mathcal{K}f)(z) = \frac{f(R(z))}{R'(z)}.$$

Let $\varphi : A_\infty \to \mathbb{C}$ denote the Fatou map defined by

$$\varphi(z) = \lim_{n\to\infty}(R^{\circ n}(z) - n), \quad z \in A_\infty.$$

Under our assumption on R, φ is holomorphic in A_∞ and stisfies function equation

$$\varphi \circ R(z) = \varphi(z) + 1, \quad z \in A_\infty$$

and

$$\varphi'(z) \neq 0 \quad \text{for} \quad z \in A_R.$$

Define a linear isomorphism $\mathcal{T} : \mathcal{O}(A_R) \to \mathcal{O}(A_R)$ by

$$(\mathcal{T}f)(z) = f(z)\varphi'(z).$$

The linear operater $\mathcal{K}$ is conjugate to $\mathcal{M} = \mathcal{T}\circ\mathcal{K}\circ\mathcal{T}^{-1}$ and $\mathcal{M} : \mathcal{O}(A_R) \to \mathcal{O}(A_R)$ is a very simple operator.

PROPOSITION 2.2

$$(\mathcal{M}h)(z) = h \circ R(z), \quad h \in \mathcal{O}(A_R).$$

PROOF By a direct computation.

$$(\mathcal{T}^{-1}h)(z) = h(z)(\varphi'(z))^{-1},$$

$$(\mathcal{K}\mathcal{T}^{-1}h)(z) = \frac{(\mathcal{T}^{-1}h)(R(z))}{R'(z)} = \frac{h(R(z))(\varphi'(R(z)))^{-1}}{R'(z)},$$

and, as we have $\varphi'(R(z))R'(z) = \varphi'(z)$ by differentiating the function equation $\varphi \circ R = \varphi + 1$,

$$(\mathcal{M}h)(z) = (\mathcal{T}\mathcal{K}\mathcal{T}^{-1}h)(z) = \frac{h(R(z))(\varphi'(R(z)))^{-1}}{R'(z)}\varphi'(z) = h(R(z)).$$

If a complex number $\nu \neq 0$ is an eigenvalue of the operator $\mathcal{M}$ and $h_\nu \in \mathcal{O}(A_R)$ is an eigenfunction associated to ν, then h_ν must satisfy the function equation

$$(\mathcal{M}h_\nu)(z) = h_\nu(R(z)) = \nu h_\nu(z).$$

The Fatou function $\varphi : A_\infty \to \mathbb{C}$ has an inverse function $\psi = \varphi^{-1}$ defined for $\{x \in \mathbb{C} \mid \Re x > r\}$ for sufficiently large r. In this region, we have

$$h_\nu(\psi(x+1)) = \nu h_\nu(\psi(x)).$$

Hence, by taking an appropriate value for $\log \nu$,

$$p(x) = e^{-x \log \nu} h_\nu(\psi(x))$$

is a periodic function of x of period 1. This function $p(x)$ must be an entire function of period 1. We obtain an expression of the eigenfunction

$$h_\nu(z) = e^{\varphi(z) \log \nu} p(\varphi(z)).$$

The eigenfunction $f_\nu \in \mathcal{O}(A_R)$ of the operator $\mathcal{K}$ corresponding to h_ν is given by

$$f_\nu(z) = \frac{e^{\varphi(z) \log \nu} p(\varphi(z))}{\varphi'(z)}.$$

PROPOSITION 2.3 Any $\nu \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathcal{L}^*$, and its eigenfunction $f_\nu \in \mathcal{O}(A_\infty)$ is given by

$$f_\nu(z) = \frac{e^{\varphi(z) \log \nu} p(\varphi(z))}{\varphi'(z)},$$

where $\varphi : A_\infty \to \mathbb{C}$ is the Fatou function and $p : \mathbb{C} \to \mathbb{C}$ is an entire periodic function of period 1 satisfying $p(\varphi(c)) = 0$ for all critical point $c \in A_\infty$.

PROOF The Fatou function φ has critical points at the critical points of R and at the backward images of these critical points. As we assumed that the critical points of R are simple and the critical points do not collide, the function f_ν is holomorphic in A_∞. In case if critical points are not simple or collision of critical points occur, we pose appropriate degenrate zero conditions upon p at the corresponding points $\varphi(c)$. There exists entire periodic functions with prescribed zeroes at the images $\varphi(c)$ of critical points. For such periodic entire functions p, functions f_ν belong to $\mathcal{O}(A_\infty)$. And as $f_\nu(R(c)) = 0$ for all critical points $c \in C(R) \cap A_\infty$, they are also eigenfunctions of $\mathcal{L}^*$.

We define a subspace of $\mathcal{O}(A_\infty)$ which is invariant under the adjoint Ruelle operator $\mathcal{L}^*$.

DEFINITION 2.4

$$\mathcal{O}_1(A_\infty) = \{f \in \mathcal{O}(A_\infty) \mid \forall t > 0, \exists M > 0, \exists r > 0,$$

$$s.t. |f(z)| < M \quad \text{for} \quad \Re z > r \text{ and } |\Im z| < t\}.$$

PROPOSITION 2.5 The space $\mathcal{O}_1(A_\infty)$ is invariant under $\mathcal{L}^*$.

PROOF As $R(z) = z + 1 + O(z^{-2})$ near the infinity, we have $R'(z) = 1 + O(z^{-1})$. Therefore, by taking sufficiently large positive number $s > (\max_{c \in C(R) \cap A_\infty} \Re c) + 1$, we can assume

$$|R(z) - z - 1| \le \frac{1}{2} \quad \text{and} \quad |R'(z) - 1| \le \frac{1}{2}$$

holds for $\Re z > s$. If $f \in \mathcal{O}_1(A_\infty)$, then for any $t > 0$, we can find positive constants M_0 and r_0 such that $|f(z)| < M_0$ holds for $\Re z > r_0$ and $|\Im z| < t + 1$. Let

$$M_1 = 2M_0 + \sum_{c \in C(R) \cap A_\infty} \left| \frac{f(R(c))}{R''(c)} \right| (1 + |c|)$$

and $r_1 = \max(s, r_0, 2)$. Then we have

$$|(\mathcal{L}^* f)(z)| \le \left| \frac{f(R(z))}{R'(z)} \right| + \sum_{c \in C(R) \cap A_\infty} \left| \frac{f(R(c))}{R''(c)} \right| \frac{1}{|z - c|}$$

$$\le 2M_0 + \sum_{c \in C(R) \cap A_\infty} \left| \frac{f(R(c))}{R''(c)} \right| (1 + |c|) \le M_1$$

for $\Re z > r_1$ and $|\Im z| < t$.

PROPOSITION 2.6 The adjoint operator $\mathcal{L}^*$ restricted to the subspace $\mathcal{O}_1(A_\infty)$ has a continuum of eigenvalues $\{\nu \in \mathbb{C} \mid 0 < |\nu| \le 1\}$. The eigenfunctions are as given in proposition 2.3.

3. Discrete eigenvalues of the operator

In this section, we apply the perturbation method described in [1] to our case. Let ℓ denote the number of critical points of R in A_∞, and let $C(R) \cap A_\infty = \{c_1, \cdots, c_\ell\}$. Define linear maps $\mathcal{G} : \mathcal{O}(A_R) \to \mathbb{C}^\ell$ and $\mathcal{F} : \mathbb{C}^\ell \to \mathcal{O}(A_R)$ by

$$\mathcal{G}f = \left(\frac{f(R(c_j))}{R''(c_j)} \right)_{j=1,\cdots,\ell}, \qquad f \in \mathcal{O}(A_R),$$

and

$$\mathcal{F}(\alpha_j) = \sum_{j=1}^{\ell} \frac{\alpha_j}{z - c_j}, \qquad (\alpha_j) \in \mathbf{C}^{\ell}.$$

The adjoint operator $\mathcal{L}^*$ can be expressed as

$$\mathcal{L}^* = \mathcal{K} - \mathcal{F}\mathcal{G}.$$

As $\ker \mathcal{G} = \{f \in \mathcal{O}(A_R) \mid f(R(c_j)) = 0, j = 1, \cdots, \ell\}$, We see that

$$\mathcal{L}^* |_{\ker \mathcal{G}} = \mathcal{K} |_{\ker \mathcal{G}}$$

and

$$\mathcal{O}(A_R)/\ker \mathcal{G} \simeq \mathbf{C}^{\ell}.$$

We define an $\ell \times \ell$ matrice $M(\lambda)$ by

$$M(\lambda) = I_{\ell} + \lambda \mathcal{G} \left(\sum_{k=0}^{\infty} \lambda^k \mathcal{K}^k \right) \mathcal{F}.$$

As

$$(\mathcal{K}^k f)(z) = \frac{f(R^{\circ k}(z))}{(R^{\circ k})'(z)}$$

the (i,j)-component of $M(\lambda)$ is given by

$$\delta_{ij} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i) - c_j)}.$$

Note that $M(\lambda)$ is holomorphic for $|\lambda| < 1$, since critical points c_i are in the parabolic basin A_∞.

PROPOSITION 3.1 If $\det M(\lambda) = 0$ holds for some λ with $0 < |\lambda| < 1$ and there exists an eigenvector $u \in \ker M(\lambda) \setminus \{0\}$ satisfying $M(\lambda)u = 0$, then

$$V = \sum_{k=0}^{\infty} \lambda^k \mathcal{K}^k \mathcal{F} u$$

satisfies

$$\mathcal{L}^* V = \frac{1}{\lambda} V.$$

Moreover, $V \in \mathcal{O}_1(A_\infty)$.

PROOF Let $u = (\alpha_j)$ and $v = \mathcal{F}u = \sum_{j=1}^{\ell} \frac{\alpha_j}{z - c_j}$. Clearly, v belongs to $\mathcal{O}(A_R)$, since

$$\mathcal{K}^k v = \mathcal{T}^{-1} \mathcal{M}^k \mathcal{T} v = \mathcal{T}^{-1}((\mathcal{T} v) \circ R^{\circ k})$$

and V converges uniformly on compact subsets of A_R. Next we show that $V \in \mathcal{O}(A_\infty)$. V may have poles at critical point c_i or at its backward images by R. The residue of $\mathcal{K}^k v$ at critical point c_i is given by

$$\mathrm{Res}_{z=c_i}\mathcal{K}^k v = \mathrm{Res}_{z=c_i}\frac{v(R^{\circ k}(z))}{(R^{\circ k})'(z)} = \mathrm{Res}_{z=c_i}\sum_{j=1}^{\ell}\frac{\alpha_j}{(R^{\circ k})'(z)(R^{\circ k}(z)-c_j)}$$

$$= \sum_{j=1}^{\ell}\frac{\alpha_j}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i)-c_j)}.$$

Hence we have

$$\mathrm{Res}_{z=c_i}V(z) = \alpha_i + \sum_{k=1}^{\infty}\sum_{j=1}^{\ell}\frac{\lambda^k\alpha_j}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i)-c_j)}$$

$$= \sum_{j=1}^{\ell}\left(\delta_{ij} + \sum_{k=1}^{\infty}\frac{\lambda^k}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i)-c_j)}\right)\alpha_j = 0.$$

Therefore V is regular at critical points c_i and consequently it is regular at the backward images of the critical points. This implies that $V \in \mathcal{O}(A_\infty)$. Furthermore V belongs also to $\mathcal{O}_1(A_\infty)$. For, as we assumed $R(z) = z + 1 + O(z^{-2})$, for any $t > 0$, we can find some $t_1 > t$ and $r > 0$ such that if $\Re z > r$ and $|\Im z| < t$ then $\frac{1}{2} < |\varphi'(z)| < \frac{3}{2}$, $\Re(R^{\circ k}(z)) > r$ and $|\Im(R^{\circ k}(z))| < t_1$ holds for $k = 1, 2, \cdots$. Let $m = \sup_{\Re z > r, |\Im z| < t_1}|v(z)\varphi'(z)|$. As

$$\mathcal{T}V = \sum_{k=0}^{\infty}\lambda^k\mathcal{T}\mathcal{K}^k v = \sum_{k=0}^{\infty}\lambda^k\mathcal{M}^k\mathcal{T}v = \sum_{k=0}^{\infty}\lambda^k(\mathcal{T}v)\circ R^{\circ k},$$

we have

$$|V(z)| \le 2\sum_{k=0}^{\infty}|\lambda|^k m = \frac{2m}{1-|\lambda|}.$$

Hence $V \in \mathcal{O}_1(A_\infty)$.

We have also

$$\lambda\mathcal{L}^*V = \lambda(\mathcal{K} - \mathcal{F}\mathcal{G})V$$

$$= \sum_{k=1}^{\infty}\lambda^k\mathcal{K}^k v - \lambda\mathcal{F}\mathcal{G}\left(\sum_{k=0}^{\infty}\lambda^k\mathcal{K}^k\right)\mathcal{F}u$$

$$= \sum_{k=1}^{\infty}\lambda^k\mathcal{K}^k v + \mathcal{F}\left(I_\ell - M(\lambda)\right)u$$

$$= \sum_{k=1}^{\infty}\lambda^k\mathcal{K}^k v + \mathcal{F}u = V.$$

Hence V is an eigenfunction of $\mathcal{L}^*$.

4. Eigenfunctions of L corresponding to the discrete eigenvalues

In this section, we consider eigenfunctions for the Ruelle operator L itself. As we saw in the previous section, the adjoint operator has a continuum of eigenvalues. In order to distinguish eigenvalues and eigenfunctions, we have to examine the eigenspaces for each eigenvalues. The Cauchy's integral formula

$$g(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta$$

indicates that rational functions of the form

$$\chi_\eta(z) = \frac{1}{z-\eta}$$

form a "basis" of the function space $\mathcal{O}_0(\overline{K})$. For $\eta \in A_\infty$, χ_η belongs to $\mathcal{O}_0(\overline{K})$. The image $L\chi_\eta$ is computed as follows.

PROPOSITION 4.1 If $\eta \in A_\infty \setminus C(R)$, then

$$(L\chi_\eta)(x) = \sum_{y \in R^{-1}(x)} \frac{1}{(R'(y))^2}\chi_\eta(y) - \sum_{c \in C(R)\cap \overline{K}} \frac{1}{R''(c)(c-\eta)}\chi_{R(c)}(x)$$

and

$$L\chi_\eta = \frac{1}{R'(\eta)}\chi_{R(\eta)} + \sum_{j=1}^{\ell} \frac{1}{R''(c_j)(c_j-\eta)}\chi_{R(c_j)}.$$

PROOF These formulas are directly verified by applying the residue formula to domains inside and outside of the contour courve γ.

Let us consider a formal sum of the following form.

$$U = \sum_{i=1}^{\ell}\sum_{k=1}^{\infty} \alpha_{i,k}\chi_{R^{\circ k}(c_i)}, \qquad \alpha_{i,k} \in \mathbb{C}.$$

The space of functions of this form is invariant under L. In this space, we can formulate a formal eigen equation

$$LU = \frac{1}{\lambda}U.$$

By a formal computation, we obtain an equation for λ as follows.

PROPOSITION 4.2 If the eigen equation has a solution, then λ satisfies $\det N(\lambda) = 0$, where $N(\lambda)$ is an $\ell \times \ell$-matrice

$$N(\lambda) = \left(\delta_{ij} + \frac{\lambda}{R''(c_i)} \sum_{k=0}^{\infty} \frac{\lambda^k}{(R^{\circ k}(R(c_j)) - c_i)(R^{\circ k})'(R(c_j))}\right).$$

PROOF This is verified by a straightforward computation.

PROPOSITION 4.3

$$\det N(\lambda) = \det M(\lambda).$$

PROOF The (i,j)-component of $M(\lambda)$ is given by

$$\delta_{ij} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(R^{\circ k})''(c_i)(R^{\circ k}(c_i) - c_j)}$$

$$= \delta_{ij} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(R^{\circ(k-1)})'(R(c_i))R''(c_i)(R^{\circ(k-1)}(R(c_i)) - c_j)}$$

$$= \delta_{ij} + \frac{\lambda}{R''(c_i)} \sum_{k=0}^{\infty} \frac{\lambda^k}{(R^{\circ k})'(R(c_i))(R^{\circ k}(R(c_i)) - c_j)}.$$

Let S denote the diagonal $\ell \times \ell$-matrice whose (i,i)-component is $\lambda/R''(c_i)$, and let W denote the $\ell \times \ell$-matrice whose (i,j)-component is

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{(R^{\circ k})'(R(c_i))(R^{\circ k}(R(c_i)) - c_j)}.$$

Then we see that

$$M(\lambda) = I_\ell + SW \quad \text{and} \quad {}^tN(\lambda) = I_\ell + WS.$$

Hence we have $\det M(\lambda) = \det N(\lambda)$.

Finally, we compute the eigenfunction for the eigenvalue λ^{-1}.

PROPOSITION 4.4 Formal eigenfunction of the Ruelle operator L is given by

$$U = \sum_{i=1}^{\ell} \sum_{k=1}^{\infty} \alpha_{i,k} \chi_{R^{\circ k}(c_i)},$$

where $(\alpha_{1,1}, \cdots, \alpha_{\ell,1})$ is a vector in the kernel of $N(\lambda)$ and

$$\alpha_{i,k} = \frac{\lambda^{k-1}\alpha_{i,1}}{(R^{\circ(k-1)})'(R(c_i))} \quad \text{for} \quad k = 1, 2, \cdots.$$

Note that the obtained eigen function converges as a meromorphic function if $|\lambda| < 1$. However, the limit function does not belong to the space $\mathcal{O}_0(\overline{K})$.

References

[1] G.M.Levin, M.L.Sodin, and P.M.Yuditski: A Ruelle Operator for a Real Julia Set, Communications in Mathematical Physics, 141, 119-132(1991).

[2] G.Levin, M.Sodin, and P.Yuditski: Ruelle operators with rational weights for Julia sets, Journal d'analyse mathématiques, Vol. 63(1994),303-331.

[3] M.Tsujii: A transversality condition for quadratic family at Collet-Eckmann parameter, Problems in Complex Dynamical Systems, RIMS Kokyuroku 1042,pp99-101,1998.

[4] S.Ushiki: Complex Ruelle Operator and Hyperbolic Complex Dynamical Systems, pp50-61,RIMS Kokyuroku 1072, "Invariants of Dynamical Systems and Applications", 1998.