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Stokes geometry of Painlevé equations with a large parameter

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Introduction

The purpose of this article is to present graphics of some examples of Stokes curves of the first, the second, and the third Painlevé equations. The notion of a Stokes curve for Painlevé transcendents was introduced by Kawai–Takei [KT1]. It plays an important rôle in WKB analysis of Painlevé transcendents with a large parameter, especially in the local reduction theorem for Painlevé transcendents [KT1], [AKT], [KT2]. It has global information of Painlevé transcendents that are obtained through Borel resummation of formal solutions to the Painlevé equations.

The J-th Painlevé equation ($J = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{IV}$) with a large parameter η is written in the form

$$(P_J): \quad \frac{d^2\lambda}{dt^2} = \eta^2 F_J(\lambda, t) + G_J(\lambda, \frac{d\lambda}{dt}, t),$$

where F_J and G_J are suitable rational functions. This article is concerned only with the case where $J = \mathrm{I}$, II, III. In each case, F_J and G_J are given as follows:

$$F_{\mathrm{I}} = 6\lambda^2 + t, \qquad G_{\mathrm{I}} = 0, \tag{1}$$

$$F_{\mathrm{II}} = 2\lambda^3 + t\lambda + c, \qquad G_{\mathrm{II}} = 0, \tag{2}$$

$$F_{\mathrm{III}} = 8\{2c_\infty\lambda^3 + \frac{c'_\infty}{t}\lambda^2 - \frac{c'_0}{t} - 2\frac{c_0}{\lambda}\}, \quad G_{\mathrm{III}} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt}. \tag{3}$$

Here c, c_∞, c'_∞, c_0, c'_0 are given constants. Stokes geometry of the other Painlevé equations will be discussed elsewhere.

This work is based on a collaboration with Professor T. Kawai and Professor Y. Takei. The author would like to express his heartiest thanks to them for stimulating discussions with them.

1. Formal solutions to Painlevé equations.

There is a formal solution to (P_J) of the form

$$\lambda_J^{(0)} = \lambda_0(t) + \eta^{-2}\lambda_2(t) + \eta^{-4}\lambda_4(t) + \cdots, \tag{4}$$

where $\lambda_0(t)$ is an algebraic function defined by

$$F_J(\lambda_0(t), t) = 0. \tag{5}$$

Such a solution is constructed in [KT1]. We call it 0-parameter solution. A family of formal solutions that contain two free parameters is constructed in [AKT]. We denote them by

$$\lambda_J(t, \alpha, \beta) = \lambda_0(t) + \eta^{-\frac{1}{2}}\lambda_{\frac{1}{2}}(t) + \eta^{-1}\lambda_1(t) + \cdots. \tag{6}$$

Here α and β are arbitrary constants and $\lambda_0(t)$ is the same as in (4). $\lambda_{\frac{k}{2}}$ $(k \geq 1)$ are constructed by the so-called multiple-scale method. For example, $\lambda_{\frac{1}{2}}$ has the form

$$\lambda_{\frac{1}{2}}(t) = \mu_J(t)(\alpha e^{\Phi_J} + \beta e^{-\Phi_J}), \tag{7}$$

where

$$e^{\Phi_J} = e^{\eta\phi_J(t)}(\theta_J(t)\eta^2)^{\alpha\beta}$$

with

$$\phi_J(t) = \int^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}dt$$

and θ_J being some function defined in terms of F_J and G_J ([AKT],[A1],[A2]). Formal solutions $\lambda_J(t, \alpha, \beta)$ are called instanton-type solutions.

2. Turning points and Stokes curves.

Let us recall the definition of turning points and of Stokes curves ([KT1]). For an instanton-type formal solution $\lambda_J(t, \alpha, \beta)$, a turning point r is, by definition, a zero point of

$$\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t).$$

That is, a turning point is a branch point of $\lambda_0(t)$. A Stokes curve for $\lambda_J(t,\alpha,\beta)$ is an integral curve of the direction field

$$\operatorname{Im}\sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t),t)}\,dt = 0$$

that emanates from a turning point of $\lambda_J(t,\alpha,\beta)$. The turning points and the Stokes curves of $\lambda_J(t,\alpha,\beta)$ should be considered on the Riemann surface of the algebraic function $\lambda_0(t)$ defined by the equation $F_J(\lambda_0(t),t)=0$. For the sake of simplicity, however, we sometimes regard them as points and curves on the base space. We also call them turning points and Stokes curves of the J-th Painlevé equation.

3. The first Painlevé equation.

Let us consider the first Painlevé equation:

$$(P_{\mathrm{I}}): \quad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2+t).$$

The leading term $\lambda_0(t)$ of an instanton-type solution is defined by the equation

$$F_{\mathrm{I}}(\lambda_0(t),t) = 6\lambda_0(t)^2 + t = 0.$$

Thus we have

$$\lambda_0(t) = \sqrt{-\frac{t}{6}}.$$

Since

$$\frac{\partial F_{\mathrm{I}}}{\partial \lambda} = 12\lambda,$$

we see that there is only one turning point $t=0$. Stokes curves are defined by

$$\operatorname{Im}\int_0^t \sqrt{12\sqrt{-\frac{t}{6}}}dt = 0,$$

namely,

$$\operatorname{Im}(-t)^{\frac{5}{4}} = 0.$$

If we write $t = |t|e^{\arg t}$ in this equation, we have

$$\arg t = \frac{3}{5}\pi,\ \frac{7}{5}\pi,\ \frac{11}{5}\pi,\ 3\pi,\ \frac{19}{5}\pi,\ldots. \tag{8}$$

Thus we have five half lines emanating from the origin. Hence we get the following picture of the first Painlevé transcendents on the base space:

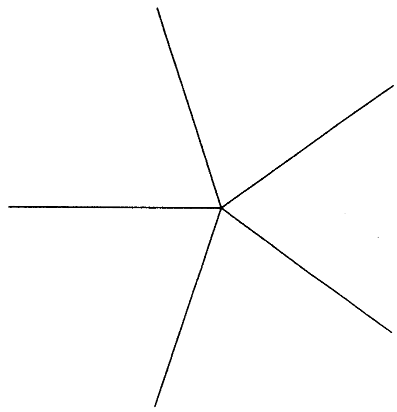

Figure 1.

The Riemann surface of $\lambda_0(t)$ is constructed of two copies of a Riemann sphere with a cut connecting 0 and ∞. Identifying two sides of four edges of the cuts as usual, we have the Riemann surface of $\lambda_0(t)$. It follows from (8) that, on the Riemann surface, Stokes lines appear every other line of the lines of Figure 1. Hence we have the following pictures.

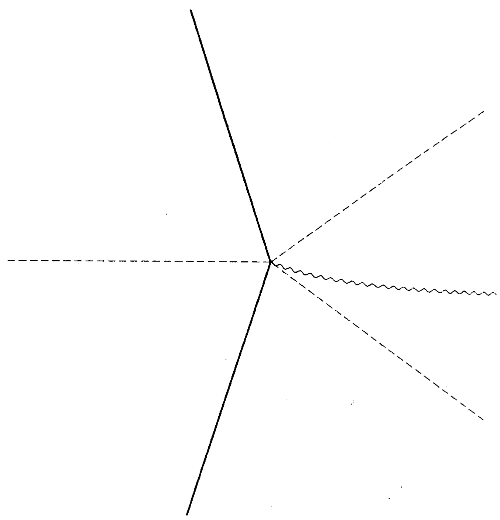

Figure 2.

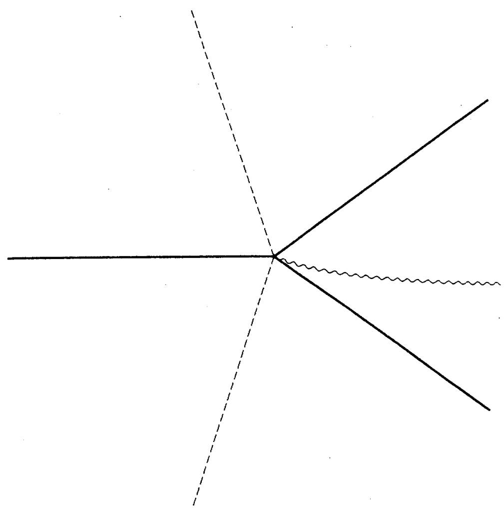

Figure 3.

Here the zigzag curves designate the cuts, the first sheet (Figure 2) is chosen to be $\sqrt{-t/6} < 0$ for $t < 0$ and the second sheet (Figure 3) $\sqrt{-t/6} > 0$ for $t < 0$. Broken lines indicate that they appear on another sheet.

4. The second Painlevé equations

The second Painlevé equations with the large parameter η have the form

$$(P_{\rm II}): \ \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c).$$

Here c is a constant. We assume that c does not equal zero. Note that under this assumption, we can reduce $(P_{\rm II})$ to the case $c = 1$ by a suitable scaling transformation of independent and dependent variables. The leading term λ_0 is defined by the following cubic equation

$$F_{\rm II}(\lambda_0(t), t) = 2\lambda_0(t)^3 + t\lambda_0(t) + c = 0. \tag{9}$$

The turning points are obtained by solving

$$\frac{\partial F_{\rm II}}{\partial \lambda}(\lambda_0(t), t) = 6\lambda_0(t)^2 + t = 0 \tag{10}$$

with respect to t. Hence we have three turning points

$$r_1 = -6\left(\frac{c}{4}\right)^{\frac{2}{3}}, \ r_2 = -6\left(\frac{c}{4}\right)^{\frac{2}{3}}\omega, \ r_3 = -6\left(\frac{c}{4}\right)^{\frac{2}{3}}\omega^2,$$

where we set $\omega = (-1+\sqrt{3}i)/2$. For the algebraic function $\lambda_0(t)$, r_j $(j = 1,2,3)$ are branch points of square-root type; two of three branches of $\lambda_0(t)$ coalesce at r_j $(j = 1,2,3)$. The value of coalescing $\lambda_0(t)$ at r_j $(j = 1,2,3)$ are

$$\lambda_0(r_1) = \left(\frac{c}{4}\right)^{\frac{1}{3}}, \ \lambda_0(r_2) = \left(\frac{c}{4}\right)^{\frac{1}{3}} \omega^2, \ \lambda_0(r_3) = \left(\frac{c}{4}\right)^{\frac{1}{3}} \omega.$$

By using Puiseux expansion of $\lambda_0(t)$ at each turning point, we can see that the local configuration of Stokes curves near the turning point is almost the same as in the case of (P_{I}). That is to say, on the base space, there are five Stokes curves emanating from the turning point.

Global configuration of Stokes curves for (P_{II}) is obtained by numerical experiment. For the case $c = 1$, we have the following configuration on the base space:

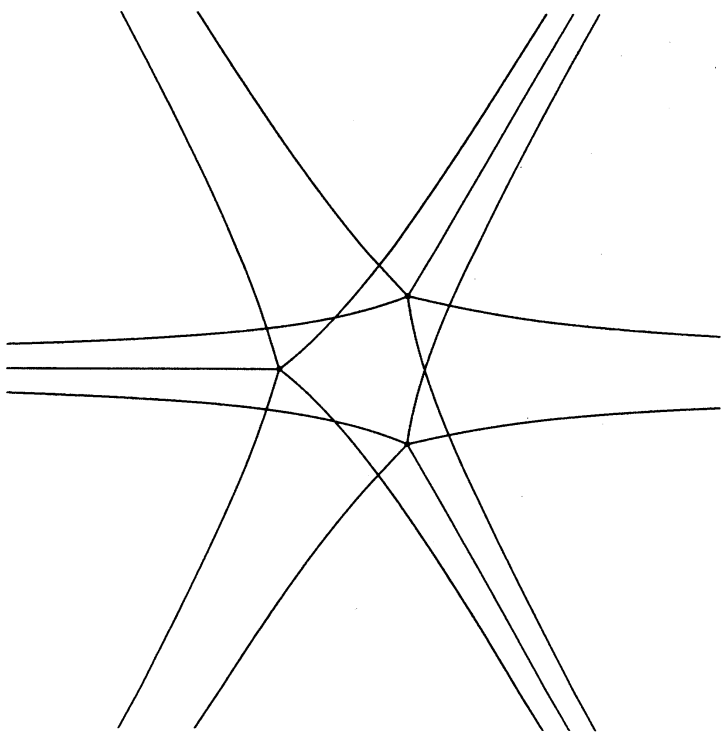

Figure 4.

Taking the sheet structure of $\lambda_0(t)$ into account, we see that the configuration of the Stokes curves on the Riemann surface of $\lambda_0(t)$ is as follows.

Figures 5, 6 and 7 show the first, the second and the third sheet, respectively.

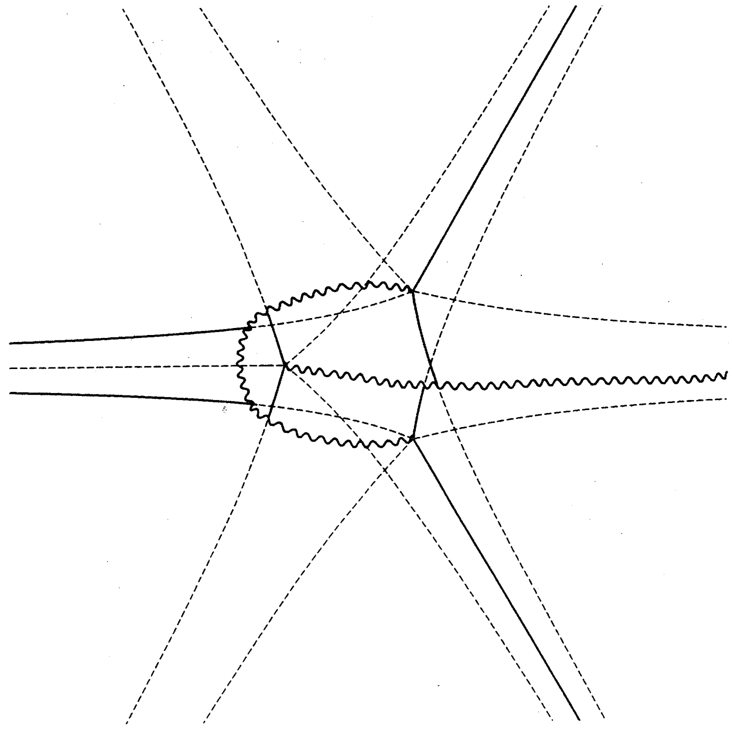

Figure 5.

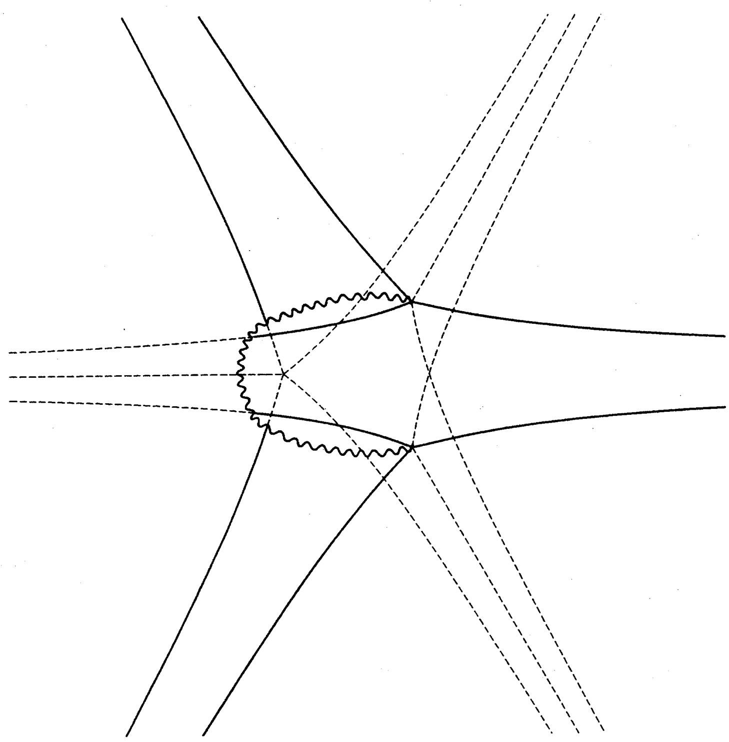

Figure 6.

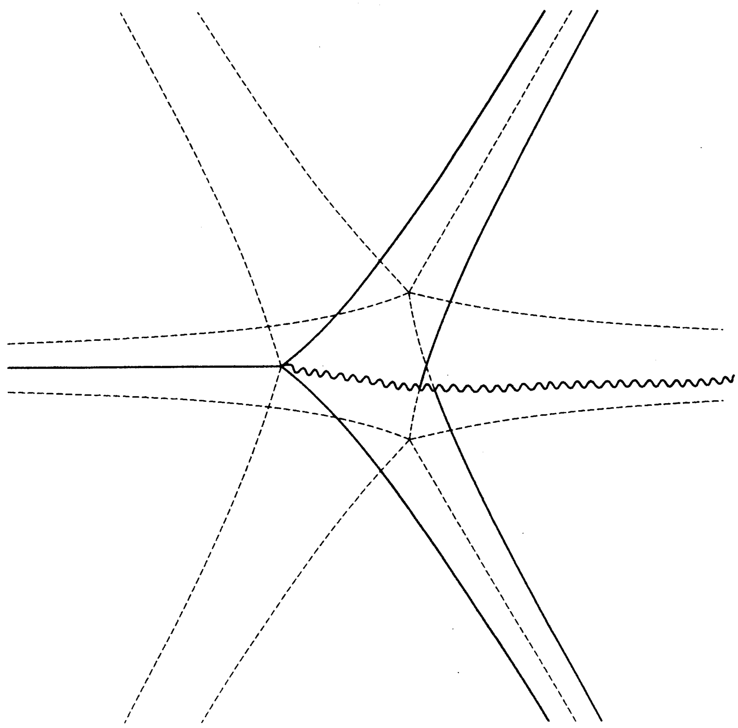

Figure 7.

Here the convention (of zigzag curves, etc.,) is the same as in the case of (P_{I}).

5. The third Painlevé equations.

Finally we consider the third Painlevé equations:

$$(P_{\mathrm{III}}): \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2 8\left[2c_\infty\lambda^3 + \frac{c'_\infty}{t}\lambda^2 - \frac{c'_0}{t} - 2\frac{c_0}{\lambda}\right].$$

The leading term $\lambda_0(t)$ of the instanton-type solution is defined by the equation

$$F_{\mathrm{III}}(\lambda_0(t), t) = \frac{8}{t\lambda_0(t)}\left[2c_\infty t\lambda_0(t)^4 + c'_\infty\lambda_0(t)^3 - c'_0\lambda_0(t) - 2c_0 t\right] = 0. \quad (11)$$

The turning points are obtained by the equation

$$\frac{\partial F_{\mathrm{III}}}{\partial\lambda}(\lambda_0(t), t) = 0. \quad (12)$$

The resultant of the numerators of (11) and (12) with respect to $\lambda_0(t)$ is a constant multiple of the left-hand side of the equation

$$t(4096c_0^3c_\infty^3t^6-768c_0^2c_\infty^2c_0'c_\infty't^4+3(9c_\infty^2c_0^{\prime 4}-2c_0c_\infty c_0^{\prime 2}c_\infty^{\prime 2}+9c_0^2c_\infty^{\prime 4})t^2-c_0^{\prime 3}c_\infty^{\prime 3})=0,$$

which is satisfied by the turning points. If the parameters c_0, c_0', c_∞, c_∞' are taken to be generic, this equation has seven distinct roots and one of them is 0. Note that the root 0 is special because the origin is a singular point of (P_{III}). Hence there are six turning points of (P_{III}) in generic case. Each turning point is a branch point of $\lambda_0(t)$ of square-root type. Hence the local configuration of Stokes curves near the turning point is the same as in the case of (P_{I}) and of (P_{II}). The origin is not a branch point of $\lambda_0(t)$ but there are two branches of $\lambda_0(t)$ which have local expansion of the form

$$\lambda_0(t)=\pm\sqrt{\frac{c_0'}{c_\infty'}}+O(t).$$

For these two branches, we have

$$\sqrt{\frac{\partial F_{\mathrm{III}}}{\partial\lambda}(\lambda_0(t),t)}=4\sqrt{\pm\sqrt{c_0'c_\infty'}}\,\frac{1}{\sqrt{t}}+O(1).$$

Hence we have to take into account two Stokes curves that are emanating from the origin defined by

$$\operatorname{Im}\int_0^t\sqrt{\frac{\partial F_{\mathrm{III}}}{\partial\lambda}(\lambda_0(t),t)}dt=\operatorname{Im}\left(8\sqrt{\pm\sqrt{c_0'c_\infty'}}\,\sqrt{t}+O(t)\right)=0.$$

Stokes curves of this kind have not been discussed in the previous works. But it is inevitable to consider the connection problem across a Stokes curve of this kind in the study of (P_{III}). Therefore the origin should be regarded as a turning point. Note that the type of degeneration of the Stokes geometry of corresponding linear equation ((SL_{III}) in the notation of [KT1]) that occurs when t is on a Stokes curve of (P_{III}) emanating from the origin is different from the case when t is on a Stokes curve emanating from a non-zero turning point. Similar situations take place for (P_{V}) and (P_{VI}). These observations were obtained in a collaboration with Kawai and Takei after my talk in this workshop. Details will be discussed elsewhere.

Now let us show an example of Stokes geometry of (P_{III}). We set $c_0 = i$, $c_0' = -2$, $c_\infty = 1$, $c_\infty' = 3$ hereafter. There are six non-zero turning points:

$$\begin{aligned} r_1 &= -0.894205 - 0.708713i, & r_2 &= -0.311643 + 0.0148773i, \\ r_3 &= -0.104626 + 0.636534i, & r_4 &= 0.104626 - 0.636534i, \\ r_5 &= 0.311643 - 0.0148773i, & r_6 &= 0.894205 + 0.708713i. \end{aligned}$$

After some numerical computation, we have the following picture of an example of Stokes curves of (P_{III}) on the base space:

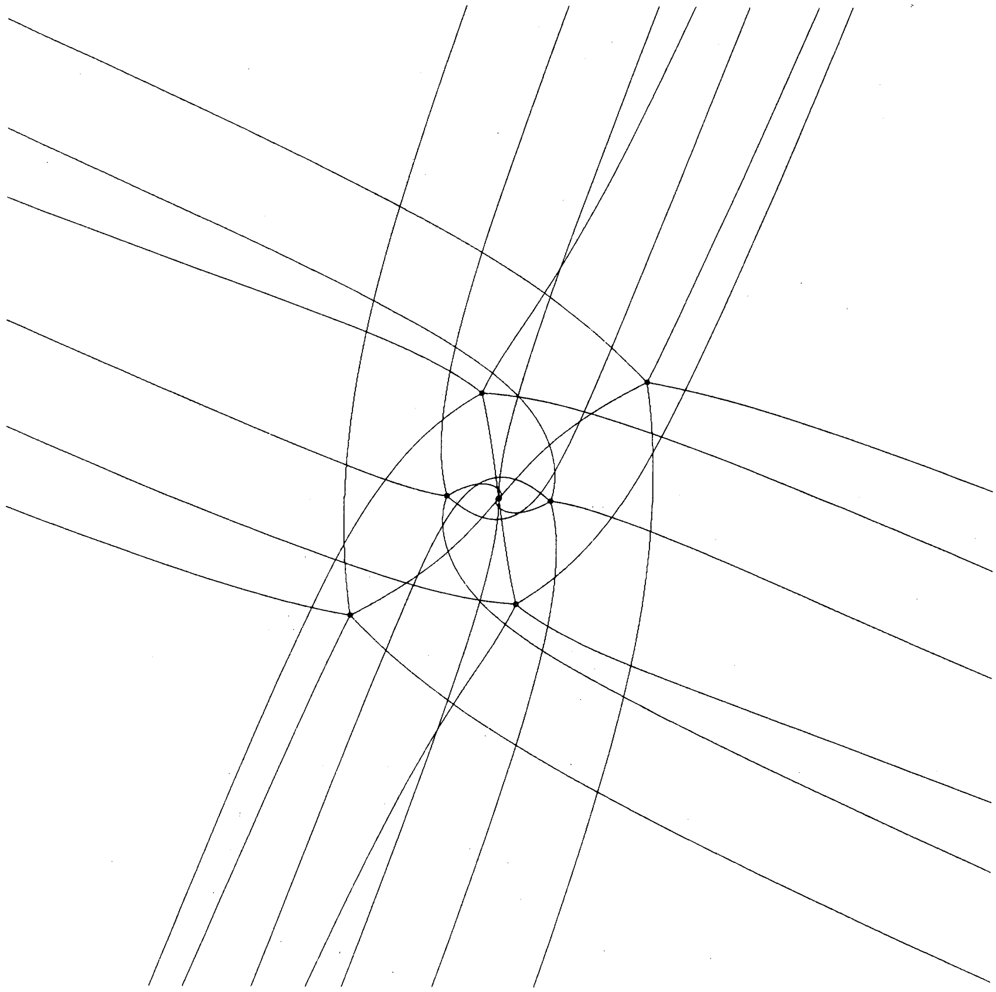

Figure 8.

To see the sheet structure of the Riemann surface of $\lambda_0(t)$, we distinguish the branches of $\lambda_0(t)$ through their local expansion at the origin and put superscripts to label them:

$$\text{First sheet}: \lambda_0^1(t) = -\frac{c'_\infty}{2c_\infty t} + O(1) = -\frac{3}{2t} + O(1),$$

$$\text{Second sheet}: \lambda_0^2(t) = -\frac{2c_0}{c'_0}t + O(t^2) = i\,t + O(t^2),$$

$$\text{Third sheet}: \lambda_0^3(t) = -\sqrt{\frac{c'_0}{c'_\infty}} + O(t) = -\frac{\sqrt{6}}{3}i + O(t),$$

$$\text{Fourth sheet}: \lambda_0^4(t) = \sqrt{\frac{c'_0}{c'_\infty}} + O(t) = \frac{\sqrt{6}}{3}i + O(t).$$

After analytic continuation of these roots to the non-zero turning points along the straight lines, we can observe, by numerical experiment, that the following relations hold:

$$\begin{aligned}
\lambda_0^1(r_1) &= \lambda_0^2(r_1), \\
\lambda_0^2(r_2) &= \lambda_0^3(r_2), \\
\lambda_0^1(r_3) &= \lambda_0^4(r_3), \\
\lambda_0^1(r_4) &= \lambda_0^3(r_4), \\
\lambda_0^2(r_5) &= \lambda_0^4(r_5), \\
\lambda_0^1(r_6) &= \lambda_0^2(r_6).
\end{aligned}$$

Thus we can take cuts on the base space to make the Riemann surface. For example, on the first sheet, we take two cuts; one is connecting r_1 and r_6 and another connecting r_3 and r_5, and so on. Figures 9–12 give the configuration of Stokes curves on each sheet. Note that, besides Stokes curves emanating from non-zero turning points, there is a Stokes curve emanating from the origin in the third sheet (Figure 11) and in the fourth sheet (Figure 12).

All numerical experiments in this article have been done by using *Mathematica* 3.01.

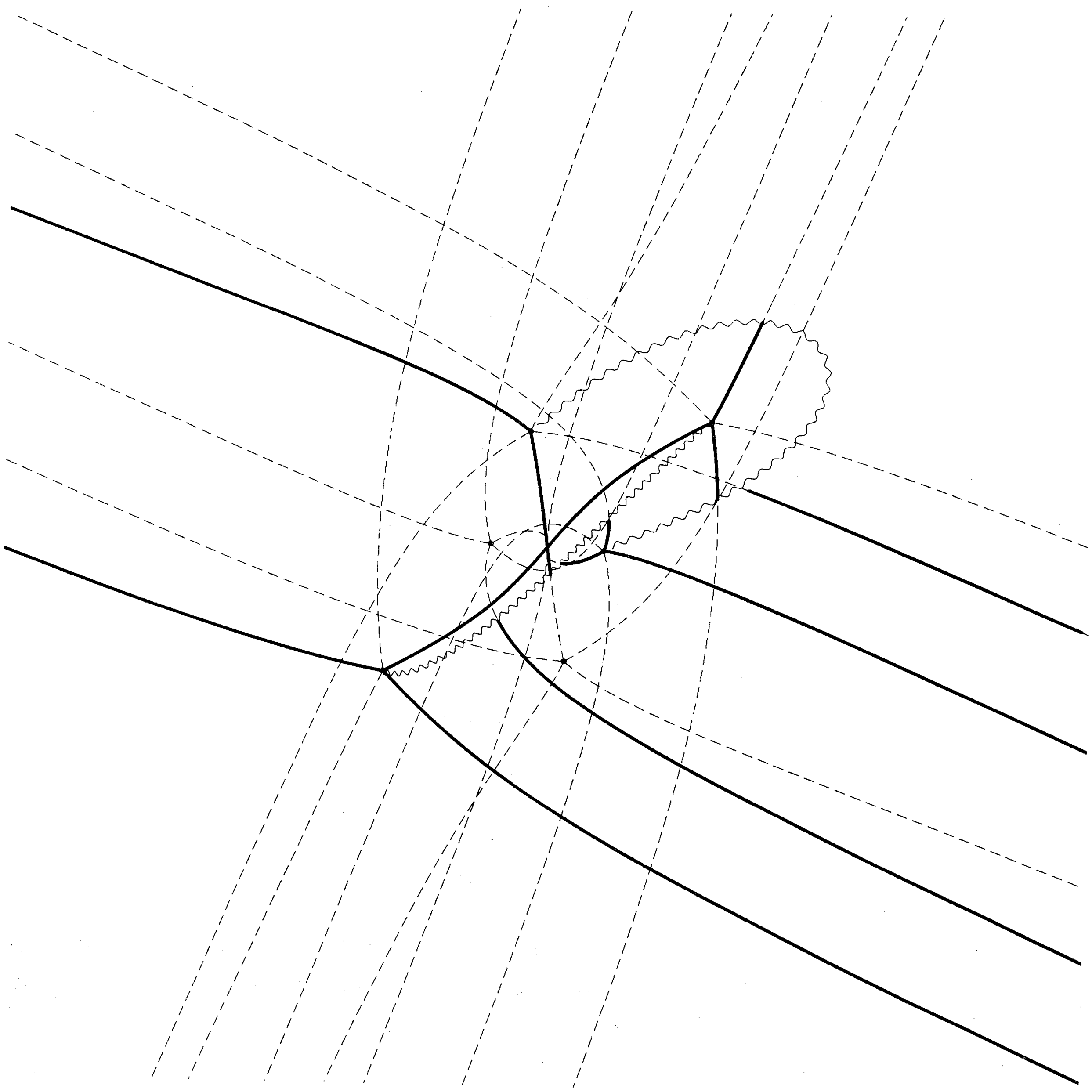

Figure 9.

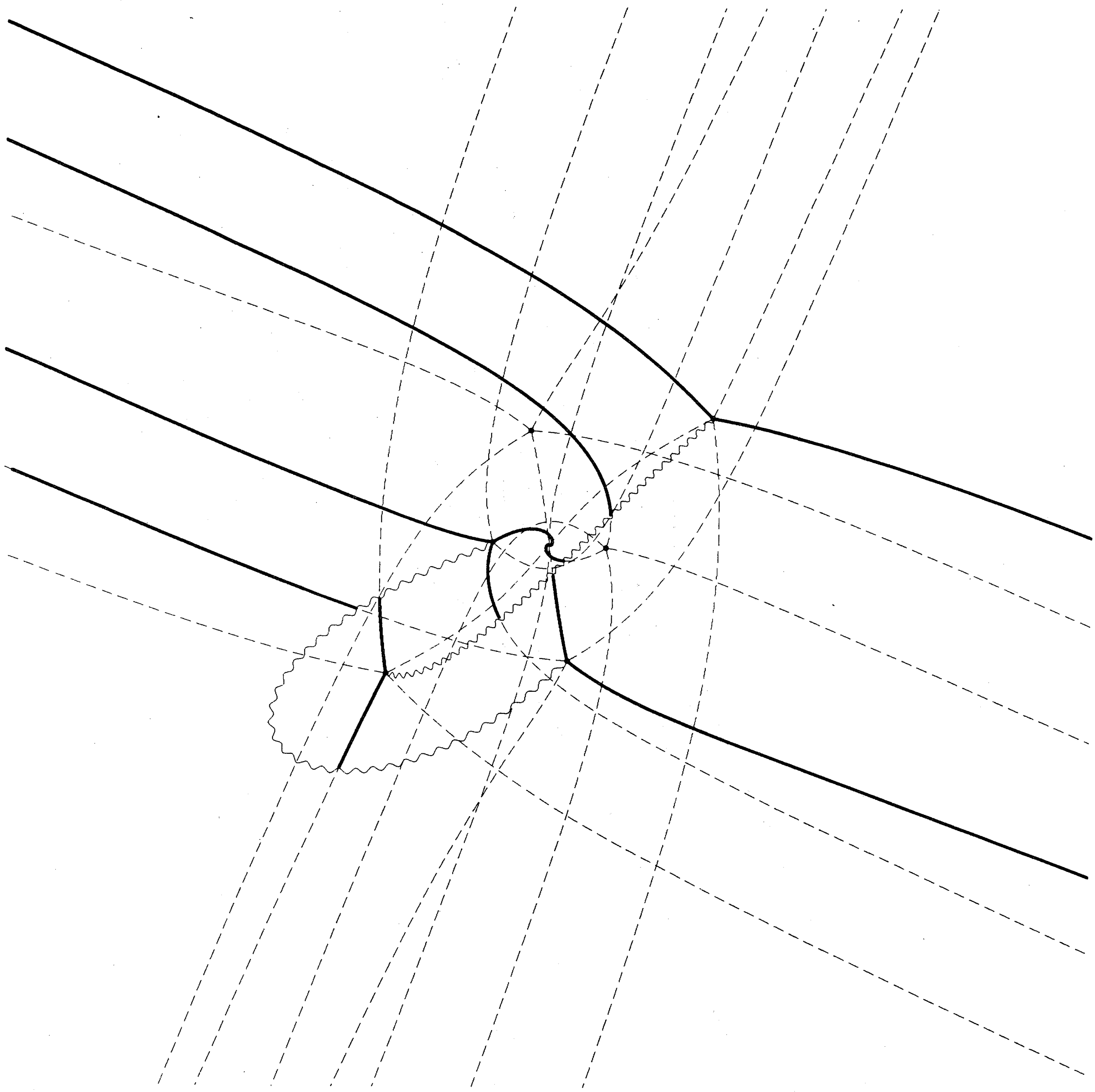

Figure 10.

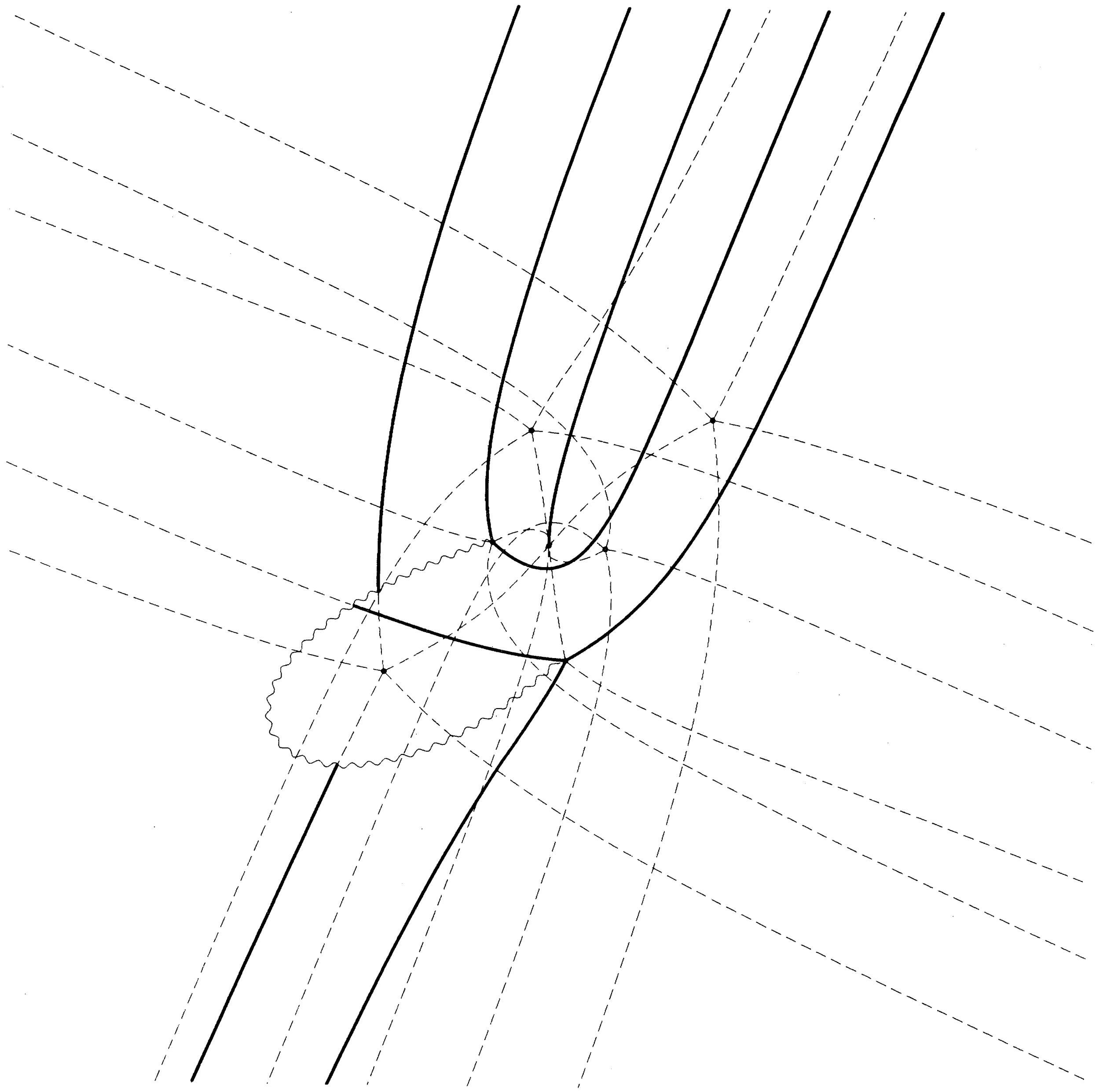

Figure 11.

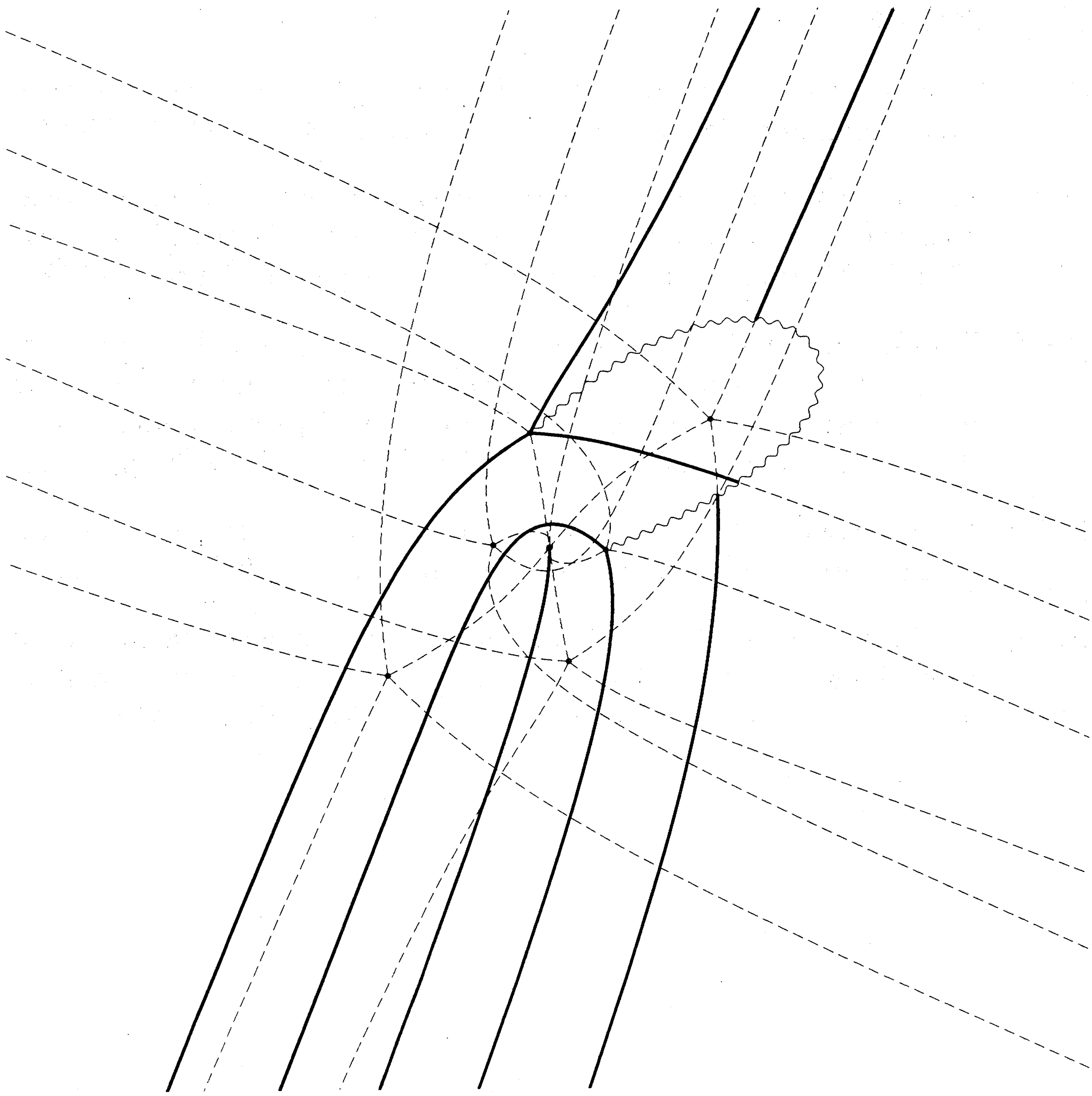

Figure 12.

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