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Asymptotic analysis of the modified Bessel function with respect to the parameter

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1 Introduction

This is an intermediate report of a joint work with William Boyd at Bristol University. As the work is not yet completed, this talk is given in the author's responsability.

The aim of this talk is to give some observation on adjacency of saddles concerning hyperasymptotic analysis to the modified Bessel function.

In the paper [3], W.Boyd derived a new representation of the gamma function, exploiting hyperasymptotic analysis, that is, reformulation of the method of steepest descents by M. Berry and C. Howls in [1]. There appeared a series of infinitely many saddles lying on an adjacent contour in the integral representation of the gamma function. He expected generalization of his analysis and proposed, as an example, the modified Bessel function $K_{\nu}(\nu z)$ when ν tends to infinity ([3] p628).

In this note, I explain the feature of the case of the modified Bessel function, which turns out to be more complicated than that of the gamma function in [3]. In fact, there appear two series of infinitely many saddles $w^{(2n)}(z)$ and $w^{(2n+1)}(z)$, $n \in \boldsymbol{N}$. The fundamental step in the hyperasymptotic analysis of integrals with saddles is to determine adjacency of saddles. In this report, our observation on adjacency relies on computer graphics of Stokes lines and steepest descent curves. Our conjectural conclusion is:

The complex z-plane is divided into the two regions, the unbounded domain (called radial) and the bounded domain (called spiral) separated by a piecewise analytic bounded closed curve (Figure 2(b)) [1]

$$\Re\left(\sqrt{1+z^2}+\log\frac{1-\sqrt{1+z^2}}{z}\right)=0.$$

(i) Suppose z is in the unbounded (radial) domain. For any fixed p, only $w^{(\pm 1+p)}(z)$ are adjacent to $w^{(p)}(z)$.
(ii) Suppose z is in the bounded (spiral) domain. For any fixed p, $w^{(2n+1+p)}(z)$ are adjacent to $w^{(p)}(z)$ for all n. $w^{(2n+p)}(z)$ are on the adjacent contour emanating from $w^{(p)}(z)$ for all n.

Hyperasymptotic analisys has been developped by M.Berry, C.Howls, W.Boyd, F.W.J.

[1] Cf. the eye-shaped domain in Olver's [7], Chap.10, **8.2**, where the asymptotic analysis of the modified Bessel function of large order is given by the Liouville-Geen approximation (WKB method).

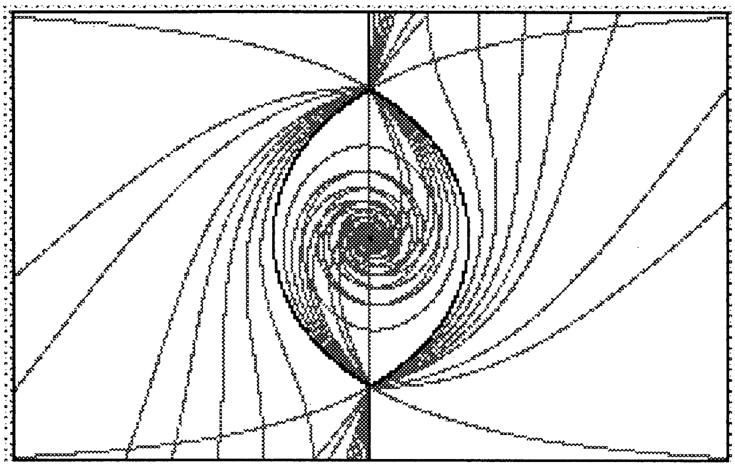

Figure 1: The Stokes lines for vaious phases of ν. The bounded Stokes lines from i to $-i$ for ph $(\nu) = \pm\frac{1}{2}\pi$ are the boundary of the bounded (spiral) and the unbounded (radial) domains.

Olver, A. Olde Daalhuis and others (*See* Boyd [4], Howls [6] also the reports given by C. Howls and A. Olde Daalhuis in this symposium). E. Delabaere (e.g. [5]) gave an interpretation by resurgent function theory. It seems an interesting problem to give rigorous analysis of this example in view point of hyperasymptotic analysis or resurgent function theory.

We review the several notations. The modified Bessel function $K_\nu(z)$ is the solution to the equation

$$\frac{d^2K}{dz^2} + \frac{1}{z}\frac{dK}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)K = 0, \tag{1.1}$$

defined by $K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu z}$ with $I_\nu(z) = e^{-i\nu\pi/2}J_\nu(iz)$. When $\Re z$ is positive, it has an integral representation

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\nu t - z\cosh t}dt. \tag{1.2}$$

In this paper, we shall discuss the modified Bessel function $K_\nu(\nu z)$ and related variants for large complex ν and fixed complex z.

Our starting point is the integral reprentation (see e.g. [7] p.250, (8.03))

$$K_\nu(\nu z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\nu\{t + z\cosh t\}}dt. \tag{1.3}$$

This is an integral, as we shall see, with infinite number of saddles The extended method of steepest descents [3], [1] is applicable. On the other hand, if we put $F(z) = K_\nu(\nu z)$ for fixed ν, $F(z)$ satisfies an second order differential equation

$$\frac{d^2F}{dz^2} + \frac{1}{z}\frac{dF}{dz} - \nu^2\left(1 + \frac{1}{z^2}\right)F = 0, \tag{1.4}$$

with a large complex parameter ν. WKB analysis is applicable for asymptotic analysis with respect to ν.

These two view points and their interplay are charactersitic in our arguments.

Acknowledgement: The author wish to express his appreciation to members at the School of Mathematics, University of Bristol and also at the Laboratory of J.-A. Dieudonné, University of Nice for their interest and hospitality during his stay on sabbatical leave.

2 Infinite number of saddles and their adjacency

2.1 Saddle points

We rewrite the integral representation of the modified Bessel function

$$K_\nu(\nu z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\nu p(z,w)}dw, \tag{2.5}$$

where

$$p(z,w) = w + z\cosh w. \tag{2.6}$$

We assume always $z \neq 0$. This is a phase function of w with a nonzero parameter z and we often omit z to denote it simply by $p(w)$. The equation $p'(w) = 0$ reduces to a quadratic equation of e^w. We have

$$e^w = \frac{-1+\sqrt{1+z^2}}{z}, \quad -\frac{1+\sqrt{1+z^2}}{z}. \tag{2.7}$$

Therefore, we have two series of an infinite number of saddles

$$w^{(2n)} = \log\frac{-1+\sqrt{1+z^2}}{z} + 2n\pi i, \tag{2.8}$$

$$w^{(2n+1)} = \log\frac{1+\sqrt{1+z^2}}{z} + (2n+1)\pi i, \tag{2.9}$$

where $n \in \boldsymbol{N}$. We easily see that $p'(w) = p''(w) = 0$ if and only if $w = (n+\frac{1}{2})\pi i$ and $z = (-1)^n i$. In this case, $p'''(w) = 0$ is not satisfied. Hence, only $\pm i$ are the two double saddles and the others are all simple.

The points $z = \pm i$, where two simple saddles coalesce, are considered the turning points also in view point of steepest descents method.

We note that $e^{-w} = \frac{1\pm\sqrt{1+z^2}}{z}$ with the same choice of the double signatures as in e^w.

We call their images by the map $p(\cdot)$ singularities on the complex ξ-plane. We have two series of the infinite number of singularities $p^{(n)} = p(w^{(n)})$:

$$p^{(2n)} = \log\frac{-1+\sqrt{1+z^2}}{z} + \sqrt{1+z^2} + 2n\pi i, \tag{2.10}$$

$$p^{(2n+1)} = \log\frac{1+\sqrt{1+z^2}}{z} - \sqrt{1+z^2} + (2n+1)\pi i. \tag{2.11}$$

We note that

$$p^{(2n)} + p^{(2m+1)} = (2(n+m)+1)\pi i. \tag{2.12}$$

Conjecture. Any two singulairities $p^{(n)}$ and $p^{(m)}$ for different n, m do not coalesce on the ξ-plane, when $z \neq \pm i$. See Appendix A.

2.2 Steepest descents

For simplicity, we omit z in the function $p(z,w)$. We put

$$\theta = \mathrm{ph}\,(\nu) \text{ and } p^{(n)} = p(w^{(n)}). \tag{2.13}$$

We define

$$p^{(nm)} = p^{(m)} - p^{(n)}, \quad \theta^{(nm)} = \mathrm{ph}\,(p^{(nm)}). \tag{2.14}$$

Following Berry and Howls's computations in our context (see also Boyd [3]), we consider

$$S^{(n)}(\nu) = \nu^{\frac{1}{2}} \int_{C^{(n)}(\theta)} \exp(-\nu[p(w) - p^{(n)}])dw, \tag{2.15}$$

where $C^{(n)}(\theta)$ is the path of steepest descent for θ passing through $w^{(n)}$. We define Stokes lines in view point of steepest descents method by the locus of $\{z \in \boldsymbol{C};\ \Im[\nu p^{(n)}(z)] = \Im[\nu p^{(m)}(z)]\}$ for some different integers n and m. It is readily reduced to the case where n and m have different parity, say, n is even and m is odd.

Proposition 2.1 *Stokes lines in view point of steepest descents method satisfies the same differential relation satisfied by those defined by the WKB method.*

We give description of paths of steepest descents. We introduce real variables, $z = x + iy$ for the independent variable of the differential equation, θ for the phase of the asymptotic parameter ν and $w = u + iv$ for the integration variable.

Proposition 2.2 *The paths of steepest descents or ascents satisfy the differential equation*

$$g(u,v)\dot{u} + f(u,v)\dot{v} = 0 \tag{2.16}$$

where

$$\begin{aligned} f(u,v) &= \cos\theta + \frac{1}{2}\left\{e^{u}(x\cos(\theta+v) - y\sin(\theta+v))\right. \\ &\qquad \left. + e^{-u}(x\cos(\theta-v) - y\sin(\theta-v))\right\}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} g(u,v) &= \sin\theta + \frac{1}{2}\left\{e^{u}(x\sin(\theta+v) + y\cos(\theta+v))\right. \\ &\qquad \left. - e^{-u}(x\sin(\theta-v) - y\cos(\theta-v))\right\}. \end{aligned} \tag{2.18}$$

REMARK 1. If we choose β such that $\cos\beta = x/\sqrt{x^2+y^2}$ and $\sin\beta = y/\sqrt{x^2+y^2}$, we have

$$f(u,v) = \cos\theta + \frac{\sqrt{x^2+y^2}}{2}\left\{e^{u}\cos(\theta+v+\beta) + e^{-u}\cos(\theta-v+\beta)\right\} \tag{2.19}$$

$$g(u,v) = \sin\theta + \frac{\sqrt{x^2+y^2}}{2}\left\{e^{u}\sin(\theta+v+\beta) - e^{-u}\sin(\theta-v+\beta)\right\}. \tag{2.20}$$

REMARK 2. The differential relation is invariant for translation of 2π in v-variable. The steepest descent figures show this 2π periodicity in v direction.

We give approximate description of paths of the steepest near simple saddles by Taylor expansion. This approximation is used as initial step in the numerical integration of the differential relation.

Let $w^{(n)}$ be a simple saddle. Then,

$$\begin{aligned} p(w) &= p(w^{(n)}) + p'(w^{(n)})(w - w^{(n)}) + \frac{p''}{2!}(w - w^{(n)})^2 + O\left((w - w^{(n)})^3\right) \\ &= p(w^{(n)}) + \frac{(-1)^n}{2}\sqrt{1+z^2}(w - w^{(n)})^2 + O\left((w - w^{(n)})^3\right). \end{aligned}$$

Hence, if w in on the locus of $\Im[p(w)] = \Im[p(w^{(n)})]$, we have approximately,

$$\theta + 2\mathrm{ph}(w - w^{(n)}) + \mathrm{ph}\left(\nu p''(w^{(n)})\right) = m\pi,$$

where m is an integer. The linear approximation of the paths of the steepest near $w^{(n)}$ is given by

$$\mathrm{ph}(w - w^{(n)}) = -\frac{\theta}{2} - \frac{\mathrm{ph}((-1)^n\nu\sqrt{1+z^2})}{2} + \frac{m}{2}\pi, \quad m = 0, 1, 2, 3, \tag{2.21}$$

which does not depend on n as a set of four lines.

We define the domain $\Delta^{(n)}$. We consider all the steepest-descent paths, for different θ, which emanate from the saddle point $w^{(n)}$. They sweep out the domain surrounded by a finite or infinite number of steepest-descent paths $C^{(m)}(-\theta^{(nm)})$ passing through $w^{(m)}$, called *adjacent contour*. This domain is denoted by $\Delta^{(n)}$. The saddles $w^{(m)}$ are called adjacent to the saddle $w^{(n)}$. They are hit by the steepest descent path $C^{(n)}(-\theta^{(nm)})$ issuing from the $w^{(n)}$.

2.3 Review of hyperasymptotic representation

We recall the hyperasymptotic formula given by M.Berry - C. Howls [1]. We use the same notations as in previous sections.

Let $\Gamma^{(n)}$ be a finite loop in $\Delta^{(n)}$ surrounding a part of $C^{(n)}$ including the two zeros of the denominator of the integrand.

The basis for hyperasymptotics is the following resurgence formula:

$$\begin{aligned} S^{(n)}(\nu) &= \sum_{r=0}^{N-1} \frac{a_r^{(n)}}{\nu^r} + R_N^{(n)}(\nu), \\ a_r^{(n)} &= \frac{\Gamma(r+\frac{1}{2})}{2\pi i}\int_{\Gamma^{(n)}} \frac{1}{[p(w) - p^{(n)}]^{r+1/2}}dw, \\ R_N^{(n)}(\nu) &= \frac{1}{2\pi i\nu^N}\sum_m \frac{1}{\left(p^{(nm)}\right)^N}\int_0^\infty \frac{\sigma^{N-1}e^{-\sigma}}{1 - \frac{\sigma}{\nu p^{(nm)}}} S^{(m)}\left(\frac{\sigma}{p^{(nm)}}\right) d\sigma, \end{aligned} \tag{2.22}$$

where $w^{(m)}$ are adjacent saddles to $w^{(n)}$.

This is a review of general theory in our context.

Then, we can make use of periodic structure in our integrals with saddles.

Proposition 2.3

$$S^{(2n)}(\nu) = S^{(0)}(\nu) \tag{2.23}$$
$$S^{(2n+1)}(\nu) = S^{(1)}(\nu) \tag{2.24}$$

We notice

$$p^{(2n,2n\pm1)} = -2p^{(0)} \pm \pi i, \tag{2.25}$$
$$p^{(0,\pm(2m+1))} = -2p^{(0)} \pm (2m+1)\pi. \tag{2.26}$$

In this note, we will not discuss the analytic formulas in detail.

3 Stokes lines

In this section, we discuss Stokes lines of the equation in view point of WKB analysis.

We assume the WKB formal expansion

$$F(z) = e^{-\nu S(z)} \sum_{n=0}^{\infty} a_n(z)\nu^{-n}. \tag{3.1}$$

Substituting this into the equation and rearranging the result by the power of ν, we obtain the relation of the highest order

$$\nu^2 \left\{ \left(\frac{dS}{dz}\right)^2 - \left(1 + \frac{1}{z^2}\right) \right\} = 0. \tag{3.2}$$

Hence, we have

$$S(z) = \pm \int_{z_0}^{z} \frac{\sqrt{1+\zeta^2}}{\zeta} d\zeta, \tag{3.3}$$

where z_0 is an arbitrary initial point except the origin. Then we have $a_0(z) = C(1+z^2)^{-1/4}$, although we do not need it for our analysis. We choose i or $-i$ as z_0, which is one of the turning points of the equation and define the Stokes lines from the turning point z_0. We consider three cuts on the complex plane, two along a half straight line starting from the turning points and the other from the origin. We put on this cut plane

$$S_j(z) = (-1)^{j-1} \int_{z_0}^{z} \frac{\sqrt{1+\zeta^2}}{\zeta} d\zeta, \text{ for } j = 1,\, 2, \tag{3.4}$$

where $\sqrt{1} = 1$ in the integrand. Then, Stokes lines of the equation issuing from the turning point z_0 are defined as the analitically continued locus

$$\{z \in \boldsymbol{C};\ \Im\left[\nu S_1(z)\right] = \Im\left[\nu S_2(z)\right]\}, \tag{3.5}$$

that is,

$$\{z \in \boldsymbol{C};\ \Im\left[e^{i\theta} \int_{z_0}^{z} \frac{\sqrt{1+\zeta^2}}{\zeta} d\zeta\right] = 0\}, \tag{3.6}$$

where by definition

(3.7) $$\theta = \mathrm{ph}(\nu).$$

Stokes lines are locally smooth curves except the turning points as singularities. Denoting locally one of the Stokes lines by $z(\sigma)$ with one dimensional real parameter σ, we have a differential relation of $z(\sigma)$.

Proposition 3.1 *Outside the turning points, any analytically continued Stokes line $z(\sigma)$ satisfies*

(3.8) $$\Im\left[e^{i\theta}\dot{z}(\sigma)\frac{\sqrt{1+z(\sigma)^2}}{z(\sigma)}\right] = 0.$$

This differential relation will be numerically integrated and give graphical imformation of the Stokes lines. We put

(3.9) $$\frac{\sqrt{1+z^2}}{z} = P(x,y) + iQ(x,y).$$

Then the differential relation has the form

(3.10) $$\dot{y}(P(x,y)\cos\theta - Q(x,y)\sin\theta) + \dot{x}(P(x,y)\sin\theta Q(x,y)\cos\theta) = 0.$$

In (x,y)-coordinates, the points $(0,1),(0,-1),(0,0)$ are singularities of the vector field, corresponding to the turning points $z = i, -i$ and the pole $z = 0$. All the pictures of Stokes lines are obtained by approximate integration of the vector field by the Runge-Kutta method, starting from a small neighbourhood of the turning points.

On the other hand, the $\frac{\sqrt{1+z^2}}{z}$ in (3.6) has a primitive

$$\sqrt{1+z^2} + \log\frac{1-\sqrt{1+z^2}}{z}.$$

We fix the branches as $\sqrt{1} = 1$ and $\log\frac{1-\sqrt{1+z^2}}{z} = \mp\frac{\pi}{2}i$ when $z = \exp(\pm\frac{\pi}{2}i)$. Therefore, the equation in (3.6) for $z_0 = i$ is

(3.11) $$\Im\left[e^{i\theta}\left(\sqrt{1+z^2} + \log\frac{1-\sqrt{1+z^2}}{z} + \frac{\pi}{2}i\right)\right] = 0,$$

and the other one for $z_0 = -i$ is

(3.12) $$\Im\left[e^{i\theta}\left(\sqrt{1+z^2} + \log\frac{1-\sqrt{1+z^2}}{z} - \frac{\pi}{2}i\right)\right] = 0.$$

Graphics of Stokes lines from i and from $-i$ show point-symmetry with respect to the origin $z = 0$. This symmetry reflects that if z satifies the first equation, then $ze^{\pi i}$ satisfies the (analytically continued around the origin) second equation and that the converse is also true.

We can verify analytically several facts seen by the figures obtained by numerical integration.

(S1) If z tends to ∞ along a Stokes line, $\theta+\mathrm{ph}(z)$ is asymptotically 0 or π mod 2π.
(S2) Approximate description near the turning points: we have asymptotically

$$\mathrm{ph}(z-i)=\frac{\pi}{6}-\frac{2}{3}\theta+\frac{2}{3}n\pi, \tag{3.13}$$

where n is an arbitrary integer.

For example, if $\theta=0$, the asymptotic directions of Stokes lines issuing from i are

$$\frac{\pi}{6},\frac{5}{6}\pi,-\frac{\pi}{2}.$$

We have the symmetric result for the case $z_0=-i$. They are used as the first step from the turning points for the numerical integration of the differential relation (3.11).
(S3) Approximate description near the pole is a conformal image of a logarithmic spiral.
(S4) Existence of finite closed Stokes lines when $\theta=\pm\frac{\pi}{2}$. In this case, the two equations (3.11) and (3.12) are equal to

$$\Re\left[\sqrt{1+z^2}+\log\frac{1-\sqrt{1+z^2}}{z}\right]=0. \tag{3.14}$$

The two Stokes lines connecting $\pm i$ consist of the boundary of a bounded domain. We call this bounded domain *spiral* , since it is filled with spiral Stokes lines issuing from $\pm i$ as θ changes from 0 to $\pm\frac{\pi}{2}$ (See Figure 1, Figure 2(b)). The interior of the complement will be called *radial.*

4 Discussion on adjacency

Integrating numerically by a simple version of Runge-Kutta method the differential relations in the previous sections, we can observe steepest descent curves and Stokes lines by computer graphics ([8],[9]).

When we follow faithfully the definition of adjacency, we have to verify whether steepest descent curves issuing from one saddle $w^{(n)}$ hit or not another saddle for a certain phase of ν. Since adjacency of saddles $w^{(m)}$ to $w^{(n)}$ is transformed into visibility between the corresponding two singularities $p^{(m)}$ and $p^{(n)}$ on the Borel plane, the critical phase of ν is the angle $-\theta^{(nm)}$, if the they are on the same Riemann sheet over the Borel plane.

Let the steepest descent $C^{(n)}(-\theta^{(nm)})$ passing through $w^{(n)}$ hit the saddle $w^{(m)}$. This measns that we have Stokes phenomenon with respect to θ near $-\theta^{(nm)}$ while z is fixed. The same steepest descent picture means also that we have Stokes phenomenon with respect to z when the phase $-\theta^{(nm)}$ is fixed. Hence, in order to know adjacency, we have only to verify whether z is on the Stokes line by the exact WKB method ([10]) with the phase $-\theta^{(nm)}$.

We observe that if z is inside the spiral domain, z is passed over infinitely many times by spiral Stokes lines when θ changes from 0 to $\frac{\pi}{2}$ or $\frac{-\pi}{2}$. This must induce infinite number of adjacent saddles when z is in the spiral domain. On the other hand, the domain outside the spiral one is called the *radial* domain, since any point in this domain is swept out only

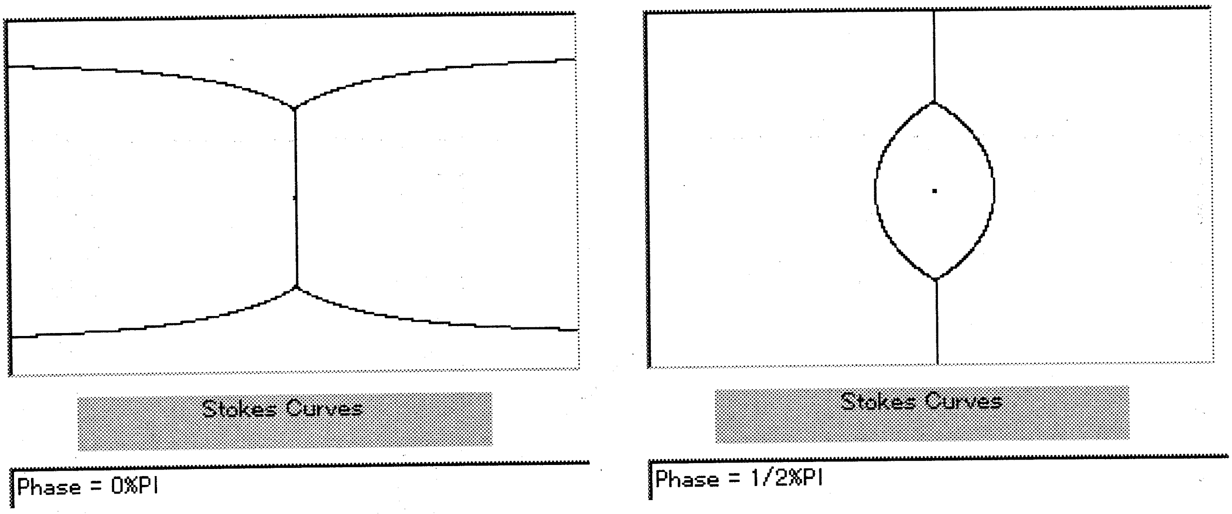

Figure 2: (a) and (b).

four times modulo 2π by the radial Stokes lines issuing from $\pm i$ as θ changes. We are convinced of the following conjectural conclusion through the observation of graphics.

(i) Suppose z is in the radial domain. For any fixed p, only $w^{(\pm 1+p)}(z)$ are adjacent to $w^{(p)}(z)$.

(ii) Suppose z is in the spiral domain. For any fixed p, $w^{(2n+1+p)}(z)$ are adjacent to $w^{(p)}(z)$ for all n. $w^{(2n+p)}(z)$ are on the adjacent contour emanating from $w^{(p)}(z)$ for all n.

A similar situation to the first case (i) has already appeared in Boyd [2]. The second situation seems new and very much different from the Gamma function case.

When $z = 0.662743..$ with the phase $\frac{1}{2}\pi$, all saddles are connected by one zig-zag steepest descent curve. This picture recall us of a double Stokes phenomenon ([8],[9]).

5 Examples of adjacent saddles

5.1 An example with z in the unbounded domain: $z = 2$

We take $z = 2$ as an example of a point in the unbounded domain

To verify adjacency of saddles, we don't need to check all angles. (It is impossible for the computer!) We have only to verify the steepest curves when $\theta = \mathrm{ph}(\ p^{(mn)})$. Notice that there exist "horizontal" steepest descent curves prevent from adjacency of saddles except the pair $w^{(n)}$ and $w^{(n\pm 1)}$. The reason why $w^{(n)}$ is adjacent only to $w^{(n\pm 1)}$ is seen by the fact that the point $z = 2$ is exactly on a Stokes line for $\pm\theta^{(n\pm 1,n)}$ i.e. ± 41.83 and ± 138.16 deg (rounded).

Here, we have approximately

$$\begin{aligned} w^{(0)} &= \log\frac{-1+\sqrt{5}}{2} = -0.481211825, \\ w^{(1)} &= \log\frac{1+\sqrt{5}}{2} + \pi i = 0.481211825 + 3.141592653i, \end{aligned}$$

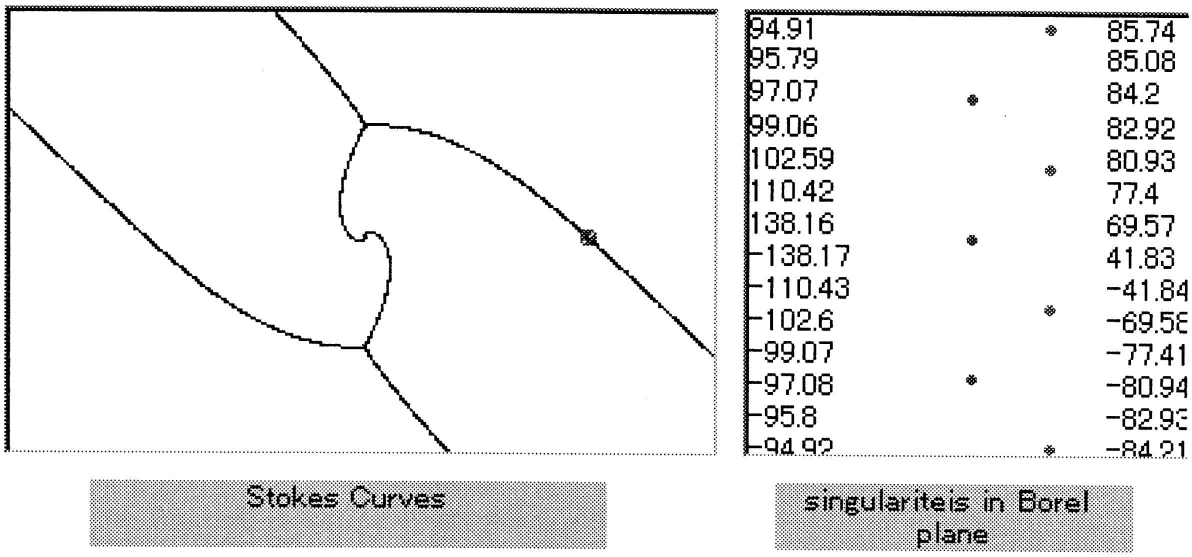

Figure 3: The left shows the point $z = 2$ is exactly on the radial Stokes line for $\theta = 41.84$ deg (rounded). The right is the image of the saddles on the Borel plane. The figures in the right box are degrees of the angle between the singularities.

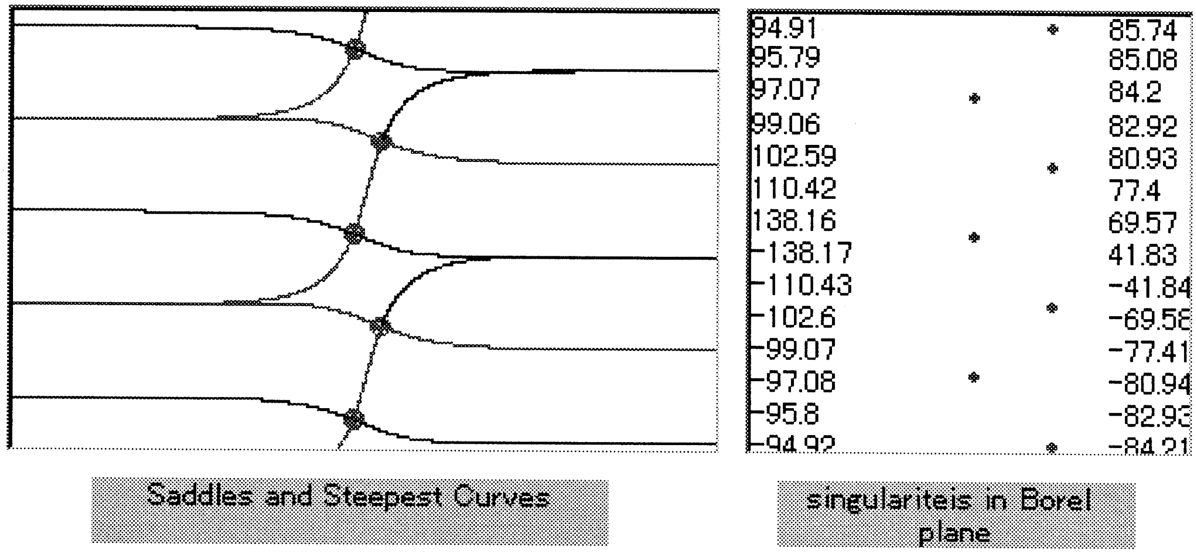

Figure 4: The left is a figure of saddles and steepest curves for $\theta^{(n,n-1)} = -41.84$ deg (round) when $z = 2$. The right is the image of the saddles on the Borel plane. The figures in the right box are degrees of the angle between the singularities.

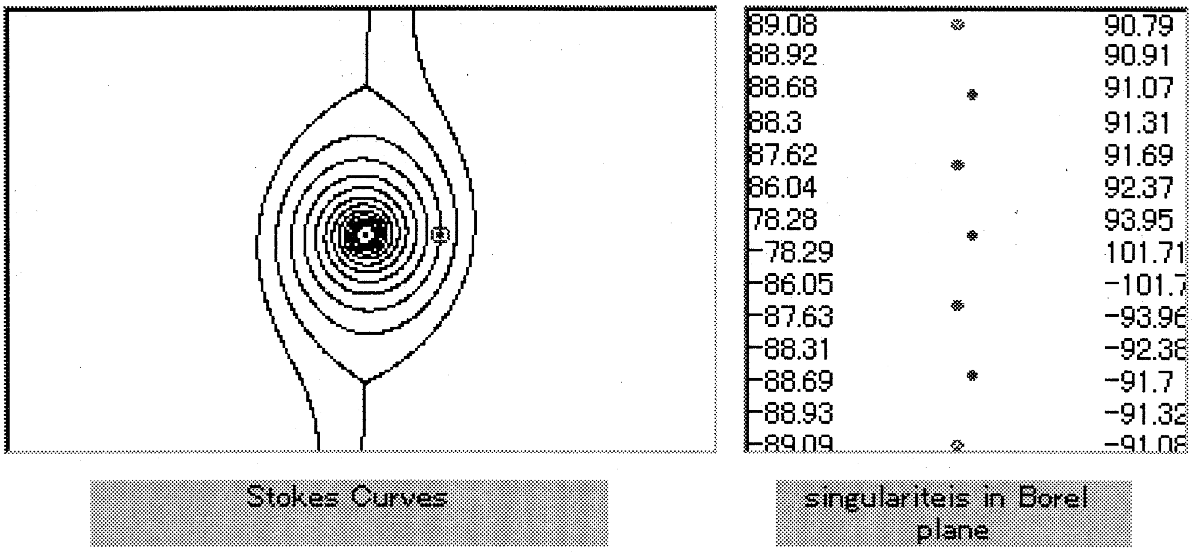

Figure 5: The left shows the point $z = 0.5$ is on the spiral Stokes line for $\theta^{(n,n+3)} = 86.04\,\mathrm{deg}$. The right is the image of the saddles on the Borel plane. The figures in the right box are degrees of the angles between the singularities.

$$
\begin{aligned}
p^{(0)} &= \log\frac{-1+\sqrt{5}}{2} + \sqrt{5} = 1.754856153, \\
p^{(1)} &= \log\frac{1+\sqrt{5}}{2} - \sqrt{5} + \pi i = -1.754856152 + 3.141592653i.
\end{aligned}
$$

5.2 An example when z in the bounded domain : $z = 0.5$

We take $z = 1/2$ as an example of a point in the bounded domain.

It seems that all angles ± 78.28, ± 86.04, $87.62\ldots.$ and the limit $\pm 90\,\mathrm{deg}$ give adjacency. Here, we have approximately

$$
\begin{aligned}
w^{(0)} &= \log(-2+\sqrt{5}) = -1.443635475, \\
w^{(1)} &= \log(2+\sqrt{5}) + \pi i = 1.443635475 + 3.141592653i, \\
p^{(0)} &= \log(-2+\sqrt{5}) + \frac{\sqrt{5}}{2} = -0.32560148, \\
p^{(1)} &= \log(2+\sqrt{5}) - \frac{\sqrt{5}}{2} + \pi i = 0.32560148 + 3.141592653i.
\end{aligned}
$$

We show the picture for $86.04\,\mathrm{deg}$.

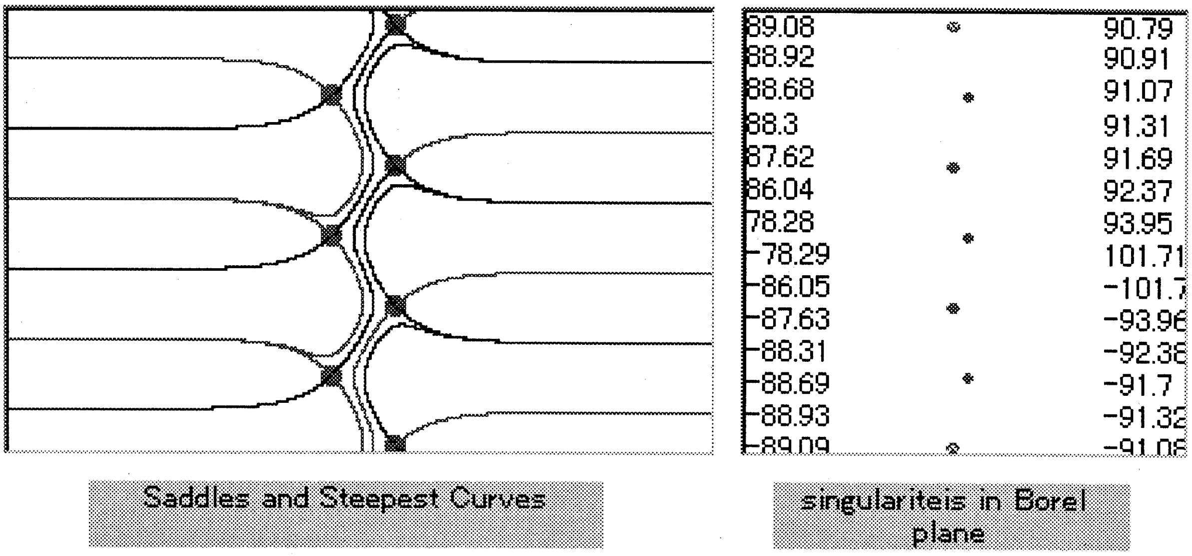

Figure 6: The left is a figure of saddles and steepest curves for $\theta^{(n,n+3)} = 86.04$ deg (roughly rounded). The steepest descents creep up from the saddles $w^{(n)}$ to hit the adjacent saddles $w^{(n+3)}$.

A Transcendental equations

Conjecture. Any two singulairities $p^{(n)}$ and $p^{(m)}$ for different n, m do not coalesce on the ξ-plane, when $z \neq \pm i$.

More precisely, we put cuts in z-plane from i to ∞ and from $-i$ to ∞ along the imaginary axis to uniformize $\sqrt{1+z^2}$ with $\sqrt{1} = 1$. We should consider the possibility $p^{(2n)} = p^{(2m+1)}$ for some integers n and m. It means

$$\log \frac{-1+\sqrt{1+z^2}}{-1-\sqrt{1+z^2}} + 2(n-m)\pi i + 2\sqrt{1+z^2} = 0,$$

which is equivalent to

$$2\sqrt{1+z^2} - 2 - z^2 = z^2 e^{-2\sqrt{1+z^2}}.$$

By change of variable $s = \sqrt{1+z^2}$, this is

$$(s^2-1)e^{-4s} + 2(s^2+1)e^{-2s} + (s^2-1) = 0,$$

which reduces to the equation

$$(s + \tanh s)(s - \tanh s) = 0. \tag{A.1}$$

Our conjecture is that $s = 0$ is a unique solution of (A.1)corresponding to $z^2 = -1$.

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