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An Overview of The Studies on Catalytic Stochastic Processes*

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1 Prologue

The aim of this expository article is to give an overview of comparatively recent results on critical continuous super-Brownian motions X in catalytic media, which have been studied and developed by chiefly D.A. Dawson and K. Fleischmann (cf. [DFG95]). Here catalytic media means that the branching rate ρ for measure-valued processes is given by a generalized function. The most typical example is an extremely simplified one-dimensional case, namely, $\rho=\delta_c$. In this case, branching occurs only at position $c\in\mathbf{R}$, in which there is the so-called single point catalyst with infinite rate; while outside c only the heat flow is predominant.

Generally, the superprocess is a measure-valued branching process, and it may be obtained from certain specific branching particle systems via high density limit procedure [D93]. It is known that many new distinct remarkable phenomena are observable for the single point catalytic super-Brownian motion (SBM).

(a) Jointly continuous super-Brownian motion local times

$$y:=\{y_t(a);\ t>0,\ a\in\mathbf{R}\}$$

exists, but $y(c):=\{y_t(c);\ t>0\}$ is only singularly continuous for the catalyst point c.

(b) The super-Brownian local time $y(c)$ at c can be alternatively constructed at the total occupation time measure of a one-sided super-$\frac{1}{2}$ stable motion on $\mathbf{R}_+$.

(c) With an excursion type formula, we can define the mass density field

$$x:=\{x_t(a);\ t>0,\ a\neq c\}$$

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of single point catalytic super-Brownian motion X by employing $y(c)$, with the result that we can show that it solves the heat equation and also that it is C^{∞}.

So much for interesting properties that the single point catalytic super-Brownian motion X possesses, another stimulating problem is the construction of higher dimensional catalytic super-Brownian motions with absolutely continuous states, in contrast to the constant medium case. In so doing, the main analytical tool would be the theory of nonlinear reaction diffusion equations in which δ-functions enter in various ways.

Remark. The intuitive interpretation behind the phenomenon described above in (a) is that the catalyst normally kills off the mass by the infinite branching rate, however, branching does occur occasionally, at exceptional times of full dimension.

2 From Branching Diffusion to SBM

2.1 The Feller Branching Diffusion

A. The BGW Process. First of all we consider the idealized simplest model where the space consists of a single point, say, $E = \{x\}$. This means that the positions of the particles are not distinguishable and their motions are completely neglected. Let

$$z^{(\rho)} = \left\{ z_t^{(\rho)};\ t \geq 0 \right\}$$

be a continuous-time critical binary Bienaymé-Galton-Watson (BGW) process with branching rate $\rho > 0$. The model starts at time $t = 0$, and $z_0 > 0$ indicates the number of initial particles of the model. Here particles evolve independently in accordance with the law that each particle dies with rate ρ and splits into two particles with rate ρ; i.e.,

$$1 \to \begin{cases} 0, & \text{with rate } \rho, \\ 2, & \text{with rate } \rho. \end{cases} \tag{1}$$

Note that this $z^{(\rho)}$ is nothing but a birth and death process.

B. The Scaled BGW Process. Now let $N \gg 1$ ($N \in \mathbf{N}$), and let $\zeta_0 > 0$ be fixed. Put

$$z_0 := [N\zeta_0].$$

This time, let the model start with z_0 initial particles, and give each particle the small mass $1/N$ (because N is a large number). We consider the situation of speeding up the process by switching ρ to the large branching rate $N\rho$. Then we define the scaled BGW process by

$$\left\{ \zeta_t^{(N)} \right\}_{t \geq 0} := \left\{ \frac{1}{N} z_t^{(N\rho)} \,|\, z_0 = [N\zeta_0] \right\}_{t \geq 0}.$$

Then this $\zeta^{(N)}$ determines a sequence of $\mathbf{R}_+$-valued continuous-time processes.

C. The Feller Branching Diffusion. We consider the limit procedure of the above-mentioned scaled model. It is well-known (cf. Feller (1951), [Fe51]) that by passage to the limit as $N \to \infty$, the scaled BGW model $\{\zeta_t^{(N)}\}$, $t \geq 0$ converges in distribution to the critical branching diffusion process $\zeta = \{\zeta_t\}$:

$$\zeta^{(N)} \Longrightarrow \zeta \qquad (N \to \infty).$$

The limiting process ζ is called Feller's branching diffusion. This diffusion process ζ on $\mathbf{R}_+$ can also be obtained as a solution of the stochastic differential equation of the form

$$d\zeta_t = \sqrt{2\rho\zeta_t} dB_t, \quad t > 0, \quad \zeta_0 \geq 0, \tag{2}$$

where $B = \{B_t\}$ is a one-dimensional standard Brownian motion (BM).

2.2 Super-Brownian Motion

D. A Prototype of Superprocess. Since the superprocess is a certain measure-valued process, adding a spatial concept we may introduce

$$X_t := \zeta_t \, \delta_0, \qquad t \geq 0.$$

The location 0 just corresponds to the single point of the space in question. In the above, thought out is a new notion to combine the population mass ζ_t together with the location. Note that there is no notational distinction between the δ-measure and the Dirac δ-function as its generalized derivative. We can interpret this X_t as a superprocess in zero dimensions. Symbolically, we may have the following formal expression

$$dX_t = \sqrt{2\rho X_t} \, dB_t, \quad t > 0, \quad X_0 \geq 0. \tag{3}$$

E. Branching BM in R^d and Diffusion Approximation. Let $\{z_t^{(\rho)}; t \geq 0\}$ be a critical binary BGW process given in "A" of §2.1. Suppose now that the particles act according to law of independent standard Brownian motions in $\mathbf{R}^d$. Here newly born particles start at their parents' position. Then we get the critical binary branching Brownian motion in $\mathbf{R}^d$ with branching rate ρ

$$\Phi_t^{(\rho)} := \sum_{i=1}^{z_t^{(\rho)}} \delta_{w_t^i}, \qquad t \geq 0,$$

where $z_t^{(\rho)}$ is the number of particles at time t, acting as the BGW process, and w_t^i denotes the position of the i-th particle at time t, which arises from some Brownian path where these paths are not independent any more. The state $\Phi_t^{(\rho)}$ is described in terms of counting measures at time t. We consider the same type of diffusion approximation. Let N ($\gg 1$) be the number of the initial particles situated at x, and let the model start with N particles.

Let each particle branch with the large rate $N\rho$ and possess the small mass $1/N$. That is to say, this leads to the scaled process

$$X_t^{(N)} := \left\{ \frac{1}{N} \Phi_t^{(N\rho)} \,|\Phi_0 = N\delta_x \right\}.$$

By passage to the limit $N \to \infty$, we obtain

$$X_t^{(N)} \Longrightarrow X_t, \qquad (N \to \infty)$$

(cf S. Watanabe (1968), [W68]), where the limiting process X_t is called the critical continuous super-Brownian motion with branching rate ρ (cf. Dawson-Watanabe superprocess [Dy94]). Consequently, the measure X_t describes the population at time t, and can be interpreted as a cloud of mass. Heuristically, $X = \{X_t\}$ can be regarded as a solution of symbolic stochastic equation

$$dX_t = \frac{1}{2} \Delta X_t dt + \sqrt{2\rho X_t} dW_t, \quad t > 0, \quad X_0 = \delta_x, \quad x \in \mathbf{R}^d, \tag{4}$$

where Δ is the Laplacian and $\dot{W}$ is a space-time white noise.

Remark. (i) The equation (4) consists of two components, in other words, it seems to be a combined version of Eq.(2) and Eq.(3). This suggests that, as to the $(2\rho X_t)^{1/2}\, dW_t$ part, the population grows at each point $x \in \mathbf{R}^d$ according to Feller's branching diffusion, while, as to the $(1/2)\, \Delta X_t\, dt$ part, the population mass is smeared out by the heat flow.
(ii) It is known that in the case $d = 1$, X lives on the space of absolutely continuous measures with probability one, i.e., there exists the density field $x_t(a)$ such that

$$X_t(da) = x_t(a) da, \qquad t > 0.$$

Then $x_t(a)$ solves the stochastic partial differential equation (SPDE):

$$dx_t(a) = \frac{1}{2} \Delta x_t(a) dt + \sqrt{2\rho x_t(a)}\, dW_t(a), \quad T > 0, \quad a \in \mathbf{R}. \tag{5}$$

Note that Δ acts on the space variable a only. In fact, Eq.(5) has a rigorous meaning for the one-dimensional case, which is due to e.g. Konno-Shiga(1988) [KS88].

3 Catalyst and Catalytic Media

3.1 Particular Situation in Irregular Media

Consider the SBM with branching rate ρ (= constant) > 0, which was introduced in the previous section. If this parameter ρ is not a constant any longer, and if it varies in space and take different values respectively at each point, namely,

$$\rho = \rho(a), \qquad a \in \mathbf{R}^d,$$

then the situation will be described by the so-called model in a varying medium. Then the situation implies that there exists some place in the model in which the branching occurs with a larger rate than at others. If a generalization of ρ is taken into consideration, it is possible to think of extensions of ρ to wider classes of functions, in accordance with the inclusion of functional spaces, such as

$$C^{\infty} \subset C^{m} \subset \mathcal{B},$$

where $\mathcal{B}$ denote the space of measurable functions. Furthermore, ρ may possibly be a generalized function. Such a model is called the model in an irregular medium. One of the typical examples in the irregular medium model is the case

$$\rho = \delta_c \quad (\textit{ a Ditac } \delta - \textit{function }).$$

That is to say, it exhibits the particular situation that branching occurs only at the single point c with an infinite rate [DFG95].

Definition 1 (Single Point Catalyst) *In this case, we say that a point catalyst is located at c.*

This single point catalyst controls the branching at c, whereas only the heat flow acts and is dominant outside c.

3.2 Approximation and Local Time

Now let us consider some approximation of a Dirac δ-function by step function φ_{ε} with width $\varepsilon > 0$ and height $1/\varepsilon$. As the particle level, this means that a particle may branch only if it lies within the ε-vicinity of the point c. Then the integral

$$\int_0^t \mathbf{I}\left\{ |w_s - c| \leq \frac{\varepsilon}{2} \right\} ds$$

represents the occupation time by time t. This gives us the interpretation that the particle will branch with rate $1/\varepsilon$ during that period. Then the Brownian local time $L^c(t)$ (at c) arises naturally in the limiting procedure [IW81]:

$$\frac{1}{\varepsilon}\int_0^t \mathbf{I}\left\{ |w_c - c| \leq \frac{\varepsilon}{2} \right\} ds \to L^c(t), \qquad (\varepsilon \to 0).$$

On this account, it seems quite reasonable to think of point catalysts only in dimension one, since Brownian local times at points make proper sense only in dimension one. (cf. [DFG95])

3.3 Studies on Catalysts

Some other related studies or fields on catalyst are the followings [DF97]:

1. Fractal catalysts from a physical point of view (Sapoval, 1991), where the objects are higher dimensional catalytic media.

2. Catalytic reaction diffusion equations via PDE method (Bramson-Neuhauser, 1992), (Chan-Fung, 1992), where the nonlinear differential equation of the form $-\partial_t u = (1/2)\Delta u + \rho_t \cdot R(u)$ is considered.

3. Catalytic chemical systems (Chadam-Yin, 1994), where various kinds of chemical reactions are considered.

4. Catalytic biological systems, where considered is the model of biochemical reactions that glycosis enzymes are serving as catalysts on a filament network.

One of the main reasons to study branching models in varying media, in particular, in irregular media is to search for new mathematical phenomena caused by the medium itself. Superprocess in irregular media was introduced by Dawson-Fleischmann-Roelly (1990), [DFR91]. The next theorem is a simple example of a new effect.

Theorem 1 *Let $X = \{X_t\}$ be a super-Brownian motion. The total mass process $\{X_t(R^d);\ t \geq 0\}$ is a Feller branching diffusion if and only if the medium is constant, i.e., $\rho =$ const.*

4 Single Point Catalytic SBM

We can observe many interesting and stimulating phenomena in the mathematical model of one-dimensional super-Brownian motion X with a single point catalyst $\rho = \delta_c$, which is the extremely simplified model among catalytic superprocesses. First of all, we observe some basic properties on the density field of single point catalytic SBM.

Let $c \in \mathbf{R}$ be the catalyst's position and fixed once and for all. A single point catalyst $\rho = \delta_c$ provides with the point-catalytic medium for superprocess. We denote by $M_F = M_F(\mathbf{R})$ the set of finite measures on $\mathbf{R}$. We consider below the one-dimensional SBM

$$X = \{X_t, \mathbf{P}_\mu, \mu \in M_F\}$$

with a single point catalyst $\rho = \delta_c$, where μ is the initial state X_0 of X, namely, $X_0 = \mu$ holds a.s., and $\mathbf{P}_\mu$ is the law of X. The existence of such a finite measure-valued Markov process X is intuitively based on the existence of Brownian local times in one dimension. There are two distinct methods to show its existence. One is a strong construction of X by regularization of δ_c, which is due to Dawson-Fleischmann (1991) [DF91]. The other is a quite different method to construct X by making use of the Brownian local time at c via additive functional approach, which is due to Dynkin (1991) [Dy91].

Let $C(\mathbf{R}_+, M_F)$ denote the space of weakly continuous finite measure-valued trajectories satisfying $X_t(\{c\}) = 0$ for all $t > 0$. $\mathcal{G}_+$ is the space of all non-negative continuous functions φ on $\mathbf{R}$ having a Gaussian decay, i.e.,

$$|\varphi(z)| \exp\{c_\varphi z^2\}, \qquad z \in \mathbf{R},$$

is bounded for some constant $c_\varphi > 0$. $\{S_t;\, t \geq 0\}$ denotes the Brownian semigroup, i.e., it is the Markov semigroup with generator $(1/2)\ \Delta$ and transition density $p = p(t,z)$. The next theorem asserts the path continuity of catalytic SBM.

Theorem 2 ([DF94], Theorem 1.2.1, p.6) *The time-homogeneous Markov process* $X = \{X_t, \mathcal{F}_t,\ t \geq 0,\ P_\mu,\ \mu \in M_F\}$ *determined by equation*

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,z) = \dfrac{1}{2}\Delta u(t,z) - \delta_c(z) u^2(t,z), & t > 0, \quad z \in R, \\ \qquad\qquad\qquad u(0,z) = \varphi(z), & z \in R, \quad \varphi \in \mathcal{G}_+, \end{cases} \tag{6}$$

via the Laplace transition functional

$$E\{\exp\langle X_t, -\varphi\rangle | X_s = \mu\} = \exp\langle \mu, -u(t-s)\rangle, \quad 0 \leq s \leq t, \ z \in \mathcal{G}_+, \ \mu \in M_F, \tag{7}$$

can be constructed on $C(R_+, M_F)$.

Moreover, the following expectation and covariance formulae hold.

Proposition 1 *For* $0 \leq s \leq t$, $\mu \in M_F$, $\varphi, \psi \in \mathcal{G}$,

$$E_\mu\langle X_t, \varphi\rangle = \langle \mu S_t, \varphi\rangle, \tag{8}$$

$$Cov_\mu[\langle X_s, \varphi\rangle, \langle X_t, \psi\rangle] = 2\int \mu(da) \int_0^s p(r, c-a) S_{s-r}\varphi(c) S_{t-r}\psi(c) dr. \tag{9}$$

The next result shows that the catalytic SBM $\{X_t;\, t > 0\}$ lives on the space of absolutely continuous measures, and also that the density field $\{x_t\}$ can be chosen properly to be jointly continuous on $\{t > 0\} \times \{z \neq c\}$.

Theorem 3 ([DF94], Theorem 1.2.2, p.7) *(a) There is a version of* X *(with* $\rho = \delta_c$, $c \in R$ *) such that there exists a sample jointly continuous random field* $x = \{x_t(z);\, t > 0,\, z \neq c\}$ *satisfying*

$$X_t(dz) = x_t(z)dz, \qquad for \quad all \quad t > 0, \quad P_\mu - a.s., \quad \mu \in M_F.$$

(b) The state x_t *at time* $t > 0$ *of the time-homogeneous Markov process* x *has the Laplace functions*

$$E_\mu \exp\left\{-\sum_{i=1}^k x_t(z_i)\theta_i\right\} = \exp\langle \mu, -u(t)\rangle, \qquad t > 0, \quad \theta_i \geq 0, \quad z_i \neq c, \quad 1 \leq i \leq k,$$

where u *(* ≥ 0 *) solves the equation*

$$\frac{\partial}{\partial t} u = \frac{1}{2}\Delta u - \delta_c u^2, \qquad u|_{t=0+} = \sum_{i=1}^k \theta_i \delta_{z_i}.$$

Moreover, for fixed $t > 0$ and $a, c \in \mathbf{R}$ the following estimate is obtained:

$$Var[x_t(z)|X_0 = \delta_a] \sim \text{const.}|\log|z-c|| \quad \text{as} \quad z \to c.$$

Thus immediately we get

Proposition 2 (Blow-Up of the Variance)

$$Var_\mu[x_t(z)] \to \infty, \qquad (z \to c).$$

The variance of the continuous density $x_t(z)$ blows up as z approaches the catalyst's position c. Roughly speaking, it suggests that the density of X_t is highly fluctuating in the vicinity of the catalyst. However, opposed to this, the following is true.

Theorem 4 (Vanishing Density at the Catalyst's Position) *For fixed $t > 0$, by passage to the limit $z \to c$,*

$$x_t(z) \to 0, \quad in \quad P_\mu - probability, \quad \mu \in M_F.$$

5 Super-Brownian Local Time

By the sample path continuity of the X-process stated in the previous section, we may introduce the occupation time process $Y = \{Y_t;\, t \geq 0\}$ related to X, that is,

$$\langle Y_t, \varphi \rangle := \int_0^t \langle X_s, \varphi \rangle ds, \qquad \varphi \in \mathcal{G}_+.$$

Since Y is smoother than X by the integration,

$$y_t(z) := \int_0^t x_t(z) ds, \qquad t \geq 0, \quad z \neq c, \tag{10}$$

is jointly continuous on $\mathbf{R}_+ \times \{z \neq c\}$, $\mathbf{P}_\mu$-a.s., $\mu \in M_F$. This y_t yields an occupation density field of Y, which is also called the super-Brownian local time (SBLT) related to X. Our main concern is to study the behavior of SBLT when approaching the catalyst's position. The following theorem implies the non-degeneracy of SBLT at c.

Theorem 5 ([DF94], Theorem 1.2.4, p.8) *(a) There is a version of X such that the occupation density field y of Y defined by Eq.(10) extends continuously to all of $R_+ \times R$. (b) Moreover, the following moment formulae hold for $0 \leq s \leq t < s' \leq t'$, $z, z' \in R$, $\mu \in M_F$:*

$$E_\mu[y_t(z)] = \int \mu(da) \int_0^t p(s, z-a) ds, \tag{11}$$

$$Var_\mu[y_t(z) - y_s(z)] = 2 \int \mu(da) \int_0^t d\tau p(\tau, c-a) \left\{ \int_{\tau \vee s}^t p(r-\tau, z-c) dr \right\}. \tag{12}$$

The expectation formula implies that even at the catalyst's position the occupation density $y_t(c)$ cannot be identically 0, which is in contrast to the a.s. vanishing random density $x_t(c)$ at c for fixed t in the sense of Theorem 4. Note also that the variance of y remains finite even at the catalyst's position, opposed to the blow-up effect described in Proposition 2.

6 Singularity at The Catalyst

The occupation density field y is monotone increasing in the time variable t, and hence for each $z \in \mathbf{R}$ it defines some locally finite continuous random measure λ^z on $\mathbf{R}_+$:

$$\lambda^z(dt) := dy_t(z), \qquad z \in R.$$

We call it the occupation density measure at z. By the definition (10), these measures λ^z are a.s. absolutely continuous as long as $z \neq c$.

Theorem 6 ([DFLM95], Theorem 1.1.4, p.38) *Assume that $X_0 = \delta_c$. The occupation density measure λ^c at the catalyst's position is with probability one a singular diffuse random measure on R_+.*

The approach to the above theorem, adopted in [DFLM95], is very unique. They first consider an enriched version of X, namely, the historical point catalytic super-Brownian motion $\tilde{X} = \{\tilde{X}_t;\ t \geq 0\}$. Here the state $\tilde{X}_t$ at time t keeps track of the entire history of the population masses alive at t and their family relationships. In addition, it arises as the diffusion limit of the reduced branching tree structure associated with the approximating branching particle system (cf. Dawson-Perkins (1991) [DP91]; and also see Dynkin (1991) [Dy91a]).

In this setting, the occupation density measure $\lambda^c(dr)$ is replaced by $\tilde{\lambda}^c\ (d[r, w])$, where

$$\tilde{\lambda}^c([r_1, r_2] \times B)$$

exposes the contribution to the occupation density increment $\tilde{\lambda}^c\ ([r_1, r_2])$ due to paths in the subset B of c-Brownian bridge paths w on $[0, r]$, which start at time 0 at c and also end up in c at time r, $(r_1 \leq r \leq r_2)$.

Remark. The above-mentioned c-Brownian bridge paths w on $[0, r]$ can be interpreted as the trajectories of particles in question that contributed to the occupation density increment $\lambda^c\ ([r_1, r_2])$.

We fix a closed finite time interval $I := [0, T]$, $0 < T < \infty$. For a path $w \in \mathcal{C} = C(I, \mathbf{R})$, we write $\mathcal{C}^t$ for the set of all stopped paths $\tilde{w}_t$. Given a path $w \in \mathcal{C}$, we interpret

$$\tilde{w} := \{\tilde{w};\ \ t \in I\ \}$$

as a path trajectory. In addition, we introduce the set

$$\mathcal{C}^{t,z} := \{w \in \mathcal{C}^t,\ \ w_t = z\ \}, \qquad t \in I, \quad z \in R,$$

of continuous paths on I stopped at time t at z. For $s \in I$, the starting measure $\mu \in M_F^s$ of $\tilde{X}$ at time s is a unit measure δ_{w^*} concentrated at $w^* \in \mathcal{C}^{s,c}$, a path stopped at time s at the catalyst. Also, let

$$\mathcal{C}^{r,c}_{s,w^*} := \{w \in \mathcal{C}^{r,c};\ w \quad \text{is} \quad \text{a} \quad \text{continuous} \quad \text{extension} \quad \text{of} \quad w^*\ \}$$

for $0 \le s \le t \le T$, $w^* \in \mathcal{C}^{s,c}$. We define a subset $A\hat{\times}B^{\cdot}$ of $A \times B$ by

$$A\hat{\times}B^{\cdot} := \{[a,b];\ a \in A,\ b \in B^a\} = \bigcup_{a \in A} \{a\} \times B^a.$$

Analogously to the standard theory (e.g. [DP91]), even in [DFLM95] the infinite divisibility of the law of the random measure $\tilde{\lambda}^c$ also allows us to use the framework of the so-called Lévy-Khintchine representation. That is,

Lemma 1 ([DFLM95], Lemma 3.3.1, p.47) *For $s \in I$ and $w^* \in \mathcal{C}^{s,c}$, there is a unique σ-finite measure Q_{s,w^*} defined on the set of all nonvanishing finite measures χ on $[s,T] \hat{\times} \mathcal{C}^{\cdot,c}_{s,w^*}$ such that*

$$\tilde{P}_{s,w^*} \exp\langle \tilde{\lambda}^c_{s,T}, -\psi \rangle = \exp\left\{ - \int (1 - \exp\langle \chi, -\psi \rangle) Q_{s,w^*}(d\chi) \right\}, \tag{13}$$

for $\psi \in \mathcal{B}([s,T]\hat{\times}\mathcal{C}^{\cdot,c}_{s,w^*}, R)$.

Moreover, we may disintegrate the Lévy-Khintchine measure Q_{s,w^*} relative to its intensity measure $\bar{Q}_{s,w^*}$ (cf. Lemma 3.3.6 in [DFLM95]) to obtain its Palm distribution $Q^{r,w}_{s,w^*}(d\chi)$. Roughly speaking, $Q^{r,w}_{s,w^*}(d\chi)$ is the law of a canonical cluster χ. Then it is easy to show that a Palm representation formula holds in terms of the Brownian local time measure L^c (w, dt) at c of the given bridge path w. That is to say,

Theorem 7 ([DFLM95], Theorem 3.3.9, p.48) *Let $s \in I$ and $w \in \mathcal{C}s, c$. For $\bar{Q}_{s,w^*}$-almost all $[r,w] \in [s,T] \hat{\times} \mathcal{C}^{\cdot,c}_{s,w^*}$, the Palm distribution $Q^{r,w}_{s,w^*}$ has Laplace functional*

$$\int \exp\langle \chi, -\psi \rangle Q^{r,w}_{s,w^*}(d\chi) = \exp\left\{ -2 \int_s^r u_{\psi,c}(t, w_{\cdot\wedge t}, T) L^c(w, dt) \right\}, \tag{14}$$

for $\psi \in \mathcal{B}_+([0,T]\hat{\times}\mathcal{C}^{\cdot,c}, R)$, where $u_{\psi,c}(\cdot,\cdot,T)$ is the unique bounded non-negative solution of a historical version of the nonlinear singular equation

$$-\frac{\partial}{\partial r} u = \frac{1}{2}\Delta u + f\delta_c - \delta_c u^2, \qquad r > 0.$$

On this account, Theorem 6 can be readily obtained by employing the aforementioned results. More precisely, it is interesting to note that the key step of the proof of Theorem 6 is to demonstrate that the random measure $\chi(d[r', w'])$ distributed according to $Q^{r,w}_{s,w^*}$ has with probability 1 at the Palm point r an infinite left upper density with respect to the Lebesgue measure dr' (cf. Theorem 4.2.2, p.52, [DFLM95]). The mathematical tools exploited in [DFLM95] are of strong independent interest for the author in connection with historical stochastic calculus (e.g. [P95]).

So much for Theorem 6, we lastly introduce another different behavior of the super-Brownian local time measure λ^a at the catalyst c. Recall the definition of the Hausdorff-Besicovitch dimension $d^* = \dim(A) \in [0,1]$ of a subset A of $\mathbf{R}$. It is defined by the requirement that

$$\liminf_{\delta \to 0+} \left\{ \sum_k (\operatorname{diam}(B_k))^\rho;\ \bigcup_k B_k \supset A,\ \operatorname{diam}(B_k) < \delta \right\}$$

equals $+\infty$ for $\rho \in (0, d^*)$ wheras it vanishes for $\rho \in (d^*, 1]$. Here $\{B_k\}$ is a countable covering of A by closed intervals B_k with diameter smaller than δ. Then the following result holds:

Theorem 8 ([DF94], Theorem 1.2.5, p.9) *Assume $X_0 = \delta_c$. The occupation density measure λ^c at the catalyst's position has a.s. carrying Hausdorff-Besicovitch dimension one.*

Consequently, the super-Brownian local time is singular continuous at c (cf. Theorem 6). Recall that this is in a sharp contrast to the constant medium case Eq.(5) in Section 2. But nevertheless production of population mass occurs on a time set of full dimension (cf. Theorem 8).

Remark. It is quite interesting to compare the result obtained in Theroem 8 with the usual Brownian local time, which determines a singular random measure with carrying dimension 1/2. See, for instance, Itô-McKean (1974) [IMc74; §2.5, pp.50-54.].

7 Total Mass Extinction

In [FLG95] Fleischman and Le Gall (1995) has proposed a new approach to SBM X with a single point catalyst δ_c as branching rate, and has proved that the occupation density measure λ^c of X at the catalyst c is distributed as the total occupation time measure of U, and also that X_t is determined from λ^c by an explicit representation formula, where U is a superprocess with constant branching rate and spatial motion by the 1/2-stable subordinator. Moreover, a new derivation of the singularity of the measure λ^c is provided in [FLG95] as well.

Recall that the stable subordinator with index 1/2 is the Lévy process on the real line whose transition probabilities are given by

$$q(s, b) := \mathbf{I}_{\{b>0\}} \frac{s}{\sqrt{2\pi b^3}} \exp\left\{-\frac{s^2}{2b}\right\}, \qquad s > 0, \quad b \in \mathbf{R}.$$

Notice that $q(s, \cdot)$ can also be interpreted as the density function of the first hitting time of the point s by a linear Brownian motion started at the origin. The next result is a representation of the mass density field x via the SBLT measure λ^c.

Theorem 9 ([FLG95], Theorem 1 (b), p.67) *With P_{δ_c}-probability one the mass density field x can be represented as*

$$x_t(a) = \int_0^t q(t - s, |a - c|)\lambda^c(ds), \qquad t > 0, \quad a \neq c. \tag{15}$$

Now assume for the moment that X starts off at time 0 with the Lebesgue measure denoted by dm, namely, $X_0(dz) = m(dz)$. Then we already know that X_t suffers local extinction. That is, as $t \to \infty$,

$$\langle X_t, \varphi \rangle \to 0 \qquad \text{stochastically,} \quad \text{for each} \quad \varphi \in \mathcal{G}_+.$$

In fact, according to [DF94], we have

Proposition 3 ([DF94], proposition 1.3.1, p.11) *For all $\varphi \in \mathcal{G}_+$, we have $\int u(t,z)dz \to 0$ as $t \to \infty$, where u (≥ 0) is the solution to the integral equation*

$$u(t,z) = S_t\varphi(z) - \int_0^t p(t-r, c-z)u^2(r,c)dr, \qquad t \geq 0, \quad z \in R.$$

Actually, the representation formula Eq.(15) in the above theorem is a very powerful tool and has interesting applications. For example, the next result is a complement to the above-mentioned local extinction proposition by a total extinction property, which is due to [FLG95].

Proposition 4 (Total Mass Extinction) *(a) The total mass of X at time t is expressed by*

$$X_t(R) = \int_0^t \sqrt{\frac{2}{\pi(t-s)}} \lambda^c(ds).$$

(b) This total mass is strictly positive for every $t \geq 0$ a.s. and $X_t(R)$ converges to 0 in P_{δ_c}-probability as $t \to \infty$.

That is to say, although $X_t(\mathbf{R}) > 0$ holds for $t \geq 0$, a.s., the probability that some total mass survives as $t \to \infty$ becomes very small.

8 Support Property

We consider the closed supports of the states of super-Brownian motions X in catalytic media. It is known (e.g. [Is88]) that the support property is valid in the constant medium case, i.e.,

Theorem 10 (Iscoe, 1988) *Let X be a super-Brownian motion without catalyst in the constant medium (i.e., the case $\rho = const.$). If the initial mesure $X_0 = \in M_F$ has compact support, then so too does X_t $(t > 0)$, whatever the dimension is.*

On the contrary, for catalytic SBM we have

Theorem 11 (Dawson-Mueller, 1993) *Let X be a single point catalytic SBM. If $X_0 \neq 0$, then the support of X_t, $t > 0$ is the whole space R.*

The above theorem indicates that the compact support property is obviously violated. The question arises whether one can formulate criteria for the compact support property to hold for super-Brownian motions in catalytic media.

9 Epilogue

Dawson and Fleischmann (1997) have recently studied a catalytic SBM in a super-Brownian medium. As a matter of fact, in [DF97] a continuous super-Brownian motion X^ρ is constructed in which branching occurs only in the presence of catalysts which evolve themselves as a continuous super-Brownian motion ρ with constant branching rate. More precisely, there the Brownian collision local time plays an important role, that is, the collision local time $L_{[W,\rho]}$ of an underlying Brownian motion path W with the catalytic mass process ρ governs the branching of new system. Furthermore, in the one-dimensional case, new types of limit behaviors are discovered. In fact, almost sure convergence of the total mass process is proved with preservation of the mean and also with a non-degenerate limit, and for the catalytic SBM starting with a Lebesgue measure m, stochastic convergence of X^ρ to m is proved as well when time t tends to infinity. For the details, see [DoKj99] which is an expository article of [DF97].

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