数理解析研究所講究録
1089 巻 1999 年 42-60

A Version of Evans-Perkins Type Stochastic Representation Formula for Historical Superprocesses*

Isamu DÔKU (道工 勇)
Department of Mathematics, Saitama University
Urawa 338-8570, Japan

1 Introduction

The purpose of this article is to introduce a version of Evans-Perkins type stochastic representation formula for a generalized $\{\gamma, a, b, g\}$-historical superprocess (see the definition in §2). Here by Evans-Perkins type formula we mean an explicit stochastic integral representation for historical functional of a certain class, which is similar to and is a historical process counterpart of Itô-Clark formula (e.g. [U95, p.42]) in elementary stochastic calculus. The key idea of demonstration of the Itô-Clark type formula for historical superprocess is to derive a variant of stochastic integration by parts with respect to the historical process in the Perkins sense [P92].

The review of the Evans-Perkins theory [EP95] is a good point to start. There are two reasons why their integration by parts formula is so important. For one thing, it can provides with a new formula of transformations of stochastic integrals closely connected with the so-called historical processes. In addition, a generalization of formula itself is of independent interest, and it is very useful as a theoretical tool of stochastic calculus in the theory of measure-valued processes. For another, it has an extremely remarkable meaning on an applicational basis. By making use of the formula S.N. Evans and E.A. Perkins (1995) have succeeded in deriving a kind of Itô-Wiener chaos expansion for functionals of superprocesses [EP95].

S.N. Evans and E.A. Perkins have showed that any L^2 functional of superprocess may be represented as a constant C_0 plus a stochastic integral with respect to the associated orthogonal martingale measure M (e.g. [EP94]). Recently they have obtained the explicit representations involving multiple stochastic integrals for a quite general functional

*Research supported in part by JMESC Grant-in-Aids SR(C) 07640280 and CR(A) 09304022.

of the so-called Dawson-Watanabe superprocesses. Actually, the results are obtained in the setting of the historical process associated with the superprocess [EP95].

2 Notation and Preliminaries

Let $C = C^d = C([0,\infty),\mathbf{R}^d)$ denote the space of $\mathbf{R}^d$-valued continuous paths on $\mathbf{R}_+ = [0,\infty)$ with the compact-open topology. $\mathcal{C} = \mathcal{B}(C)$ is its Borel σ-field and

$$\mathcal{C}_t = \mathcal{B}_t(C) = \sigma(y(s),\ s \le t)$$

denotes its canonical filtration. For $y, w \in C^d$ and $s \ge 0$, we define the stopped path by $y^s(t) = y(t \wedge s)$ and let

$$y/s/w = \begin{cases} y(t), & \text{for} \quad t < s, \\ w(t-s), & \text{for} \quad t \ge s. \end{cases} \tag{1}$$

$M_F(C)$ is the space of finite measures on C with the topology of weak convergence and we define

$$M_F(C)^t := \left\{ m \in M_F(C);\ \ y = y^t,\ \ m - a.s.\ y \right\}, \qquad t \ge 0.$$

If P_x denotes Wiener measure on $(C, \mathcal{B}(C))$ starting at x, $\tau \ge 0$, and $m \in M_F(C)^\tau$, define $P_{\tau,m} \in M_F(C)$ by

$$P_{\tau,m}(A) := \int_C P_{y(\tau)}(\{w;\ \ y/\tau/w \in A\}) dm(y).$$

Let

$$\Omega_H[\tau,\infty) := \left\{ H \in C([\tau,\infty), M_F(C));\ \ H_t \in M_F(C)^t,\ \ \forall t \ge \tau \right\},$$

and put $\Omega_H := \Omega_H[0,\infty)$. We write $\mathcal{H}$ for the totality of Borel sets of Ω_H. We use the notation $H_t(\omega) = \omega(t)$ for $\omega \in \Omega_H$ as for the canonical realization of historical process.

Fix $0 \le t_1 < \cdots < t_n$ and $\psi \in C_b^2(\mathbf{R}^{nd})$. For $y \in C$ we set

$$\begin{aligned} \bar{y}(t) &= (y(t \wedge t_1), \cdots, y(t \wedge t_n)), \\ \bar{\psi}(y) &\equiv \bar{\psi}(t_1, \cdots, t_n)(y) = \psi(y(t_1), \cdots, y(t_n)), \end{aligned}$$

and $\tilde{\psi}(t,y) = \bar{\psi}(y^t)$. ψ_i (resp. ψ_{ij}) stands for the first (resp. second) order partials $\partial_i\psi$ (resp. $\partial^2_{ij}\psi$) of ψ. $\nabla\bar{\psi} : [0,\infty) \times C \to \mathbf{R}^d$ is the $(\mathcal{C}_t)$-predictable process whose j-th component at (t,y) is given by

$$\sum_{i=0}^{n-1} \mathbf{I}(t < t_{i+1}) \psi_{id+j}(\bar{y}(t)).$$

While, for $1 \le i, j \le d$, $\bar{\psi}_{ij} : [0,\infty) \times C \to \mathbf{R}$ is the $(\mathcal{C}_t)$-predictable process defined by

$$\bar{\psi}_{ij}(t,y) := \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \mathbf{I}(t < t_{k+1} \wedge t_{l+1}) \partial_{kd+i}\partial_{ld+j}(\bar{y}(t)).$$

Let us define the domains

$$\begin{aligned} D_0 &:= \bigcup_{n=1}^{\infty} \left\{\bar{\psi}(t_1,\cdots,t_n);\ 0 \le t_1 < \cdots < t_n,\ \bar{\psi} \in C_0^{\infty}(\mathbf{R}^{nd})\right\} \bigcup \{1\}, \\ \tilde{D}_0 &:= \left\{\tilde{\psi};\ \tilde{\psi}(t,y) = \bar{\psi}(y^t) \quad \text{for some} \quad \bar{\psi} \in D_0\right\}. \end{aligned}$$

Let $\bar{\Omega} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge\tau}, \mathbf{P})$ be a filtered probability space and let $(\omega, y) = (\omega, y_1, \cdots, y_d)$ denote sample points in $\hat{\Omega} = \Omega\times C^d$. Here $\tau \ge 0$ is fixed. When f is a function on $[\tau, \infty) \times \hat{\Omega}$ taking values in a normed linear space $(E, \|\ \|)$, then a bounded $(\mathcal{F}_t)$-stopping time T is a reducing time for if and only if

$$\mathbf{I}(\tau < t \le T)\, \|f(t,\omega,y)\|$$

is uniformly bounded. In addition we say that a sequence $\{T_n\}$ reduces f if and only if each T_n reduces f and $T_n \nearrow \infty$ holds $\mathbf{P}$-a.s. We say that f is locally bounded if such a sequence $\{T_n\}$ exists. We assume that

(LB) $\gamma \in [0,\infty), a \in S^d, b \in \mathbf{R}^d$ and $g \in \mathbf{R}$ are $(\hat{\mathcal{F}}_t^*)$-predictable processes on $[\tau, \infty) \times \hat{\Omega}$ such that $\Lambda = (\gamma, a, b,\ g\gamma^{-1}\ \mathbf{I}(\hat{g} \neq))$ is locally bounded.

Notice that the above assumption implies that g is locally bounded.

Now we introduce the martingale problem formulation of historical processes in stochastic calculus on historical trees (cf. [P92], [P95]). For $\tau \ge 0$ and $m \in M_F(C)^\tau$, we define

$$A_{\tau,m}\tilde{\psi}(t,y) \equiv A(\tilde{\psi})(t,y) := \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(t,,y)\tilde{\psi}_{ij}(t,y) + b(t,\omega,y)\cdot\nabla\tilde{\psi}(t,y) + g(t,\omega,y)\bar{\psi}(y^t)$$

for $\bar{\psi} \in D_0$. We write $\langle \mu, f\rangle$ or sometimes $\mu(f)$ for the integral $\int f d\mu$ when μ is a measure and f is a suitable μ-integrable function. Suggested by [DkTn98], we may define

Definition 1 (cf. [P95], §2) *A predictable process $K = \{K_t, t \ge \tau\}$ on $\bar{\Omega}$ with sample paths a.s. in $\Omega_H[\tau,\infty)$ is a generalized $\{\gamma,\ a, b,\ g\}$-historical process (GHP) (or $(A,\ -\gamma\lambda^2/2)$-historical process) if and only if $K_t \in M_F(C)^t$ for all $t \ge \tau$, a.s. and $P[K_\tau(1)] < \infty$, and if there exists a probability measure $\mathcal{P}$ on $\Omega_H[\tau,\infty)$ such that it satisfies the martingale problem (MP) with initial data $\{\tau, m\}$ and $\{\gamma, a, b, g\}$: for $\forall\ \bar{\psi} \in D_0$,*

$$Z_t(\bar{\psi}) = \langle K_t, \bar{\psi}\rangle - \langle m, \bar{\psi}\rangle - \int_\tau^t \langle K_s, A(\tilde{\psi})(s)\rangle ds, \quad t \ge \tau, \tag{2}$$

is a continuous $(\mathcal{F}_t)$-local martingale satisfying $Z_\tau(\bar{\psi}) = 0$ and

$$\langle Z(\bar{\psi})\rangle_t = \int_\tau^t \int \gamma(s,\omega,y)\bar{\psi}(y)^2 K_s(dy)ds, \qquad \forall t \ge \tau, \quad a.s.$$

Remark. The existence and uniqueness of the law of K is essentially due to [F88] (cf. [DIP89]).

Set $T_s = [s, \infty)$, and in particular $T_0 = [\tau, \infty)$. Define $C(M_F(C)) := C(T_0; M_F(C))$, and we write $C(t) = (\tau, t] \times C$ for the integral domain. When $\mathcal{F}$ is the σ-field or the usual filtration, then $f \in \mathcal{F}$ indicates that the function f is $\mathcal{F}$-measurable and $\mathcal{P}(\mathcal{F})$ is the totality of $(\mathcal{F})$-predictable functions, and $b\mathcal{P}(\mathcal{F})$ denotes the whole space of functions that are all bounded elements of $\mathcal{P}(\mathcal{F})$. We use the symbol $U(M_F(C))$ for an admissible subset of the space $C(C(M_F(C)); \mathbf{R})$; more precisely $U(M_F(C))$ is the totality of real valued continuous functions F on $C(M_F(C))$ such that for some compactly supported finite measure $L(dt)$ on T_0, the estimate

$$|\Delta F(h, g)| \leq \int_{T_0} g(t, C) L(dt)$$

holds for all $h, g \in C(M_F(C))$, where we define $\Delta F(x, y) := F(x + y) - F(x)$.

3 Predictable Representation Property

Let $\{T_N\}$ be a reducing sequence. Take a sequence $\{\bar{\psi}_n\}$, $\bar{\psi}_n \in D_0$ such that $\bar{\psi}_n$ converges bounded pointwise (*bp* for short) to ψ, namely,

$$\bar{\psi}_n \to \psi, \quad bp \quad (n \to \infty).$$

An application of dominated convergence theroem together with the local boundedness of γ implies that

$$\langle Z(\bar{\psi}_n - \bar{\psi}_m) \rangle_t \to 0 \quad \text{as} \quad n, m \to \infty$$

for $\forall\, t \geq \tau$, a.s. Therefore we obtain

Proposition 1 *There is an a.s. continuous adapted process $\{Z_t(\psi);\, t \geq \tau\}$ such that*

$$\sup_{\tau \leq t \leq N} \left| Z_t(\bar{\psi}_n) - Z_t(\psi) \right| \to 0$$

holds in probability (w.r.t. P) as $n \to \infty$ for $\forall N > \tau$.

To proceed our discussion, we need the following lemmas.

Lemma 1 (cf. Corollary 2.2, p.11, [P95]) *Let T be a reducing time for (γ, g). Then we have*
(a) $0 < P[K_T(1)] \leq P[\sup_{\tau \leq t \leq T} |K_t(1)| + \langle Z(1) \rangle_T] < \infty$.
(b) If $P[K_\tau(1)^p] < \infty$ for $p \in N$, then

$$P\left\{ \left(\sup_{\tau \leq t \leq T} |K_t(1)| \right)^p + \langle Z(1) \rangle_T^{\frac{p}{2}} \right\} < \infty.$$

Lemma 2 (cf. [EP94, p.123]) *D_0 is dense in $b\mathcal{B}(C)$ relative to the bounded pointwise convergence topology.*

We may use Lemma 1 to obtain

$$\sup_{\tau \le t \le T_N} |Z_t(\bar{\psi}_n) - Z_t(\psi)| \to 0 \quad \text{in} \quad L^2$$

as $n \to \infty$, for $\forall N \in \mathbf{N}$. Clearly $Z_t(\psi)$ is a continuous $(\mathcal{F}_t)$-local martingale whose quadratic variation process is given by

$$\langle Z(\psi) \rangle_t = \int_\tau^t \int_C \gamma(s, \omega, y) \psi(y)^2 K_s(dy) ds. \tag{3}$$

By virtue of Lemma 2, it is a routine work to show that this Z_t extends to an orthogonal martingale measure

$$\{ Z_t(\psi); \ t \ge \tau, \ \psi \in b\mathcal{B}(C) \ \}.$$

Consequently, the mapping $t \mapsto Z_t(\psi)$ is a continuous local martingale satisfying Eq.(3) for eaxh $\psi \in b\mathcal{B}(C)$, and $\psi \mapsto Z_{t \wedge T_N}(\psi)$ is an L^2-valued measure on $\mathcal{B}(C)$ for each $t \ge \tau$, $N \in \mathbf{N}$. By a trivial localization argument, we may define the stochastic integral

$$Z_t(\psi) = \int_\tau^t \int \psi(s, \omega, y) dM(s, y) \tag{4}$$

($\exists$ an orthogonal martingale measure $M = M^K$ in the sense of Walsh [W86, Chapter 2]) such that

$$\langle Z(\psi) \rangle_t = \int_\tau^t \langle K_s, \gamma(s, \omega) \psi(s, \omega)^2 \rangle ds, \tag{5}$$

$\forall t \ge \tau$, a.s., as long as ψ belongs to $L^2_{loc}(K, \mathbf{P})$. Here $L^2_{loc}(K, \mathbf{P})$ denotes the L^2 space of $(\mathcal{F}_t \times \mathcal{C})_{t \ge \tau}$-predictable functions f and

$$\int_\tau^t \int \gamma(s, y) f(s, y)^2 K_s(dy) ds < \infty$$

for $\forall t \ge \tau$, $\mathbf{P}$-a.s.

We write $f \in L^2(K, \mathbf{P})$ (resp. $L^2_\infty(K, \mathbf{P})$) if, in addition,

$$\mathbf{P}\left\{ \int_\tau^t \int \gamma(s, \omega, y) f(s, \omega, y)^2 K_s(dy) ds \right\} < \infty, \qquad \forall t > 0,$$

respectively,

$$\mathbf{P}\left\{ \int_\tau^\infty \int \gamma(s, \omega, y) f(s, \omega, y)^2 K_s(dy) ds \right\} < \infty.$$

Theorem 1 (Predictable Representation Property) *If $V \in L^2(\Omega, \mathcal{F}, P)$, then there is an f in $L^2_\infty(K, P)$ such that*

$$V = P[V] + \int_\tau^\infty \int f(s, \omega, y) dM^K(s, y), \qquad P - a.s. \tag{6}$$

The proof of Theorem 1 will be given in the suceeding section.

Remark. The predictable representation property was proved by Evans-Perkins (1994) [EP94, Theorem 1.1] for the $(Y, -\lambda^2/2)$- superprocess with a Hunt process Y as its underlying process. In [EP95] a variant of the stochastic integral representation formula of the above type was proved for the $(Y, -\lambda^2/2)$- historical process with a Markov process Y.

4 Proof of Theorem 1

If $f \in b\mathcal{B}(C)$, then the moment $\mathbf{P}[K_t(f)]$ is uniformly bounded as t ranges over a compact subset of $[\tau, \infty)$. We have the following explicit formula for the moment, namely,

Lemma 3 $P[K_t(f)] = P_{\tau,\nu}[f(Y^t)]$ *holds for every* f *in* $b\mathcal{B}(C)$ *under* $\nu \in M_F(C)^\tau$, *where* Y^t *is the corresponding stopped path-valued process.*

We set $\hat{E} := \{(s, y) \in [\tau, \infty) \times C;\ y^s = y\ \}$ and define a measure $Q_{s,y}$ on $(C, \mathcal{C})$ by

$$Q_{s,y}(A) := P_{y(s)}\{w \in C;\ (y/s/w) \in A\ \}, \quad A \in \mathcal{C}, \quad (s, y) \in \hat{E}.$$

Then a similar argument as in [F88] (cf. Theorem 2.1.3, [DP91]) allows us to show

Proposition 2 *Assume that* $T_{s,t}f(y) := P_{y(s)}[f(y/s/Y^{t-s})]$ *satisfies the semigroup property for* $(s, y) \in \hat{E}$, $t \geq s$, *and* $f \in b\mathcal{B}(C)$. *Then we have*

$$P[\exp\{-\langle K_t, f\rangle\}] = \exp\{-\langle m, V_{\tau,t}f\rangle\},$$

for all $f \in bp\mathcal{B}(C)$ *and* $m \in M_F(C)$. *Moreover,* $\{V_{\tau,t}\}$ *forms a semigroup on* $bp\mathcal{C}$, *and* $V_{s,t}f(y) \equiv v_{s,t}(y)$ *is Borel measurable as a function of* (s, y, t) *in* $\hat{E} \times [\tau, \infty)$ *with* $t \geq s$, *and is the unique solution of*

$$v_{s,t}(y) = P_{y(s)}[f(y/s/Y^{t-s})] - \frac{1}{2}\int_\tau^{t-s} P_{y(s)}\left[\gamma(u, \omega, y)v_{u+s,t}(y/s/Y^u)\right] du.$$

Proof of Lemma 3. According to the same discussion as in Theorem 2.1.5 [DP91, p.19], we can deduce from Proposition 2 that under $\nu \in M_F(C)^\tau$

$$\mathbf{P}[\langle K_t, f\rangle] = \langle \nu, G_{\tau,y}f\rangle, \qquad ------(*)$$

where $G_{s,t}f(y) = Q_{s,y}[f(Y_t)]$. A simple computation reads

$$\begin{aligned}
\langle \nu, G_{\tau,t}f\rangle &= \int_C Q_{\tau,y}[f(Y_t)]\nu(fy) \\
&= \int_C \left\{\int_{C^t} f(Y_t)P_{y(\tau)}\{w \in C;\ (y/\tau/w) \in d\zeta\}\right\}\nu(dy) \\
&= \int_{C^t} f(Y^t)\int_C P_{y(\tau)}\{(y/\tau/Y) \in d\zeta\}\nu(dy) \\
&= \int_{C^t} f(Y^t)P_{\tau,\nu}(dy) = P_{\tau,\nu}[f(Y^t)],
\end{aligned}$$

because we made use of the Fubini theorem in the second line. By (*), this concludes the proof. Q.E.D.

Suggested by the argument [MP92, pp.331-332] (also see [EP95, pp.1779-1780]), we define

$$\begin{aligned}
F^{\tau,\nu} &:= \{\varphi \in b(\mathcal{B}([\tau, \infty)) \times \mathcal{C}); \varphi(t, y) = \varphi(t, y^t) \text{ for all } t \geq \tau, \\
&\text{the map } t \mapsto \varphi(t, Y) \text{ is } P_{\tau,\nu} - a.s. \text{ right continuous, } \forall t \geq \tau\}
\end{aligned}$$

under $\nu \in M_F(C)^t$, and $\tilde{F}^{\tau,\nu}$ is the set of bounded functions ψ in $\mathcal{B}([\tau,\infty)) \times \mathcal{F} \times \mathcal{C}$ such that

$$\psi(\cdot,\omega,\cdot) \in F^{\tau,\nu}, \quad P-\text{a.s.},$$

and the condition (C) is compatible with the definition of K in §2.

(C) For $H_t \in M_F(C)^t$, **P**-a.s. for all $t \geq \tau$ with Y as its corresponding path-valued process, and for all $\varphi \in F^{\tau,\nu}$,

$$M_t(\varphi) := \langle H_t, \varphi(t,\cdot)\rangle - \langle \nu, \varphi(\tau,\cdot)\rangle - \int_{(\tau,t]} \langle H_s, \psi(s,\omega,\cdot)\rangle ds, \quad t \geq \tau, \quad \text{under } \nu \in M_F(C)^\tau,$$

is a continuous $(\mathcal{F}_t)_{t\geq\tau}$ martingale for which $M_\tau(\varphi) = 0$ and

$$\langle M.(\varphi)\rangle_t = \int_{(\tau,t]} \int_C \gamma(s,\omega,y)\varphi(s,y)^2 H_s(dy)ds.$$

Let $A^{\tau,\nu}$ denote the set of pairs (φ,ψ) in $F^{\tau,\nu} \times \tilde{F}^{\tau,\nu}$ such that

$$Z_t := \varphi(t,Y) - \varphi(\tau,Y) - \int_{(\tau,t]} \psi(s,Y)ds, \qquad t \geq \tau,$$

is a $(\bar{\mathcal{C}}_t^\nu)_{t\geq\tau}$-martingale under $P_{\tau,\nu}$, where $\bar{\mathcal{C}}_t^\nu$ is the σ-field generated by $\mathcal{C}_{t+}$ and the $P_{\tau,\nu}$-null sets in C.

Proposition 3 *There exists for each $n \in N$ a function $g_n = g_n(t,\omega,y)$ in $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$ such that*

$$V = P[V] + \lim_{n\to\infty} \int_{(\tau,\infty)} \int_C g_n(s,\omega,y)dM^K(s,y),$$

with $L^2(P)$-convergence.

Proof. Recall the condition (C). By virtue of Theorem 2 and Proposition 2 of Jacod (1977) [J77] (e.g. [EP94, p.124] or [EP95, p.1796]), we can deduce that for each $n \in \mathrm{N}$ there exist suitable pairs

$$(\varphi_n^1,\psi_n^1),\cdots,(\varphi_n^{N(n)},\psi_n^{N(n)}) \in A^{\tau,m},$$

(relative to K_t), $\xi_n^1,\cdots,\xi_n^{N(n)} \in b\mathcal{P}(\mathcal{F}_t)$, and $\{t_n\}_n \subset (\tau,\infty)$ such that $t_n \nearrow \infty$ (as $n \to \infty$) and

$$V = \mathbf{P}[V] + \lim_{n\to\infty} \int_{(\tau,t_n]} \int_C \sum_k \xi_n^k(s,\omega)\varphi_n^k(s,y)dM^K(s,y),$$

where the convergence is in $L^2(\mathbf{P})$. Moreover, we can choose a bounded $(\mathcal{C}_t)_{t\geq\tau}$- predictable function η such that

$$\int\int_{C(t)} \xi(s,\omega)\varphi(s,y)dM^K(s,y) = \int\int_{C(t)} \xi(s,\omega)\eta(s,y)dM^K(s,y), \qquad P-a.s., \quad \forall t \geq \tau,$$

for each $(\varphi, \psi) \in A^{\tau,m}$ and each ξ in $b\mathcal{P}(\mathcal{F}_t)$, and also that the y-section

$$\{(s,y) \in [\tau,\infty) \times C;\ \ \varphi(s,y) \neq \eta(s,y)\ \}$$

is a countable set. By the property of stochastic integral and the Fubini type theorem, we readily obtain

$$\begin{aligned} P\left|\int\int_{C(t)} \xi\varphi dM^K - \int\int_{C(t)} \xi\eta dM^K\right|^2 &= P\left[\int\int_{C(t)} \gamma\xi(s)^2\{\varphi(s,y) - \eta(s,y)\}^2 dK_s ds\right] \\ &\leq C_0 \cdot P\left[\int_\tau^t \int_C \{\varphi(s,y) - \eta(s,y)\}^2 K_s(dy) ds\right] \\ &= C_0 \int_\tau^t P\left[K_s(|\varphi_s - \eta_s|^2)\right] ds. \end{aligned}$$

for some constant C_0. By Lemma 3, the last term in the above can be replaced by

$$\int_\tau^t P_{\tau,m}\left[|\varphi(s,Y^s) - \eta(s,Y^s)|^2\right] ds,$$

which, indeed, becomes null if we apply the Fubini theorem again because we employed the condition

$$\int_{(\tau,t]} \{\varphi(s,Y) - \eta(s,Y)\}^2 ds = 0, \qquad \forall t > \tau,\ \ P_{\tau,m} - a.s.$$

So that, by making use of the above-mentioned η, we have only to set

$$g_n(s,\omega,y) = \sum_k \xi_n^i(s,\omega)\eta_n^i(s,y)$$

for each n. This completes the proof. Q.E.D.

By virtue of the arguments in the proof of Proposition 3, we have that

$$\begin{aligned} 0 &= \lim_{n,k\to\infty} \left\|\int\int_{C(\infty)} g_n(s,\cdot,y) dM^K(s,y) - \int\int_{C(\infty)} g_k(s,\cdot,y) dM^K(s,y)\right\|^2_{L^2(P)} \\ &= \lim_{n,k\to\infty} \|g_n(\cdot) - g_k(\cdot)\|^2_{L^2_\infty(K,P)}. \end{aligned}$$

Hence there exists a limit function f in $L^2_\infty(K,\mathbf{P})$ such that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mathbf{P}\left[\int_\tau^\infty \int_C \gamma(s,\omega,y)\{g_n(s,\omega,y) - f(s,\omega,y)\}^2 K_s(dy) ds\right] \\ &= \lim_{n\to\infty} \mathbf{P}\left|\int\int_{C(\infty)} g_n(s,\omega,y) dM^K(s,y) - \int\int_{C(\infty)} f(s,\omega,y) dM^K(s,y)\right|^2. \end{aligned}$$

Immediately this implies from Proposition 3 that

$$\begin{aligned} V &= \mathbf{P}[V] + \lim_{n\to\infty} \int\int_{C(\infty)} g_n(s,\omega,y) dM^K(s,y) \\ &= \int\int_{C(\infty)} f(s,\omega,y) dM^K(s,y). \end{aligned}$$

which completes the proof of Theorem 1.

5 Canonical Measure and Campbell Measure

For $y \in D = D(\mathbf{R}_+; \mathbf{R}^d)$, we define $y^{t-}(s)$ as $y(s)$ itself if $s < t$ and as $y(t-)$ if $s \geq t$. $Q(s, y)$ is a σ-finite measure on $C(M_F(D))$ such that

$$Q\Big(s, y^{s-};\ \{h \in C(M_F(D));\ \tau \leq \exists t \leq s,\ \ h(t) \neq 0\}\Big) = 0,$$

which can be defined by the canonical measure $R(\tau, t, y; d\zeta)$ [D93] associated with the law of $K_t = K(t)$ and the path restriction mapping π (cf. §2, pp.1781-1782 in [EP95]) together with a discussion involved with the Dawson-Perkins theory(1991) (e.g. Theorem 2.2.3(pp.27-28) and Proposition 3.3(pp.38-39) in [DP91]). Here R is characterized by

$$\log \mathcal{P}_{s,\delta_y}[\exp\langle K_t, -\varphi\rangle] = \int_{M_F(M_F(C))} \left(\mathrm{e}^{-\langle \zeta, \varphi \rangle} - 1\right) R(s, t, y; d\zeta)$$

(cf. Lemma 1 in [Dk99c]; see also [DP91, Proposition 3.3, pp.38-39]). Let F be a real valued Borel function on $C(M_F(C))$. Assume that

$$I^Q_{s,y}[\Delta F](h) := \int_{C(M_F(C))} \Delta F(h, g) Q(s, y^{s-}; dg) \tag{7}$$

is well-defined and bounded below for all $s > \tau$, $y \in C$, and $h \in C(M_F(C))$. For a bounded $(\mathcal{F}_t)$-stopping time T, we define the Campbell measure P_T associated with $K(t)$ by

$$P_T(A \times B) := \mathbf{P}(K(T, A) \cdot \mathbf{I}_B\{K(T)\})/m(C) \tag{8}$$

for any $A \times B \in (C \times \Omega, \mathcal{C} \times \mathcal{F})$ (cf. [P95], p.21; or [DP91], p.62). Notice that $K_\tau = m$. Since the mapping $(s, y, \omega) \mapsto I^Q_{s,y}[\Delta F](K(\omega))$ is bounded below and measurable with respect to the product of the predictable σ-field associated with the filtration $(\mathcal{C}_t)$ and the σ-field $\mathcal{F}$, we can apply Lemma 2.2(p.1783) [EP95] together with the projection operation argument and the predictable section theorem (e.g. Theorem 2.14(p.19) or Theorem 2.28(p.23), [JS87]; see also [E82], pp.50-52), to deduce that there exists a $(\mathcal{C}_t \times \mathcal{F}_t)_{t \geq \tau}$-predictable function $Pr[F](s, y, \omega) : (\tau, \infty) \times C \times \Omega \to \mathbf{R}$ such that

$$P_T\{I^Q[\Delta F](T)/(\mathcal{C} \times \mathcal{F})_T\} = Pr[F](T, \omega, y) \tag{9}$$

holds P_T-a.s. for all bounded $(\mathcal{F}_t)$-predictable stopping times $T > s$. It is quite interesting to note that in particular

$$\mathbf{P} \int_C I^Q[\Delta F](T, y) K(T, dy) = \mathbf{P} \int_C Pr[F](T, y) K(T, dy).$$

We shall introduce an approximation map. For each $l \in \mathbf{N}$, let us choose a partition $\Delta(l) = \{t^{(l)}(j);\ 1 \leq j \leq k[l]\}$ such that $\tau = t^{(l)}(0) < t^{(l)}(1) < \cdots < t^{(l)}(k[l]) < \infty$,

$$\lim_{l \to \infty} \{\sup_k \Delta t[l; k]\} = 0 \ \text{ and } \ \lim_{l \to \infty} t^{(l)}(k[l]) = +\infty.$$

The approximation map $W[l]$ from $C(M_F(C))$ into $C(M_F(C))$ is defined by

$$W[l](g)(t) := \{Sb(t^{(l)}(i+1)) \cdot g(t^{(l)}(i)) - Sb(t^{(l)}(i)) \cdot g(t^{(l)}(i+1))\}\Delta t[l;i]^{-1}$$

if $t \in [t^{(l)}(i), t^{(l)}(i+1))$, and $:= g(t^{(l)}(k[l]))$ if $t \geq t^{(l)}(k[l])$, for any element g of $C(M_F(C))$ with $Sb(k) = k - t$. Immediately we get

Lemma 4 (cf. Lemma 4, [DK98a]) *Let F be an element of $C(C(M_F(C)); R)$. Then for all $g \in C(M_F(C))$*

$$\lim_{l \to \infty} (F \circ W[l])(g) = F(g).$$

6 Random Measures and Assumptions

We shall introduce the assumptions for our main results (Theorem 2, Theorem 3 and Theorem 4) which are stated in the succeeding section. C^t denotes the image of C under the map: $y \mapsto y^t$. We define a measure $K^*[s,t]$ on C^s by $K^*[s,t](F) := K_t(\{y: y^s \in F\})$. Then the measure $K^*[s,t]$ is atomic with a finite set of atoms, and we write $L[s,t](\subset C^s)$ for the locations of these atoms. For $s \in (a,b]$, let $\lambda_s[\varphi]$ be the random measure on C that places mass $\varphi(s,y)$ at each point y in $(L[b,c])^s = L[s,c]$. With some localization arguments in stochastic calculus, the Perkins-Girsanov theorem of Dawson type [P95] guarantees the existence of a probability measure $\mathbf{Q}_N$ on $(\Omega, \mathcal{F})$ such that

$$\left.\frac{d\mathbf{Q}_N}{d\mathbf{P}}\right|_{\mathcal{F}_t} = \exp\left\{ \int_\tau^{t\wedge T_N} \int g\gamma^{-1}(s)\mathbf{I}(g(s) \neq 0) dM^K(s,y) - \frac{1}{2}\int_\tau^{t\wedge T_N} \int g^2\gamma^{-1}(s)\mathbf{I}(g(s)\neq 0)K_s(dy)ds \right\}.$$

For brevity's sake we rather write $\mathcal{E}(t\wedge T_N)$ than the above. On this account, $K_{\cdot\wedge T_N}$ satisfies the martingale problem (MP)$[\gamma_N, a_N, b_N, 0]$ instead of (MP)$[\gamma, a, b, g]$, where we set $f_N := f \cdot \mathbf{I}(\tau < t \leq T_N)$. Moreover, for $s \in (a,b]$, $y \in C^s$, the symbol $\mathcal{M}[s,y]$ denotes the mapping of the set of functions $\{m : (\tau, \infty) \to M_F(C)\}$ into itself and is defined as follows: i.e., $\{\mathcal{M}[s,y]m\}_t(F)$ is equal to $m_t(F)$ if $t < s$, or is equal to $m_t(\{y' \in F : (y')^s \neq y\})$ if $t \geq s$.

Let us now introduce assumptions for our principal results.

(A.1) $g : [\tau, \infty) \times \Omega \times C \to \mathbf{R}$ is a $(\mathcal{F}_t \times \mathcal{C}_t)^*$-predictable process such that $g\gamma^{-1} \cdot \mathbf{I}(g \neq 0)$ is locally bounded.

(A.2) For any predictable function f on $[\tau, \infty) \times I \times C^* \times \Omega$, the counting measure n^* satisfies

$$\mathbf{P}\int_{C^*} n^*((s,t] \times I) G_t(dx) = m(C^*)(t-s)$$

where G_t is a marked historical process corresponding to K and N_t is the martingale measure associated with G_t (cf. §7 for details).

(A.3) There exists a random measure Λ_φ on $(\tau, \infty) \times C$ such that

$$\int\int_{C(\infty)} f(s,y)\Lambda_\varphi(ds \otimes dy) = \int_{a+}^{b}\int_C f(s,y)\lambda_s[\varphi](dy)ds$$

holds for any suitable predictable function f.

(A.4) $\Psi(s,y)\mathcal{E}(t \wedge T_N)^{-1}$ is uniformly bounded in s, K_s-a.e. y, $\mathbf{Q}_N$-a.s.

(A.5) There exists some constant C_0 (> 0) such that

$$\int\int_{C(t)} \Psi(s,y)^2 \ \mathcal{E}(t \wedge T_N)^{-2} \ \gamma(s,y) \ K_s(dy)ds \leq C_0$$

holds $\mathbf{Q}_N$-a.s., for all $t \geq \tau$.
Note that we shall assume (A.1)-(A.5) hereafter all through the whole paper.

7 Stochastic Integration Formulae : Main Results

The followings are our main results in this paper. The first one is a finite dimensional version of Evans-Perkins type stochastic integration by parts formula. Let K be a predictable measure-valued process whose law is specified by a general martingale problem (MP)$[\tau, K_\tau, \gamma, a, b, g]$.

Theorem 2 (cf. [Dk98b]) *Assume that $\Phi : C(M_F(C)) \to R$ is a cylinder function with bounded representing function $\varphi : [M(C)]^k \to R$ and base $\tau < t(1) < \cdots < t(k)$, such that*

$$|\Delta\varphi(\alpha,\beta)| \leq c_0 \sum_j \beta_j(C)$$

for some positive constant c_0, for all α, $\beta = (\beta_j) \in [M(C)]^k$. Then for $t > \tau$

$$P\left\{\Phi(K)\int\int_{C(t)} \Psi(s,y)dM^K(s,y)\right\} = P\int\int_{C(t)} Pr[\Phi](s,y)\Psi(s,y)\gamma(s,y)K_s(dy)ds$$

holds where Ψ is a bounded $(\mathcal{C}_t \times \mathcal{F}_t)_{t\geq\tau}$-predictable function, K_t is a GHP, and $Pr[\Phi]$ is a predictable function determined by (9) in accordance with the given Φ.

Remark 1. The assertion of the above theorem is quite similar to Theorem 2.4(p.1785, §2, [EP95]).

Theorem 3 (Stochastic Integration By Parts) *Let $F \in U(M_F(C))$. If Ψ is an element of $b\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$, then for all $t > s$,*

$$P\left\{F(K)\int\int_{C(t)} \Psi(s,y) \ dM^K(s,y)\right\} = P\int\int_{C(t)} Pr[F](s,y)\gamma(s,y)\Psi(s,y)K_s(dy)ds. \quad (10)$$

Remark 2. Note that it is not hard to extend the assertion in Theorem 2 to the case of a more general functional $F(K)$. As a matter of fact, once the integral formula as given in Theorem 2 is established, it is a kind of routine work to generalize it(cf. §3, [Dk98a]). We shall refer to this generalization in §9.

Theorem 4 (A Variant of Evans-Perkins Type Formula) *Let* $F \in U(M_F(C))$.

$$F(K) = P[F(K)] + \int_{\tau+}^{\infty} \int Pr[F](s,y) dM^K(s,y) \tag{11}$$

where $Pr[F](s,y)$ *is a* $\mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$*-measurable version (relative to* P_T*) of*

$$P_T \left[\int_{C(M_F(C))} \Delta F(K,h) Q(s, y^{s-}; dh) \; / \; (\mathcal{D} \times \mathcal{F})_T \right].$$

8 Marked Historical Processes and Girsanov-Dawson -Perkins Theorem

Set $I = [0,1]$, $E^* = C \times I$ and $C^* = C(\mathbf{R}_+, E^*)$, and let $\mathcal{C}^*$ (resp. $\mathcal{C}_t^*$) be the Borel σ-field (resp. the canonical filtration) of C^*. Put $x = (y,n) \in E^*$. Let G be the corresponding counterpart historical process of K starting at (τ, μ), defined on the stochastic basis $(\Omega, \mathcal{H}, \mathcal{H}_t, \mathbf{P}^*)$. Suppose that $\varphi : (\tau, \infty) \times C \times \Omega \to I$ be an element of $\mathcal{P}(\mathcal{C}_t \times \mathcal{H}_t)$. Given any cadlag function $n : \mathbf{R}_+ \to I$, we can construct a σ-finite counting measure n^* on $\mathbf{R}_+ \times I$ by assigning an atom of mass one to each point (s,z) such that $n(s) - n(s-) = z \neq 0$. Put

$$A(t,x,\omega) := n^*(\{(s,z) \in [\tau, t) \times I; \quad \varphi(s,y,\omega) > z\}) \tag{12}$$

and $B(t,x,\omega) = \mathbf{I}\{A(t,x,\omega) = 0\}$. Then we can define an $M_F(C)$-valued process $K[\varphi](t)$ by

$$K[\varphi; J](t) := \int_{C^*} \mathbf{I}\{J\}(y) B(t,x) G_t(dx). \tag{13}$$

Put

$$I_1(\varphi, N) = \int \int_{C^*(t)} \varphi(s,y) dN(s,x), \quad \text{and} \quad I_2(\varphi, G) = \int \int_{C^*(t)} \gamma(s,y) \varphi(s,y)^2 G_s(dx) ds$$

with $C^*(t) = (\tau, t] \times C^*$. Then we define

$$\Lambda[\varphi](t) := \exp\Big\{ I_1(\varphi, N) - \frac{1}{2} I_2(\varphi, G) \Big\}. \tag{14}$$

Note that $\Lambda[\varphi](t)$ is a $\mathcal{H}_t$-martingale. The new probability space $(\Omega, \mathcal{H}, \mathbf{P}^*[\varphi])$ is defined by $\mathbf{P}^*[\varphi]\{F\} := \mathbf{P}^*\{F \cdot \Lambda[\varphi](t)\}$ (cf. [Dk98a]) for any $F \in b\mathcal{H}_t$ with

$$\mathcal{H} := \bigvee_{t \geq \tau} \mathcal{H}_t \tag{15}$$

(see Theorem 2.1(pp.125-126) and Theorem 2.3b(p.127), [EP94]). It is easy to show the following proposition if we apply Dawson's Girsanov theorem [D93] (see also [P95]).

Proposition 4 (cf. Theorem 5.1, p.1798, [EP95]) *The law of* $K[\varphi]$ *under* $P[\varphi]$ *is equivalent to the law of* K *under* P.

9 Sketch of Proofs of Main Theorems

§9.1 Generalization of the Cylinder Function Case: Proof of Theorem 3

As mentioned in Remark 2 of §7, the essential part of an extension of the Evans-Perkins type integration formula is compressed into the study on its finite dimensional case, namely, Theorem 2. The general case easily follows from a kind of routine work [Dk98a]. We define a real valued function L^* on $C(M_F(C))$ by

$$L^*[g] := \int_{T_0} g(t,C)L(dt) = \langle L, g(\cdot,C)\rangle. \tag{16}$$

In connection with the measure L (see §2), we introduce the finite measure $L(l) \equiv L(l,dt)$ which concentrates its mass on $\{t^{(l)}(j);\ 0 \le j \le k[l]\}$ (cf. [Dk98a, p.5]). We have $(L^* \circ W[l])[g] = \langle L(l), g(\cdot,C)\rangle$ for $g \in C(M_F(C))$. Recall that

$$\int g(t,C)\, Q(s,y;dg) = \int \xi(C)\, R(s,t,y;d\xi) = 1$$

holds (cf. Lemma 3, [Dk99a]) with ease for $s < t$ from Lemma 3.4(pp.41-43), [DP91]. Then it is easy to verify the followings:

$$\mathbf{P}\int\int_{C(t)} \{Q(s,y^{s-})L^*[g]\}K_s(dy)ds = \lim_{l\to\infty} \mathbf{P}\int\int_{C(t)} \{Q(s,y^{s-})(L^*\circ W[l])[g]\}\, K_s(dy)ds$$

holds with $g \in C(M_F(C))$ for all $t > \tau$, and

$$\begin{aligned} \mathbf{P} & \int\int_{C(t)} Pr[F](s,y)Z(s,y)K_s(dy)ds \\ = & \lim_{l\to\infty} \mathbf{P}\int\int_{C(t)} Pr[F\circ W[l]](s,y)Z(s,y)\, K_s(dy)ds. \end{aligned} \tag{17}$$

holds for all $t > \tau$ if $Z \in \mathcal{P}(\mathcal{C}_t \times \mathcal{F}_t)$. Since, for each $n \ge 1$, $\mathbf{P}\{K_t(C)^n\}$ is uniformly bounded on compact intervals, we can readily deduce that $\mathbf{P}\{(L^*\circ W[l])[K]^n\}$ is bounded in l for each $n \ge 1$. Moreover,

$$\mathbf{P}\left\{F(K)\int\int_{C(t)} \Psi(s,y)\, dM(s,y)\right\} = \lim_{l\to\infty} \mathbf{P}\left\{\left(F\circ W[l]\right)(K)\int\int_{C(t)} \Psi(s,y)\, dM(s,y)\right\}.$$

To complete the extension discussion in this section we have only to observe that $F\circ W[l]$ satisfies all the conditions of Theorem 2 (cf. Lemma 22, pp.9-10, [Dk98a]). Thus we have a finite dimensional special case of stochastic integration by parts formula related to historical processes as far as Proposition 4 in §8 is valid. Hence, combining the above results, we obtain

$$\begin{aligned} \mathbf{P}\left\{F(K)\int\int_{C(t)} \Psi(s,y)\, dM\right\} &= \lim_{l\to\infty} \mathbf{P}\left\{\left(F\circ W[l]\right)(K)\int\int_{C(t)} \Psi(s,y)\, dM\right\} \\ &= \lim_{l\to\infty} \mathbf{P}\int\int_{C(t)} Pr[F\circ W[l]]\ \gamma(s,y)\Psi(s,y)\, K_s(dy)ds \\ &= \mathbf{P}\int\int_{C(t)} Pr[F](s,y)\ \gamma(s,y)\Psi(s,y)\, K_s(dy)ds, \end{aligned}$$

which concludes Theorem 3.

§9.2 Stochastic Integration by Parts: Proof of Theorem 2

Since the complete proof is longsome and tiresome, computation in details will be sacrificed for the sake of simplicity and clearness. The basic idea is due to §7 in [Dk99a].

Thanks to (A.1), it suffices to verify the integral formula for a special $\{\gamma_N, a_N, b_n, 0\}$-historical process $K_{\cdot\wedge T_N}$ under $\mathbf{Q}_N$ instead of the generalized K (GHP) with $\mathbf{P}$. Indeed, since $d\mathbf{P} = \mathcal{E}(t\wedge T_N)^{-1}d\mathbf{Q}_N$, what we have to show is as follows:
(*The Modified Stochastic Integration By Parts Formula*)

$$\begin{aligned}
&\mathbf{Q}_N \left\{\mathcal{E}(t\wedge T_N)^{-1}\cdot\Phi(K_{\cdot\wedge T_N})\int\int_{C(t)}\Psi(s,y)dM(s,y)\right\}\\
&= \mathbf{Q}_N\left\{\mathcal{E}(t\wedge T_N)^{-1}\int\int_{C(t)} Pr[\Phi](s,y)\,\gamma(s,y)\,\Psi(s,y)K_{s\wedge T_N}(dy)ds\right\}.
\end{aligned}$$

Note that both sides above are well-defined by virtue of (A.4). Notice that Eq.(12)-(14) remains valid even for $\varphi = \Psi\cdot\mathcal{E}^{-1}$. Hence, by the auguments on exponential martingale formalism for the historical process, $\Lambda[\Psi\cdot\mathcal{E}^{-1}](t)$ is a $\mathcal{H}_t$-martingale and the measure $\mathbf{Q}_N[\Psi\cdot\mathcal{E}^{-1}]$ is given by $\mathbf{Q}_N[\{\cdot\}\Lambda[\Psi\cdot\mathcal{E}^{-1}]]$. Then it follows from Dawson's Girsanov theorem (Proposition 3 in §8) that, for any positive ε,

$$\mathbf{Q}_N\{\Phi(K_{\cdot\wedge T_N})\} = \mathbf{Q}_N[\varepsilon\Psi\mathcal{E}^{-1}]\left\{\Phi(K_{\cdot\wedge T_N}[\varepsilon\Psi\mathcal{E}^{-1}])\right\}.$$

Immediately,

$$\begin{aligned}
&\mathbf{Q}_N\left\{\Phi(K_{\cdot\wedge T_N})\cdot(\Lambda[\varepsilon\Psi\mathcal{E}^{-1}](t)-1)\right\}\\
+\;&\mathbf{Q}_N\left\{\left(\Phi(K_{\cdot\wedge T_N}[\varepsilon\Psi\mathcal{E}^{-1}])-\Phi(K_{\cdot\wedge T_N})\right)\cdot(\Lambda[\varepsilon\Psi\mathcal{E}^{-1}](t)-1)\right\}\\
=\;&\mathbf{Q}_N\left\{\Phi(K_{\cdot\wedge T_N})-\Phi(K_{\cdot\wedge T_N}[\varepsilon\Phi\mathcal{E}^{-1}])\right\}.
\end{aligned}$$

For simplicity we denote by I_1 (resp. I_2) the first (resp. second) term at the left hand side of the above equality, and put

$$I_3 = \text{the right hand side with the minus sign.}$$

Then we find that the convergence

$$\varepsilon^{-1}\cdot(\Lambda[\varepsilon\Psi\mathcal{E}^{-1}](t)-1)\to\int\int_{C(t)}\Psi(s,y)\;\mathcal{E}(t\wedge T_N)^{-1}\;dM(s,y),\qquad \mathbf{Q}_N - a.s.\quad(\varepsilon\to 0)$$

is true (cf. Lemma 8, [Dk99a]). Hence we readily obtain

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\,I_1 = \mathbf{Q}_N\left\{\Phi(K_{\cdot\wedge T_N})\cdot\int\int_{C(t)}\Psi(s,y)\;\mathcal{E}(t\wedge T_N)^{-1}\;dM(s,y)\right\}.$$

Paying attention to the fact that

$$\lim_{\varepsilon\downarrow 0} K^*[\varepsilon\Psi\mathcal{E}^{-1}; C](t) = 0, \qquad \mathbf{Q}_N - a.s.,$$

we can show that $\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} I_2 = 0$, as well.

It remains to treat the third term I_3. In order to discuss the convergence of I_3 divided by ε, we need the following:

Key Lemma (cf. Lemma 12, [Dk99a])

$$\begin{aligned}\mathbf{Q}_N \int\int\big\{\Phi(\mathcal{M}[s,y]K_{\cdot\wedge T_N}) &- \Phi(K_{\cdot\wedge T_N})\big\}\Lambda_{\Psi\cdot\mathcal{E}^{-1}}(ds\otimes dy) \\ &= - \mathbf{Q}_N \int\int Pr[\Phi]\gamma(s,y)\Psi(s,y)\mathcal{E}^{-1}(t\wedge T_N)dK_{s\wedge T_N}(y)ds.\end{aligned}$$

On the other hand, for $\varepsilon > 0$ we have

$$\begin{aligned}&\mathbf{Q}_N[\Phi(K[\varepsilon\varphi]) - \Phi(K)\ /\ \mathcal{F}] \\ = \ &\varepsilon\cdot e^{-\varepsilon\Lambda_\varphi((\tau,\infty)\times C)}\int\int_{C(\infty)}\{\Phi(\mathcal{M}[s,y]K) - \Phi(K)\}\Lambda_\varphi(ds\otimes dy) + R(\varepsilon,\Phi,\varphi) \quad (18)\end{aligned}$$

where the residue function R satisfies $|R(\varepsilon,\Phi,\varphi)| \le o(\varepsilon)$. From (18) we get the convergence

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1} I_3 = -\mathbf{Q}_N\int\int_{C(t)} Pr[\Phi]\ \gamma(s,y)\cdot\Psi\ \mathcal{E}^{-1}\ dK_{s\wedge T_N}ds. \qquad (19)$$

In fact, a simple application of the above-mentioned Key Lemma yields the required result. To complete the proof, we have only to combine the above results.

§9.3 Cluster Representation Argument: Proof of Key Lemma

For the proof of Key Lemma, although it is very technical, we are based on the cluster representation argument [D93] (see also [DP91]). For the details, we refer to the arguments stated in §8 in [Dk99a]. The following lemmas are merely essential parts of the discussion.

For any $y \in C^s$, $R(s,t,y)$ denotes the canonical measure (cf §5) in the theory of cluster random measures (e.g. [D93], [DP91]). Actually, R is a σ-finite measure such that

$$R(s,t,y;M_F(C)) = r_{s,t}.$$

Here the crucial point is that the total mass $r_{s,t}$ does not depend on y. So $r_{s,t}^{-1}dR(s,t,y)$ becomes a probability measure. It is interesting to note that K_t is a sum of independent nonzero clusters with laws $r_{s,t}^{-1}R(s,t,y;dh)$, conditional on $L[s,t]$ (see §6). Furthermore, conditional on $\mathcal{F}_s$, $L[s,t]$ can be regarded as a Poisson point process with intensity $r_{s,t}\gamma(s)K_s$. This is one of the most important points for the computation in terms of clusters growing from the points of $L[s,t_{l+1}]$ in what follows. We define a measure S by the following equation: for $\forall g \in b\mathcal{B}([M_F(C)]^{k-l} \to \mathbf{R})$,

$$\begin{aligned}\int g(\eta_{l+1},\cdots &, \eta_k)S_{s,y}(d\eta_{l+1}\otimes\cdots\otimes d\eta_k) \\ &= \int g(h(t_{l+1}),\cdots,h(t_k))\cdot\mathbf{I}\{h(t_{l+1})\neq 0\}Q(s,y;dh)\end{aligned}$$

where $Q(s,y;dh)$ is a σ-finite measure on $C(M_F(C))$ (cf. Eq.(7) in §5). $S^*_{s,y}$ is the normalization of $S_{s,y}$, given by $dS^*_{s,y} := r^{-1}_{s,t_{l+1}} dS_{s,y}$. Moreover, we define

$$\begin{aligned}\Xi(s;E) &:= \int\int\cdots(k-l)\cdots\int \varphi(K(t_1),\cdots,K(t_l),\sum_{i=1}^{m}\eta^i_{l+1},\cdots,\sum_{i=1}^{m}\eta^i_k)\\ &\times \bigotimes_{i=1}^{m} S^*_{s,y}(d\eta^i_{l+1}\otimes\cdots\otimes d\eta^i_k),\end{aligned}$$

where $E=\{y_1,\cdots,y_m\}(\neq\emptyset)$.

Take the mass φ as $(\Psi\mathcal{E}^{-1})(s,y)$ at each point y (cf. §6). For simplicity we set

$$\Delta[\Phi](\mathcal{M};s,y,K) := \Phi(\mathcal{M}[s,y]K_{\cdot\wedge T_N}) - \Phi(K_{\cdot\wedge T_N}).$$

Recall the assumption (A.3). Immediately we can get

$$\begin{aligned}&\mathbf{Q}_N \int\int_{C(\infty)} \Delta[\Phi](\mathcal{M};s,y,K)\Lambda_{\Psi\mathcal{E}^{-1}}(ds\otimes dy)\\ =\ &\mathbf{Q}_N \int_{a+}^{b}\int_C \Delta[\Phi](\mathcal{M};s,y,K)\lambda_s[\Psi\mathcal{E}^{-1}](dy)ds\\ =\ &\int_{a+}^{b} ds\mathbf{Q}_N\left\{\sum_{y\in L[s,u]}\Delta[\Phi](\mathcal{M};s,y,K)\cdot(\Psi\mathcal{E}^{-1})(s,y)\right\}.\end{aligned}$$

In the following calculation, we may take much advantage of those concepts such as i) the Markov property of K_t; ii) the infinite divisibility of the law of historical process; iii) the Poisson nature of the location $L[s,t_{l+1}]$. Hence we can proceed with the computation. In fact,

$$\begin{aligned}&\mathbf{Q}_N\left\{\sum_{y\in L[s,u]}\Delta[\Phi](\mathcal{M};s,y,K)\cdot(\Psi\mathcal{E}^{-1})(s,y)\right\}\\ =\ &\mathbf{Q}_N\left\{\mathbf{P}\left[\sum_{y\in L[s,u]}\mathbf{P}\{\Delta[\Phi]\cdot\Psi\mathcal{E}^{-1}|\mathcal{F}_s\bigvee\sigma(L[s,u])\}\middle|\mathcal{F}_s\right]\right\}\\ =\ &\mathbf{Q}_N\left\{\mathbf{P}\left[\sum_{y\in L[s,u]}\{\Xi(s;L[s,u]\setminus\{y\})-\Xi(s;L[s,u])\}\cdot\Psi\mathcal{E}^{-1}\middle|\mathcal{F}_s\right]\right\}\end{aligned}\tag{20}$$

It is easy to see the following lemma.

Lemma 5 *The last expression of (20) is equivalent to*

$$\begin{aligned}&Q_N\int_C(\Psi\mathcal{E}^{-1})(s,y)\cdot r_{s,t_{l+1}}\gamma(s,y)K_{s\wedge T_N}(dy)\Big[\exp\left(-r_{s,t_{l+1}}K_s(C)\right)\cdot\\ \times\ &\sum_{m=0}^{\infty}\frac{1}{m!}\int\int\cdots(m)\cdots\int_{[C]^m}\{\Xi(s;\{y_1,\cdots,y_m\})-\Xi(s;\{y_1,\cdots,y_m,y\})\}\cdot\\ \times\ &(r_{s,t_{l+1}})^m K_s^{\otimes m}(dy_1,\cdots,dy_m)\Big].\end{aligned}$$

A simple computation implies that the integral expression in Lemma 5 is also equal to

$$\begin{aligned} \mathbf{Q}_N & \int_C (\Psi\mathcal{E}^{-1})(s,y)\gamma(s,y)K_{s\wedge T_N}(dy)\cdot\Bigg[\int\int\cdots(k-l)\cdots\int_{[M_F(C)]^{k-l}} \\ \times & \ \mathbf{P}\{\varphi(K(t_1),\cdots,K(t_k))-\varphi(K(t_1),K(t_l),K(t_{l+1})+\eta_{l+1},\cdots,K(t_k)+\eta_k)|\mathcal{F}_s\} \\ \times & \ r_{s,t_{l+1}}\cdot S^*_{s,y^{s-}}(d\eta_{l+1}\otimes\cdots\otimes d\eta_k)\Bigg].\end{aligned} \tag{21}$$

While, taking (7), (8) in §5, the Campbell measure theory, and predictable section argument into consideration, we readily obtain

Lemma 6 *The followinf equality holds for all s,y:*

$$\begin{aligned} Pr\ & [\Phi](s,y)=\int\int\cdots(k-l)\cdots\int r_{s,t_{l+1}}\cdot S^*_{s,y^{s-}}(d\eta_{l+1}\otimes\cdots\otimes d\eta_k)\cdot \\ \times\ & P\{\varphi(K(t_1),\cdots,K(t_l),K(t_{l+1})+\eta_{l+1},\cdots,K(t_k)+\eta_k)-\varphi(K(t_1),\cdots,K(t_k))|\mathcal{F}_s\}.\end{aligned}$$

Therefore, an application of the above assertion with Lemma 5 implies

$$\begin{aligned} & -\ \mathbf{Q}_N\int\int_{C(t)} Pr[\Phi](\gamma\cdot\Psi\mathcal{E}^{-1})(s,y)dK_{s\wedge T_N}ds \\ & =\ \int_{\tau+}^t ds\left\{\mathbf{Q}_N\int_C(-Pr[\Phi])\gamma\cdot\Psi\mathcal{E}^{-1}dK_{s\wedge T_N}ds\right\}=\int_{\tau+}^t Eq.(21)ds=\int_{\tau+}^t Eq.(20)ds \\ & =\ \mathbf{Q}_N\int\int_{C(t)}\Delta[\Phi](\mathcal{M};s,y,K)\Lambda_{\Psi\mathcal{E}^{-1}}(ds\otimes dy),\end{aligned}$$

which completes the proof.

10 Evans-Perkins Type Formula: Proof of Theorem 4

Since $\mathbf{P}[K_t(C)^2]$ is uniformly bounded on compact intervals, our major premise guarantees the finiteness of the quantity $\mathbf{P}[F(K)^2]$. Therefore we can apply Theorem 1 (§3) for $F(K)$ to obtain that

$$F(K)=\mathbf{P}[F(K)]+\int_\tau^\infty\int_C f(s,y)dM^K(s,y),\mathbf{P}-a.s. \tag{22}$$

holds for some f in $L^2_\infty(K,\mathbf{P})$. While, it follows from the covariance formula in the theroy of stochastic integration that

$$\begin{aligned} \mathbf{P}\ & \left[\left(\int\int_{C(\infty)} f(s,y)dM^K(s,y)\right)\left(\int\int_{C(t)}\Psi(s,y)dM^K(s,y)\right)\right] \\ =\ & \mathbf{P}\left[\int_\tau^t\int_C f(s,y)\Psi(s,y)\gamma(s,y)K_s(dy)ds\right]\end{aligned} \tag{23}$$

for all $t>\tau$ and Ψ in $b\mathcal{P}(\mathcal{C}_t\times\mathcal{F}_t)$. Rewriting the left hand side of Eq.(23) we get

$$\mathbf{P}\left[F(K)\int_\tau^t\int_C\Psi(s,y)dM^K(s,y)\right] \tag{24}$$

by employing the predictable representation property (22). Hence we may apply Theorem 3 (§7) to rewrite (24), because the stochastic integration by parts formula is valid for any bounded $(\mathcal{C}_t \times \mathcal{F}_t)$-predictable functions. So that, from (23)

$$\mathbf{P}\int\int_{C(t)} f(s,y)\Psi(s,y)\gamma(s,y)dK_s ds = \mathbf{P}\int\int_{C(t)} Pr[F](s,y)\Psi(s,y)\gamma(s,y)dK_s ds.$$

On this account, the general theory of Hilbert spaces shows that

$$\mathbf{P}\int_\tau^t \int_C \{f(s,y) - Pr[F](s,y)\}^2\gamma(s,y)K_s(dy)ds = 0.$$

Therefore the uniqueness argument allows us to conclude that $\int\int_{C(t)} f dM$ is equivalent to $\int\int_{C(t)} Pr[F]dM$, $\mathbf{P}$-a.s. Note that $Pr[F](s,y)$ become null for K_s-a.s. y, for any $s > t$, by its construction, as long as we choose t largely enough for the support of m to be contained in $[\tau, t]$. Consequently, the above integral $\int\int Pr[F]dM$ can be replaced by $\int\int_{C(\infty)} Pr[F]dM$, which completes the proof. This goes quite similarly as in the proof of Theroem 2.5 in [EP95].

References

[D93] Dawson,D.A.: Measure-valued Markov processes, *Lecture Notes in Math.* **1541** (1993, Springer-Verlag, Berlin), 1-260.

[DIP89] Dawson,D.A., Iscoe,I and Perkins,E.A.: Super-Brownian motion: path properties and hitting probabilities, *Probab. Th. Rel. Fields* **83**(1989), 135-205.

[Dk98a] Dôku,I.: A note on a certain stochastic integration by by parts formula for superprocesses, *J. SU Math. Nat. Sci.* **47**-1(1998), 1-11.

[Dk98b] Dôku,I.: On some integration formulae in stochastic analysis, *RIMS Kokyuroku* (Kyoto Univ.) **1035**(1998), 66-81.

[Dk99a] Dôku,I.: On a certain integral formula in stochastic analysis, to appear in *Proc. of Conference on Quantum Information*, Nov.4-8, 1997, Meijo Univ., Nagoya, (1999), 20p.

[Dk99b] Dôku,I.: A variant of Itô-Clark type formula in stochastic analysis, to appear in *Proc. Colloquium on Infinite-Dim. Non-commu. Anal. Appl.*, (1999).

[Dk99c] Dôku,I.: An overview of the studies on catalytic stochastic processes, to appear in *Proc. Workshop on Stoch. Anal. M-VSP*, (1999), 14p.

[DkTn98] Dôku,I. and Tanuma,F.: Examples of representation of quadratic variation processes related to the historical martingale problem, *J. SU Math. Nat. Sci.* **47**-2(1998), 7-16.

[DP91] Dawson,D.A. and Perkins,E.A.: Historical processes, *Mem. Amer. Math. Soc.* **93**(1991), 1-179.

[E82] Elliot,R.J.:*Stochastic Calculus and Applications*, Springer-Verlag, New York,1982.

[EP94] Evans,S.N. and Perkins,E.A.: Measure-valued branching diffusions with singular interactions, *Can. J. Math.* **46**(1994), 120-168.

[EP95] Evans,S.N. and Perkins,E.A.: Explicit stochastic integral representations for historical functions, *Ann. Probab.* **23**(1995), 1772-1815.

[F88] Fitzsimmons,P.J.: Construction and regularity of measure-valued Markov branching processes, *Israel J. Math.* **64**(1988), 337-361.

[J77] Jacod,J.: A general theorem of representation for martingales, *Proc. Symp. Pure Math.* **XXXI**(1977, AMS), 37-54.

[JS87] Jacod,J. and Shiryaev,A.N.: *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, 1987.

[MP92] Mueller,C. and Perkins,E.A.: The compact support property for solutions to the heat equation with noise, *Prob. Th. Rel. Fields* **93**(1992), 325-358.

[P92] Perkins,E.A.: Measure-valued branching diffusion with spatial interactions, *Prob. Th. Rel. Fields*, **94**(1992), 189-245.

[P95] Perkins,E.A.: On the martingale problem for interactive measure-valued branching diffusions, *Mem. Amer. Math. Soc.* **115**(1995), 1-89.

[U95] Üstünel,A.S.: *An Introduction to Analysis on Wiener Space*, Lecture Notes in Math. Vol.1610, Springer-Verlag, Berlin, 1995.

[W86] Walsh,J.B.: An introduction to stochastic partial differential equations, *Lecture Notes in Math.* **1180**(1986, Springer-Verlag), 265-439.