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SINGULAR PERTURBATION FOR LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS IN N-PARTICLE SYSTEM

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Abstract. The limit behavior of solutions of singularly perturbed stochastic differential equations (SDEs) with a small parameter $\varepsilon > 0$ is discussed. The SDEs under consideration have time-dependent drift and diffusion coefficients together with the coefficient $\gamma > 0$ which represents the intensity of the interaction. For the one-dimensional SDE with mean-field of the McKean type, which is tagged to the limit behavior as $N \to \infty$ of the interacting N-particle system, it is shown that the fast component multiplied by $\sqrt{\varepsilon}$ converges in distribution to the normal distribution with mean 0 and variance depending on γ as $\varepsilon \to 0$ and that the slow component converges in mean square to the diffusion process with the diffusion coefficient depending on γ as $\varepsilon \to 0$. Here both the variance and the diffusion coefficient are given as the decreasing functions in γ.

Moreover, the interchangeability of the limits is investigated. While the N-dimensional SDE corresponding to the N-particle system involves a double limit, such as $N \to \infty$ and $\varepsilon \to 0$, the commutative role of convergences as $N \to \infty$ and $\varepsilon \to 0$ in the derivation and identification of the limit system can be shown.

1. Introduction. Let $(\Omega,\ \mathbf{F},\ P)$ be a probability space with an increasing family $\{\mathbf{F}_t,\ t \geqq 0\}$ of sub-σ-algebras of $\mathbf{F}$ and let $w(t)$ be a one-dimensional Brownian motion process adapted to $\mathbf{F}_t$. Let ε be a small parameter such that $0 < \varepsilon \ll 1$. Then we consider the following one-dimensional stochastic differential equation(SDE) of the McKean type:

$$(1.1)\qquad \begin{aligned} \varepsilon \cdot dx^\varepsilon(t) &= \left[\, a(t)x^\varepsilon(t) + b(t) - \gamma\left\{x^\varepsilon(t) - E\left[x^\varepsilon(t)\right]\right\}\right] dt + c(t)dw(t), \\ x^\varepsilon(0) &= \xi, \end{aligned}$$

where ξ is a random variable independent of $w(t)$ and $E[\ \ ]$ stands for the mathmatical expectation.

Here and hereafter, γ is a positive constant and $\{a(t),\ b(t),\ c(t)\}$ is a family of scalar functions on $R = (-\infty,\ \infty)$. Define $y^\varepsilon(t)$ by

$$(1.1)' \qquad y^\varepsilon(t) = \int_0^t x^\varepsilon(s)ds.$$

Then, the first purpose of this paper is to investigate the asymptotic behavior of $x^\varepsilon(t)$ and $y^\varepsilon(t)$ as $\varepsilon \to 0$.

Next, let N be a natural number and let $(w_i(t))_{i=1,\dots,N}$ be an N-dimensional Brownian motion process adapted to $\mathbf{F}_t$. Then we consider the following N-dimensional stochastic differential equation(SDE) with mean-field interaction:

$$(1.2)\qquad \begin{aligned} \varepsilon \cdot dx_i^\varepsilon(t) &= \left[a(t)x_i^\varepsilon(t) + b(t) - \frac{\gamma}{N}\sum_{j=1}^N \left(x_i^\varepsilon(t) - x_j^\varepsilon(t)\right)\right] dt + c(t)dw_i(t), \\ x_i^\varepsilon(0) &= \xi_i, \\ i &= 1, \dots, N, \end{aligned}$$

where $(\xi_i)_{i=1,\dots,N}$ is a random vector independent of $(w_i(t))_{i=1,\dots,N}$. By $x_i^{\varepsilon,N}(t)$ denote the solution $x_i^\varepsilon(t)$, considering the dependence on N, and define $y_i^{\varepsilon,N}(t)$ by

$$(1.2)' \qquad y_i^{\varepsilon,N}(t) = \int_0^t x_i^{\varepsilon,N}(s)ds.$$

Then, the second purpose of this paper is to investigate the asymptotic behavior of $x_i^{\varepsilon,N}(t)$ and $y_i^{\varepsilon,N}(t)$ as $\varepsilon \to 0$ and $N \to \infty$.

We note that the SDE(1.1) is tagged to the limit behavior as $N \to \infty$ of the SDE(1.2) which is the N-particle system, as follows from the next Remark 1.1.

Remark 1.1(McKean[6]). Suppose that the functions $a(t)$, $b(t)$ and $c(t)$ are continuous in $t \geqq 0$ and that the initial random variable ξ and the initial random vector $(\xi_i)_{i=1,\dots,N}$ are square integrable, for which the family $\{\xi_1, \xi_2, \dots, \xi_N\}$ is independent and indentically distributed. Fix a positive integer k and choose N so that $N > k$. Let us consider the following k-dimensional stochastic differential equation:

$$(1.3) \qquad \begin{aligned} \varepsilon \cdot dz_i^\varepsilon(t) &= [a(t)z_i^\varepsilon(t) + b(t) - \gamma\{z_i^\varepsilon(t) - E[z_i^\varepsilon(t)]\}]\,dt + c(t)dw_i(t), \\ z_i^\varepsilon(0) &= \xi_i, \\ i &= 1,\dots,k. \end{aligned}$$

Then we have

$$E\left[\sup_{0\leqq t\leqq u} (x_i^{\varepsilon,N}(t) - z_i^\varepsilon(t))^2\right] \longrightarrow 0 \quad \text{as} \quad N \to \infty \quad \text{for every} \quad u < \infty,$$

where $i = 1, \dots, k$.

We shall use the following notations.

We say a sequence $\{X_n\}_{n=1,2,\dots}$ of real-valued random variables *converges in distribution* to the real-valued random variable X, and we write

$$(1.4) \qquad X_n \xrightarrow{D} X,$$

if the distributions μ_n of the X_n converge weakly to the distribution μ of X. We write $\mu_n \Rightarrow \mu$ for weak convergence of $\{\mu_n\}_{n=1,2,\dots}$ to μ. If X_n are real-valued random variables, if μ are the corresponding distributions, and if μ is a probability measure on (R, φ) with the class φ of Borel sets in R, we say the X_n *converge in distribution* to μ, and write

$$X_n \xrightarrow{D} \mu,$$

in case $\mu_n \Rightarrow \mu$.

Notation 1.1. In particular, if real-valued random variables X_n have asymptotically a normal distribution with mean m and variance σ^2, we shall express this fact by writing

$$X_n \xrightarrow{D} \mathbf{N}(m, \sigma^2) \tag{1.5}$$

or

$$\lim_{n\to\infty} X_n = \mathbf{N}(m, \sigma^2) \quad \text{in distribution.} \tag{1.5$'$}$$

Let $x^\varepsilon(t)$ and $y^\varepsilon(t)$ be defined by (1.1) and $(1.1)'$, respectively. Then, under the assumption that $a(t) < 0$ together with suitable conditions, Theorem 2.1 states that for each $t > 0$

$$\sqrt{\varepsilon} \cdot x^\varepsilon(t) \xrightarrow{D} \mathbf{N}(0, \sigma_\gamma(t)^2) \quad \text{as} \quad \varepsilon \to 0,$$

where

$$\sigma_\gamma(t)^2 = \frac{c(t)^2}{2(\gamma - a(t))},$$

and also Theorem 2.2 states that for all $t \geqq 0$

$$E[\,(y^\varepsilon(t) - y(t))^2\,] \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0,$$

where $y(t)$ is the diffusion process with the drift coeffient $(-b(t)/a(t))$ and the diffusion coefficient $(\,c(t)/(\,\gamma - a(t)\,)\,)^2$.

Next, write $x_i^{\varepsilon,N}(t)$ for the solution $x_i^{\varepsilon}(t)$ of (1.2), and define $y_i^{\varepsilon,N}(t)$ by (1.2)$'$. Fix a positive integer k and choose N so that $N > k$. Then, under suitable assumptions, Corollary 2.1 shows that for each $t > 0$

$$\begin{aligned}\lim_{N\to\infty}\{\lim_{\varepsilon\to 0}(\sqrt{\varepsilon}\cdot x_i^{\varepsilon,N}(t))\} &= \mathrm{N}(0,\sigma_\gamma(t)^2)\\ &= \lim_{\varepsilon\to 0}\{\lim_{N\to\infty}(\sqrt{\varepsilon}\cdot x_i^{\varepsilon,N}(t))\}\quad \text{in distribution,}\end{aligned}$$

where $i = 1, \ldots, k$, and also Corollary 2.2 shows that for all $t \geqq 0$

$$\begin{aligned}\lim_{N\to\infty}\{\lim_{\varepsilon\to 0}E[(y_i^{\varepsilon,N}(t) - y_i(t))^2]\} &= 0\\ &= \lim_{\varepsilon\to 0}\{\lim_{N\to\infty}E[(y_i^{\varepsilon,N}(t) - y_i(t))^2]\}\end{aligned}$$

with the diffusion process $y_i(t)$ given in Theorem 2.4, where $i = 1, \ldots, k$.

Remark 1.2. Observe the same variance $\sigma_\gamma(t)^2$ of the limit distribution of $\sqrt{\varepsilon}\cdot x^\varepsilon(t)$ and $\sqrt{\varepsilon}\cdot x_i^{\varepsilon,N}(t)$ corresponding to the fast components as in Theorem 2.1 and Corollary 2.1, so that

$$\sigma_\gamma(t)^2 = \frac{c(t)^2}{2(\gamma - a(t))}, \qquad \text{where} \quad a(t) < 0.$$

Then we note

$$\sigma_\gamma(t)^2 \downarrow 0 \quad \text{as} \quad \gamma \uparrow \infty,$$

where γ represents the intensity of the interaction. Set

$$d_\gamma(t)^2 = \left(\frac{c(t)}{\gamma - a(t)}\right)^2,$$

which denotes the diffusion coefficient of the limit processes of the slow components $y(t)$ and $y_i(t)$ cited in Theorem 2.2 and Corollary 2.2. Then we note

$$d_\gamma(t)^2 \downarrow 0 \quad \text{as} \quad \gamma \uparrow \infty.$$

The aim of this paper begins with a slight extension of singularly perturbed initial value problems for linear ordinary differential equations with time-dependent coefficients, which are introduced in O'Malley[8, Chap.2], to the stochastic case. From different point of view, another types of singular perturbation solutions of noisy systems together with useful applications are found, for example, in Papanicolaou[9] and Hoppensteadt[4]. The early work with which Blankenship and Sachs[2] are concerned treats an analogue of SDE(1.1), except that the family $\{\gamma,\ a(t),\ b(t),\ c(t)\}$ of coefficients is replaced by $\{0,\ a(t),\ 0,\ 1\}$ and the formal white noise $dw(t)/dt$ is replaced by the stochastic process $f^{\varepsilon}(t)$ which approaches a white noise as $\varepsilon \to 0$. On the other hand, Kokotovic[5, pp.36-42] discusses the time scale modeling of networks with large scale interacting systems. Recently, Dubko[3] obtains the limit diffusion process for the slow component in the singularly perturbed linear SDEs with time-dependent coefficients, such as (1.1) and (1.1)′ with $\gamma = 0$. Further, Singh[10] treats the SDEs, such as (1.1) and (1.1)′ with $\{\gamma,\ a(t),\ b(t),\ c(t)\}$ replaced by $\{0,\ a(t),\ b(t),\ \sqrt{\varepsilon}\}$, showing the limit distribution for the fast component. Our study is motivated and inspired by the above cited works.

2. Theorems. In the processes of proving Theorems 2.1 and 2.3, we shall use the following remarks.

Remark 2.1(Arnold[1 , p.132]). Let $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ be a $d \times d$-matrix, d-vector and $d \times n$-matrix, respectively, which are bounded, measurable and real-valued functions on the interval $0 \leqq t \leqq T$. Let us consider the following d-

dimensional linear stochastic differential equation:

$$dX(t) = [\,\alpha(t)X(t) + \beta(t)\,]dt + \sigma(t)dB(t)$$

with the initial state $X(0) = X_0$, where $B(t)$ is an n-dimensional Brownian motion process, and X_0 is a square integrable random variable that is independent of $B(t)$ for $t \geqq 0$. Then the solution $X(t)$ is a Gaussian stochastic process if and only if X_0 is normally distributed or constant.

Remark 2.2(Arnold[1, p.14]). Let $\{X_n\}_{n=1,2,\ldots}$ denotes a sequence of R^d-valued random variables having d-dimensional normal distribution $\mathrm{N}(m_n,\ C_n)$ with expectation vector m_n and covariance matrix C_n. This sequence converges in distribution if and only if

$$m_n \longrightarrow m, \quad C_n \longrightarrow C, \quad \text{as} \quad n \to \infty.$$

The limit distribution is a d-dimensional normal distribution $\mathrm{N}(m,\ C)$.

We shall need the following assumptions.

Assumption 2.1.

(i) ε is a small parameter such that $0 < \varepsilon \ll 1$, and γ is a positive constant.

(ii) $a(t), b(t)$ and $c(t)$ are once continuously differentiable function on $t \geqq 0$.

(iii) There is a constant $\delta > 0$ such that $a(t) \leqq -\delta$ for $t \geqq 0$.

Assumption 2.2. The initial state ξ is a random variable independent of the Brownian motion process $w(t)$ for $t \geqq 0$, satisfying

$$E[\xi^2] < \infty.$$

Assumption 2.3. The initial state $(\xi_i)_{i=1,\dots,N}$ is a random vector independent of the Brownian motion process $(w_i(t))_{i=1,\dots,N}$ for $t \geqq 0$ such that the family $\{\xi_i : i = 1, \dots, N\}$ is independent and identically distributed, satisfying

$$E[\,\xi_i^2\,] < \infty, \quad i = 1, \dots, N.$$

Remark 2.3. Let $(z_i^\varepsilon(t))_{i=1,\dots,k}$ be the solution of SDE(1.3) with the initial state $(\xi_i)_{i=1,\dots,k}$. Then we note that the next Theorem 2.1 holds for $z_i^\varepsilon(t)$ with the same result, except that $x^\varepsilon(t)$ is replaced by $z_i^\varepsilon(t)$, where $i = 1, \dots, k$.

Our results are the following theorems.

THEOREM 2.1. *Under Assumption* 2.1, *let $x^\varepsilon(t)$ be the solution of SDE*(1.1) *with the initial state $x^\varepsilon(0) = \xi$. Suppose that ξ is a constant or a random variable independent of the Brownian motion process $w(t)$ for $t \geqq 0$ that is normally distributed. Then, for each $t > 0$*

$$\sqrt{\varepsilon} \cdot x^\varepsilon(t) \quad \xrightarrow{D} \quad \mathrm{N}(0, \sigma_\gamma(t)^2) \quad \textit{as} \quad \varepsilon \to 0,$$

where

$$\sigma_\gamma(t)^2 = \frac{c(t)^2}{2(\gamma - a(t))}.$$

PROOF. Since $E[x^\varepsilon(t)]$ has an explicit form as an solution of ODE, Remark 2.1 implies that $\sqrt{\varepsilon} \cdot x^\varepsilon(t)$ is a Gaussian stochastic process under the assumption on the initial state. Denote by $M^\varepsilon(t)$ and $V^\varepsilon(t)$ the expection and the variance of $\sqrt{\varepsilon} \cdot x^\varepsilon(t)$, that is

$$M^\varepsilon(t) = E\left[\sqrt{\varepsilon} \cdot x^\varepsilon(t)\right] \quad \text{and} \quad V^\varepsilon(t) = \varepsilon \cdot E\left[x^\varepsilon(t)^2\right] - M^\varepsilon(t)^2.$$

Then a simple calculation shows the following estimates:

$$\begin{aligned} |M^\varepsilon(t)| &\leqq \sqrt{\varepsilon}\cdot(K+K_t) \quad \text{for} \quad t \geqq 0. \\ \varepsilon\cdot E\left[x^\varepsilon(t)^2\right] &= \frac{c(t)^2}{2(\gamma-a(t))} + G^\varepsilon(t) + \varepsilon\cdot H^\varepsilon(t) \end{aligned}$$

with the functions G^ε and H^ε such that

$$|G^\varepsilon(t)| \leqq K\cdot\exp\left[-\frac{2(\delta+\gamma)}{\varepsilon}t\right] + \varepsilon\cdot K_t \quad \text{for} \quad t \geqq 0$$

and

$$|H^\varepsilon(t)| \leqq K + K_t \quad \text{for} \quad t \geqq 0.$$

Here K is a positive constant and K_t is a positive increasing function in $t \geqq 0$.

Therefore, passing to the limit as $\varepsilon \to 0$, by Remark 2.2 we can obtain the conclusion of the theorem.

THEOREM 2.2. *Under Assumption* 2.1, *let* $x^\varepsilon(t)$ *be the solution of SDE*(1.1) *with the initial state* $x^\varepsilon(0) = \xi$. *Suppose that* ξ *satisfies Assumption* 2.2. *For* $t \geqq 0$, *define* $y^\varepsilon(t)$ *and* $y(t)$ *by*

$$y^\varepsilon(t) = \int_0^t x^\varepsilon(s)ds$$

and

$$y(t) = -\int_0^t \frac{b(s)}{a(s)}ds + \int_0^t \frac{c(s)}{\gamma - a(s)}dw(s).$$

Then

$$E[(y^\varepsilon(t) - y(t))^2] \longrightarrow 0 \quad as \quad \varepsilon \to 0 \quad for \quad t \geqq 0.$$

For the N-particle system, we have the following theorems.

THEOREM 2.3. *Under Assumption* 2.1, *let* $(x_i^\varepsilon(t))_{i=1,\dots,N}$ *be the solution of SDE*(1.2) *with the initial state* $(x_i^\varepsilon(0))_{i=1,\dots,N} = (\xi_i)_{i=1,\dots,N}$. *Suppose that* $(\xi_i)_{i=1,\dots,N}$ *is a constant vector or normally distributed random vector independent of* $(w_i(t))_{i=1,\dots,N}$ *for* $t \geqq 0$, *such that* $\xi_1, \xi_2, \dots, \xi_N$ *are independent and identically distributed random variables, each with normal distribution. By* $(x_i^{\varepsilon,N}(t))_{i=1,\dots,N}$ *denote the solution* $(x_i^\varepsilon(t))_{i=1,\dots,N}$, *considering the dependence on the size parameter* N. *Set*

$$\sigma_\gamma(t)^2 = \frac{c(t)^2}{2(\gamma - a(t))}$$

and

$$\sigma_\gamma^N(t)^2 = \sigma_\gamma(t)^2 - \frac{1}{N}\left\{\frac{\gamma \cdot c(t)^2}{(\gamma - a(t)) \cdot 2a(t)}\right\}.$$

Then, for each $t > 0$

$$\sqrt{\varepsilon} \cdot x_i^{\varepsilon,N}(t) \quad \xrightarrow{D} \quad \mathbf{N}(0, \sigma_\gamma^N(t)^2) \quad \text{as} \quad \varepsilon \to 0,$$

where $i = 1, \dots, N$.

THEOREM 2.4. *Under Assumption* 2.1, *let* $(x_i^\varepsilon(t))_{i=1,\dots,N}$ *be the solution of SDE*(1.2) *with the initial state* $(x_i^\varepsilon(0))_{i=1,\dots,N} = (\xi_i)_{i=1,\dots,N}$. *Suppose that* $(\xi_i)_{i=1,\dots,N}$ *satisfies Assumption* 2.3. *By* $(x_i^{\varepsilon,N}(t))_{i=1,\dots,N}$ *denote the solution* $(x_i^\varepsilon(t))_{i=1,\dots,N}$ *emphasizing the dependence on the size parameter* N. *For* $t \geqq 0$, *define* $y_i^{\varepsilon,N}(t)$, $y_i(t)$ *and* $y_i^N(t)$, *where* $i = 1, \dots, N$, *as follows:*

$$y_i^{\varepsilon,N}(t) = \int_0^t x_i^{\varepsilon,N}(s)ds.$$

$$y_i(t) = -\int_0^t \frac{b(s)}{a(s)}ds + \int_0^t \frac{c(s)}{\gamma - a(s)}dw_i(s).$$

$$y_i^N(t) = y_i(t) - \frac{1}{\sqrt{N}}\int_0^t \frac{\gamma \cdot c(s)}{a(s) \cdot (\gamma - a(s))}d\overline{w}(s),$$

where $\overline{w}(t)$ *is the one-dimensional Brownian motion process defined by*

$$\overline{w}(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} w_j(t).$$

Fix a positive integer k *and choose* N *so that* $N > k$. *Then*

$$\lim_{\varepsilon \to 0} E\left[(y_i^{\varepsilon,N}(t) - y_i^N(t))^2\right] = 0 \quad \mathit{for} \quad t \geqq 0$$

and also

$$\lim_{N \to \infty} \left\{ \lim_{\varepsilon \to 0} E\left[(y_i^{\varepsilon,N}(t) - y_i(t))^2\right] \right\} = 0 \quad \mathit{for} \quad t \geqq 0,$$

where $i = 1, \dots, k$.

Appealing to the above theorems, we obtain the following corollaries.

COROLLARY 2.1. *Under Assumption* 2.1, let $(x_i^\varepsilon(t))_{i=1,\dots,N}$ *be the solution of SDE*(1.2) *with the initial state* $(x_i^\varepsilon(0))_{i=1,\dots,N} = (\xi_i)_{i=1,\dots,N}$. *By* $(x_i^{\varepsilon,N}(t))_{i=1,\dots,N}$ *denote the solution* $(x_i^\varepsilon(t))_{i=1,\dots,N}$, *emphasizing the dependence on the size parameter* N. *Suppose that* $(\xi_i)_{i=1,\dots,N}$ *satisfies the same assumptions as in Theorem* 2.3. *Fix a positive integer* k *and choose* N *so that* $N > k$. *Put*

$$\sigma_\gamma(t)^2 = \frac{c(t)^2}{2(\gamma - a(t))}.$$

Then, for each $t > 0$

$$\begin{aligned} \lim_{N \to \infty} \left\{ \lim_{\varepsilon \to 0} (\sqrt{\varepsilon} \cdot x_i^{\varepsilon,N}(t)) \right\} &= \mathbf{N}(0, \sigma_\gamma(t)^2) \\ &= \lim_{\varepsilon \to 0} \left\{ \lim_{N \to \infty} (\sqrt{\varepsilon} \cdot x_i^{\varepsilon,N}(t)) \right\} \qquad \mathit{in\ distribution}, \end{aligned}$$

where $i = 1, \dots, k$.

COROLLARY 2.2. *Under Assumption* 2.1, *let* $(x_i^\varepsilon(t))_{i=1,\dots,N}$ *be the solution of SDE*(1.2) *with the initial state* $(x_i^\varepsilon(0))_{i=1,\dots,N} = (\xi_i)_{i=1,\dots,N}$. *Suppose that* $(\xi_i)_{i=1,\dots,N}$ *satisfies Assumption* 2.3. *By* $(x_i^{\varepsilon,N}(t))_{i=1,\dots,N}$ *denote the solution* $(x_i^\varepsilon(t))_{i=1,\dots,N}$, *emphasizing the dependence on the size parameter* N. *For* $t \geqq 0$, *set*

$$y_i^{\varepsilon,N}(t) = \int_0^t x_i^{\varepsilon,N}(s)ds$$

and let $y_i(t)$ *be the process defined in Theorem* (2.4). *Fix a positive integer* k *and choose* N *so that* $N > k$. *Then*

$$\begin{aligned}\lim_{N\to\infty}\left\{\lim_{\varepsilon\to 0} E\left[(y_i^{\varepsilon,N}(t) - y_i(t))^2\right]\right\} &= 0 \\ &= \lim_{\varepsilon\to 0}\left\{\lim_{N\to\infty} E\left[(y_i^{\varepsilon,N}(t) - y_i(t))^2\right]\right\} \quad \textit{for} \quad t \geqq 0,\end{aligned}$$

where $i = 1, \dots, k$.

3. Singular perturbation methods. Let ε be a small parameter such that $0 < \varepsilon \ll 1$, $f(x,y)$ and $g(x)$ be scalar functions on R^2 and R^1, respectively, and also let c be a positive constant. Then, for an equation of the form

$$\frac{d^2x}{dt^2} + g(x) = \varepsilon \cdot f\left(x, \frac{dx}{dt}\right) + \sqrt{\varepsilon} \cdot \frac{dw}{dt},$$

where dw/dt is a formal white noise, the *averaging principle* of Papanicolaou[9] applies. Now, our oscillator is of the type

$$\varepsilon \cdot \frac{d^nx}{dt^n} = f(x, \frac{dx}{dt}) + c\frac{dw}{dt},$$

where $n = 1$ and 2. The work of Van der Pol[11] corresponds to the *relaxation oscillations* in case of $n = 2$, which influences on the analysis of stochastic oscillators as in Narita[7]. Our paper is partially motivated by the *initial value problems* for $n = 1$.

For simplicity, let us consider the deterministic equation (1.1) with $\gamma = 0$ and $c(t) \equiv 0$, under Assumption 2.1:

$$\begin{aligned}\varepsilon \cdot dx^\varepsilon(t) &= [a(t)x^\varepsilon(t) + b(t)]\,dt, \\ x^\varepsilon(0) &= \xi.\end{aligned}$$

Write down the exact solution and integrate by parts, noting that

$$\exp\left[\frac{1}{\varepsilon}\int_s^t a(r)dr\right] \leqq \exp\left[-\frac{1}{\varepsilon}\delta(t-s)\right] \quad \text{for} \quad 0 \leqq s \leqq t.$$

Then we can see that the solution tends to $-b(t)/a(t)$ as $\varepsilon \to 0$ for $t > 0$ since $a(t) \leqq -\delta < 0$. On the other hand, for $\varepsilon = 0$ we obtain the <u>*reduced system*</u>

$$0 = a(t)x^0(t) + b(t) \quad \text{or} \quad x^0(t) = -b(t)/a(t).$$

Obviously, this approximate solution does not satisfy the initial value $x^\varepsilon(0) = \xi$.

Assumption 2.1 plays an essential role in our analysis of stochastic differential equations. The <u>*reduced system for*</u> the fast process $x^\varepsilon(t)$ of SDE(1.1) as $\varepsilon \to 0$ can be derived with the following result:

$$(3.1) \qquad 0 = a(t)x^0(t) + b(t) - \gamma\left\{x^0(t) - E[x^0(t)]\right\} + c(t)\frac{dw}{dt}.$$

This suggests that $x^\varepsilon(t)$ blows up to white noise dw/dt as $\varepsilon \to 0$. Therefore, some transformation of the space-time parameter is necessary as $\varepsilon \to 0$. Theorem 2.1 results from the limit behavior of the scaled process $\sqrt{\varepsilon}\cdot x^\varepsilon(t)$ as $\varepsilon \to 0$. Moreover, a glance at the mathmatical expectation on (3.1) shows

$$E\left[x^0(t)\right] = -\frac{b(t)}{a(t)},$$

so that

$$x^0(t) = -\frac{b(t)}{a(t)} + \frac{c(t)}{\gamma - a(t)}\frac{dw}{dt}.$$

Accordingly, the slow process $y^0(t)$, which is defined by

$$y^0(t) = \int_0^t x^0(s)ds \quad \text{for all} \quad t \geqq 0,$$

satisfies the same SDE with the process $y(t)$ introduced in Theorem 2.2.

Theorems 2.3 and 2.4 follow from rigorous estimates of moment bounds for SDEs (1.2) and (1.2)$'$ which depend on a small parameter ε and a size parameter N.

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