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On the connection problem for Painlevé equations

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1 Introduction

In our series of articles ([KT1], [AKT2], [KT2]) T. Kawai (RIMS, Kyoto Univ.), T. Aoki (Kinki Univ.) and the author have developed the exact WKB analysis for Painlevé equations with a large parameter. Making use of several results obtained there, we want to discuss the global behavior of solutions of Painlevé equations in this report.

Thanks to the well-known Painlevé property, the analytic continuation of any solution of a Painlevé equation is unique except at fixed singular points of the equation. This immediately implies that the (nonlinear) monodromy group can be naturally defined for Painlevé equations. At the same time the (nonlinear) Stokes multipliers are also defined at each irregular-type singular point. The goal of our study is to give an explicit representation of these objects by using the exact WKB analysis. As the final answer has not been obtained yet, we report only our present situation as well as some intermediate results here.

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2 Formal solutions of Painlevé equations

To discuss the monodromy group and the Stokes multipliers, we need formal solutions of Painlevé equations. In this section we review the construction of formal solutions of Painlevé equations with 2 free parameters.

First of all, let us list up the Painlevé equations (P_J) $(J = \mathrm{I}, \ldots, \mathrm{VI})$ with a large parameter η.

Table 1

$$
\begin{aligned}
(P_{\mathrm{I}}) \quad \frac{d^2\lambda}{dt^2} &= \eta^2(6\lambda^2 + t). \\
(P_{\mathrm{II}}) \quad \frac{d^2\lambda}{dt^2} &= \eta^2(2\lambda^3 + t\lambda + c). \\
(P_{\mathrm{III}}) \quad \frac{d^2\lambda}{dt^2} &= \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2\left[16c_\infty\lambda^3 + \frac{8c'_\infty\lambda^2}{t} - \frac{8c'_0}{t} - \frac{16c_0}{\lambda}\right]. \\
(P_{\mathrm{IV}}) \quad \frac{d^2\lambda}{dt^2} &= \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + \eta^2\left[\frac{3}{2}\lambda^3 + 4t\lambda^2 + \left(2t^2 + 8c_1\right)\lambda - \frac{8c_0}{\lambda}\right]. \\
(P_{\mathrm{V}}) \quad \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right) \\
&\quad + \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(c_0 + c_\infty) - \frac{c_0}{\lambda^2} - \frac{c_2 t}{(\lambda-1)^2} - \frac{c_1 t^2(\lambda+1)}{(\lambda-1)^3}\right]. \\
(P_{\mathrm{VI}}) \quad \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\
&\quad + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right. \\
&\quad \left. + \eta^2\left\{(c_0 + c_1 + c_t + c_\infty) - \frac{c_0 t}{\lambda^2} + \frac{c_1(t-1)}{(\lambda-1)^2} - \frac{c_t t(t-1)}{(\lambda-t)^2}\right\}\right].
\end{aligned}
$$

Note that each Painlevé equation has the following structure in common:

$$\frac{d^2\lambda}{dt^2} = G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) + \eta^2 F_J(\lambda, t),$$

where F_J and G_J are rational functions. In view of this expression of equations, we easily find that (P_J) has the following formal power series solutions denoted by $\lambda_J^{(0)}(t)$:

$$\lambda_J^{(0)}(t) = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \eta^{-2}\lambda_2(t) + \cdots, \tag{1}$$

where the top term $\lambda_0(t)$ satisfies

$$F_J(\lambda_0(t), t) = 0 \tag{2}$$

and the other $\lambda_j(t)$ $(j \geq 1)$ can be determined in a recursive manner. Furthermore, we can construct the following formal solutions $\lambda_J(t; \alpha, \beta)$ of (P_J) containing 2 free parameters (α, β):

$$\lambda_J(t; \alpha, \beta) = \lambda_0(t) + \eta^{-1/2}\lambda_{1/2}(t, \eta) + \eta^{-1}\lambda_1(t, \eta) + \cdots, \tag{3}$$

where $\lambda_0(t)$ is a solution of (2) and $\lambda_{j/2}(t,\eta)$ $(j \geq 1)$ is of the following form:

$$\begin{aligned}
\lambda_{1/2}(t,\eta) &= \mu_J(t)\left(\alpha\exp\Phi_J + \beta\exp(-\Phi_J)\right), \\
\lambda_{j/2}(t,\eta) &= \sum_{k=0}^{j} b^{(j/2)}_{j-2k}(t)\exp\left((j-2k)\Phi_J\right) \qquad (j \geq 2),
\end{aligned}$$

where

$$\Phi_J(t,\eta) = \eta\int^t \sqrt{\frac{\partial F_J}{\partial\lambda}(\lambda_0(s),s)}ds + \alpha\beta\log(\theta_J(t)\eta^2)$$

and $\mu_J(t)$ and $\theta_J(t)$ are some appropriate functions of t. (For their precise expressions, see [T1].)

The construction of the 2-parameter formal solution $\lambda_J(t;\alpha,\beta)$ was first done in [AKT2] by using the multiple-scale analysis. On the other hand, to discuss global problems we need to analyze its behavior near fixed singular points of (P_J) and for that purpose it is more convenient to employ another construction based on a singular-perturbative reduction of the associated Painlevé Hamiltonian system to its "Birkhoff normal form" (cf. [T2]). Here let us explain an outline of the latter construction.

As is well-known, the Painlevé equation (P_J) is equivalent to the Painlevé Hamiltonian system

$$(H_J) \qquad d\lambda/dt = \eta\partial K_J/\partial\nu, \qquad d\nu/dt = -\eta\partial K_J/\partial\lambda$$

(cf., e.g., [O]). For this Hamiltonian system there exists a formal power series solution $(\lambda_J^{(0)}(t), \nu_J^{(0)}(t))$ corresponding to the solution (1) of (P_J). We consider the following localization of (H_J) at this formal power series solution:

$$(4) \qquad \lambda = \lambda_J^{(0)}(t) + \eta^{-1/2}U, \qquad \nu = \nu_J^{(0)}(t) + \eta^{-1/2}V,$$

that is, we transform the unknown function of (H_J) from (λ,ν) to (U,V). Then we find that (U,V) must satisfy another Hamiltonian system:

$$(5) \qquad dU/dt = \eta\partial H_J/\partial V, \qquad dV/dt = -\eta\partial H_J/\partial U.$$

The system (5) can be reduced to its "Birkhoff normal form" in a singular-perturbative manner. To be more specific, we can prove

Proposition 1 *There exists a formal canonical transformation $(U,V) \longmapsto (\tilde{U},\tilde{V})$ of the form*

$$(6) \qquad \begin{cases} U = u_0(\tilde{U},\tilde{V}) + \eta^{-1/2}u_1(\tilde{U},\tilde{V}) + \cdots, \\ V = v_0(\tilde{U},\tilde{V}) + \eta^{-1/2}v_1(\tilde{U},\tilde{V}) + \cdots, \end{cases}$$

where u_j and v_j are homogeneous polynomials of degree $j+1$ in $(\tilde{U}, \tilde{V})$ (whose coefficients are formal power series in $\eta^{-1/2}$ with coefficients being functions of t), so that the Hamiltonian system (8) may be taken into the following normal form:

$$d\tilde{U}/dt = \eta \partial \tilde{H}_J / \partial \tilde{V}, \qquad d\tilde{V}/dt = -\eta \partial \tilde{H}_J / \partial \tilde{U}, \tag{7}$$

where

$$\tilde{H}_J = \sum_{l=0}^{\infty} \eta^{-l} f^{(l)}(t, \eta) \left(\tilde{U}\tilde{V}\right)^{l+1} \tag{8}$$

and each $f^{(l)}(t, \eta) = \sum_{j \geq 0} \eta^{-j/2} f^{(l)}_{j/2}(t)$ is a formal power series in $\eta^{-1/2}$ with coefficients being functions of t. In particular, the following equalities hold:

$$f_0^{(0)}(t) = \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}, \quad f_{1/2}^{(0)}(t) = 0.$$

Since the reduced Hamiltonian $\tilde{H}_J$ depends only on the product $\tilde{U}\tilde{V}$, the system (7) can be easily solved. As a matter of fact, the product $\tilde{U}\tilde{V}$ becomes independent of t and hence

$$\left\{ \begin{array}{rcl} \tilde{U} & = & \alpha \exp\left(\eta \int^t \sum \eta^{-l}(l+1) f^{(l)}(s, \eta)(\alpha\beta)^l ds\right) \\ \tilde{V} & = & \beta \exp\left(-\eta \int^t \sum \eta^{-l}(l+1) f^{(l)}(s, \eta)(\alpha\beta)^l ds\right) \end{array} \right. \tag{9}$$

gives a solution of (7). Substituting (9) into (6), we can obtain formal solutions of (H_J) with 2 free parameters.

The monodromy groups and Stokes multipliers of (P_J) should now be represented as a transformation in the space of parameters (α, β) contained in the formal solution $\lambda_J(t; \alpha, \beta)$. In subsequent sections let us consider the problem: *What is an explicit description of such transformations?*

3 Stokes multipliers of (P_{I})

We first study the first Painlevé equation (P_{I}). In the case of (P_{I}) $t = \infty$ is the unique singular point and it is of irregular-type. This implies that the monodromy group of (P_{I}) is trivial and hence it is sufficient to discuss the Stokes multipliers at $t = \infty$.

To compute the Stokes multipliers we apply the exact WKB analysis. The WKB-theoretic structure of (P_{I}) is quite simple; there exists only one turning point at the origin $t = 0$ and the Stokes curves consist of five straight lines $\{t | \arg t = \pi + 2n\pi/5\}$ (where n is an integer) emanating from the origin. These five lines divide the complex t-plane into five sectors. Let us take a 2-parameter solution $\lambda_{\mathrm{I}}(t; \alpha, \beta)$ in one sector and consider its analytic continuation across a Stokes curve. Such an analytically continued solution should, in general, have a different expression $\lambda_{\mathrm{I}}(t;\tilde{\alpha},\tilde{\beta})$ in an adjacent sector and the relation between the parameters (α,β) and $(\tilde{\alpha},\tilde{\beta})$ amounts to the Stokes multipliers of (P_{I}) at $t=\infty$.

To describe the relation explicitly we introduce some notations. Since there is no essential distinction among the five Stokes curves due to the symmetry of (P_{I}), we only discuss the problem on the Stokes curve $\gamma=\{t|\arg t=3\pi/5\}$. Let $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ be a solution of (P_{I}) in the sector $\{t|\pi/5<\arg t<3\pi/5\}$ and $\lambda_{\mathrm{I}}(t;\tilde{\alpha},\tilde{\beta})$ its analytic continuation across γ. Let a (resp., b) denote $\alpha e^{4i\pi\alpha\beta}$ (resp., $\beta e^{-4i\pi\alpha\beta}$) ($\tilde{a}$ and $\tilde{b}$ are defined similarly) and let us define $S_j(a,b)$ and $\tilde{S}_j(\tilde{a},\tilde{b})$ by

$$(10)\qquad \begin{cases} S_1(a,b) &= ie^{-i\pi E/2}-ibe^{-i\pi E/2}\chi(-E),\\ S_2(a,b) &= -ae^{i\pi E/4}\chi(E),\\ S_3(a,b) &= ib\chi(-E),\\ S_4(a,b) &= ie^{-i\pi E/2}+ae^{-i\pi E/4}\chi(E),\\ S_5(a,b) &= ie^{i\pi E/2}, \end{cases}$$

$$(11)\qquad \begin{cases} \tilde{S}_1(\tilde{a},\tilde{b}) &= ie^{-i\pi\tilde{E}/2},\\ \tilde{S}_2(\tilde{a},\tilde{b}) &= ie^{i\pi\tilde{E}/2}-\tilde{a}e^{i\pi\tilde{E}/4}\chi(\tilde{E}),\\ \tilde{S}_3(\tilde{a},\tilde{b}) &= i\tilde{b}\chi(-\tilde{E}),\\ \tilde{S}_4(\tilde{a},\tilde{b}) &= \tilde{a}e^{-i\pi\tilde{E}/4}\chi(\tilde{E}),\\ \tilde{S}_5(\tilde{a},\tilde{b}) &= ie^{i\pi\tilde{E}/2}-i\tilde{b}e^{i\pi\tilde{E}/2}\chi(-\tilde{E}), \end{cases}$$

where $E=-8ab=-8\alpha\beta$, $\tilde{E}=-8\tilde{a}\tilde{b}=-8\tilde{\alpha}\tilde{\beta}$ and $\chi(z)$ denotes $\sqrt{\pi}2^{z/4+1}/\Gamma(z/4+1)$ ($\Gamma(z)$: the Gamma function). Then an explicit description of the Stokes multiplier on γ is given by the following

Proposition 2

$$(12)\qquad S_j(a,b)=\tilde{S}_j(\tilde{a},\tilde{b}),$$

that is,

$$(13)\qquad S_j(\alpha e^{4i\pi\alpha\beta},\beta e^{-4i\pi\alpha\beta})=\tilde{S}_j(\tilde{\alpha}e^{4i\pi\tilde{\alpha}\tilde{\beta}},\tilde{\beta}e^{-4i\pi\tilde{\alpha}\tilde{\beta}})$$

($j=1,2,3,4,5$).

(Concerning previous results for the Stokes multipliers of (P_{I}), see [JK], [KK] etc.) If we conventionally define $S_j(a,b)$ (resp., $\tilde{S}_j(\tilde{a},\tilde{b})$) for every integer j by requiring $S_{j+5}(a,b)=S_j(a,b)$ (resp., $\tilde{S}_{j+5}(\tilde{a},\tilde{b})=\tilde{S}_j(\tilde{a},\tilde{b})$), we find the following cyclic relations:

$$\begin{aligned}1+S_{j-1}(a,b)S_j(a,b)+iS_{j+2}(a,b) &= 0,\\ 1+\tilde{S}_{j-1}(\tilde{a},\tilde{b})\tilde{S}_j(\tilde{a},\tilde{b})+i\tilde{S}_{j+2}(\tilde{a},\tilde{b}) &= 0,\end{aligned}$$

$(j=0,\pm1,\pm2,\ldots)$. These relations imply that among the five relations (13) only two of them are independent. Hence, when (α,β) is given, the relations (13) determine $(\tilde{\alpha},\tilde{\beta})$ almost completely.

Remark The Stokes multiplier on the other Stokes curves is described as follows:

$$S_j(T^k(a,b))=\tilde{S}_j(T^k(\tilde{a},\tilde{b}))\qquad(j=1,2,3,4,5),$$

where k is an appropriately chosen integer which depends on the Stokes curve in question and T is a linear transformation (in the space of parameters (a,b)) defined by $T(a,b)=(-ib,-ia)$.

Let us explain just a sketch of the proof of Proposition 2 here. A key idea is the relationship between (P_{I}) and isomonodromic deformations of some linear equation of the form

$$\left(-\frac{\partial^2}{\partial x^2}+\eta^2 Q_{\mathrm{I}}(x,t,\lambda,\nu,\eta)\right)\psi=0. \tag{14}$$

(Cf. [JMU], [O] etc.) As a matter of fact, (P_{I}) (more precisely, (H_{I})) describes the condition that the Stokes multipliers of (14) should be independent of t. By using the exact WKB analysis we can compute them in an explicit manner and consequently we find that $S_j(a,b)$ and $\tilde{S}_j(\tilde{a},\tilde{b})$ are nothing but the Stokes multipliers of (14) when t belongs to the sectors $\{t|\pi/5<\arg t<3\pi/5\}$ and $\{t|3\pi/5<\arg t<\pi\}$ respectively (cf. [T1]). Then the relation (12) immediately follows from the isomonodromy property mentioned above.

4 Connection problem for (P_J)

For the other Painlevé equations we have not yet obtained the final results. In this section we report our present situation and some intermediate results.

In the WKB-theoretic computation of monodromy groups of second-order linear ordinary differential equations the following ingredients play an important role (cf. [AKT1]):

(i) Stokes geometry, i.e., configuration of turning points and Stokes curves,

(ii) Connection formula at a (simple) turning point,

(iii) Local behavior of WKB solutions at a (regular) singular point.

The situation is more or less the same for Painlevé equations; we are required to analyze the above three properties for (P_J). As is shown in [KT2], we know that (P_{I}) gives a normal form at a simple turning point for any (P_J). To be more specific, any 2-parameter solution $\lambda_J(t;\alpha,\beta)$ of (P_J) is transformed to a 2-parameter solution $\lambda_{\mathrm{I}}(t;\alpha',\beta')$ of (P_{I}) near a simple turning point. This result implies that the connection formula for $\lambda_J(t;\alpha,\beta)$ at a simple turning point should, in principle, be obtained by Proposition 2. (Note that the relation (13) can be regarded as the connection formula for $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ at the turning point $t=0$.) In this sense the property (ii) has been well investigated for (P_J). We can also analyze the property (i) by numerical computations (though we have not yet tried seriously). Thus, what remains to be analyzed is the property (iii).

Concerning the local behavior of $\lambda_J(t;\alpha,\beta)$ at a regular-type fixed singular point of (P_J), let us consider the following particular case here: As a singular point we take $t=0$ of (P_{VI}). Further we take a solution $\lambda_0(t)$ of $F_{\mathrm{VI}}(\lambda_0(t),t)=0$ with the local behavior $\lambda_0(t)=a_1t+a_2t^2+\cdots$ (where $(1-1/a_1)^2=c_t/c_0$) near $t=0$ and the corresponding 2-parameter solution $\lambda_{\mathrm{VI}}(t;\alpha,\beta)$ of (P_{VI}) which is obtained by solving the reduced system (7)–(8) and by substituting its solution (9) into (6) and (4). Then, for the local behavior of the reduced system (7)–(8) we can prove the following

Proposition 3 *Every $f^{(l)}(t,\eta)$ in (8) has a simple pole at $t=0$ and*

$$(15)\qquad \begin{cases} \operatorname*{Res}_{t=0} f^{(0)}(t,\eta) &=& -\left(\sqrt{4c_0+\eta^{-2}}+\sqrt{4c_t+\eta^{-2}}\right), \\ \operatorname*{Res}_{t=0} f^{(1)}(t,\eta) &=& 1, \\ \operatorname*{Res}_{t=0} f^{(l)}(t,\eta) &=& 0 \qquad (l\geq 2). \end{cases}$$

(Cf. [T2].) Proposition 3 indicates that the solution $\lambda_{\mathrm{VI}}(t;\alpha,\beta)$ has a similar behavior at $t=0$ with the following local convergent solution constructed by Takano (cf. [Tka]):

$$\lambda(t;c_1,c_2)=\sum_{j,k\geq 0} a_{jk}(t)\left(c_1t^{\kappa_1+\kappa_2c_1c_2}\right)^j\left(c_2t^{-\kappa_1-\kappa_2c_1c_2}\right)^k$$

(κ_1 and κ_2 are some fixed constants determined by the equation and c_1 and c_2 are free parameters). This also suggests that our 2-parameter solutions $\lambda_J(t;\alpha,\beta)$ of (P_J) should have a nice structure at regular-type fixed singular points just like WKB solutions in the case of linear equations. However, our investigation is not sufficient to conclude that; we have several equations, several singular points, and several families of 2-parameter solutions (according to the choice of solutions of $F_J(\lambda_0(t),t)=0$). More complete description of the local behavior of $\lambda_J(t;\alpha,\beta)$ at fixed singular points will be discussed in our forthcoming papers.

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