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FOURIER TRANSFORMS AND DUALITY IN HYPERFUNCTIONS

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1 Introduction

1. The main purpose of this report is to describe some results obtained recently in the study of the Fourier-inverse transform in hyperfunctions and in a duality theory for hyperfunctions. The main motivation for studying this type of arguments comes from an attempt to construct a theory of the Fourier transform for second hyperfunctions which should be parallel to the theory of the Fourier transform for hyperfuntions described in [L2], [L3].We denote by $\mathcal{B}(U)$ the hyperfunctions on the open set U in R^n and by $\mathcal{A}(U)$ the subset of real-analytic functions. Germs of hyperfunctions or real-analytic functions at $0 \in R^n$ shall be denoted by $\mathcal{B}$, respectively by $\mathcal{A}$. Our results work best for the factor-space $\mathcal{B}/\mathcal{A}$.

Let us briefly recall the setting for the Fourier-inverse transform described in [L2].We denote by $\mathcal{H}$ the set of even sublinear functions $l : R^n \to R_+$ and, for given $l \in \mathcal{H}$ and $\varepsilon > 0$, by $L^2(C^n, l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$, the space of functions $\mu : C^n \to C$ which are locally in L^2 and satisfy

$$\mu \exp[-l(Re\,\zeta) + \varepsilon|Im\,\zeta|] \in L^2(C^n).$$

(Similarily, $L^2(R^n, l(\xi))$ denotes the space $\{f \in L^2_{loc}(R^n); f \exp[-l(\xi)] \in L^2(R^n)\}$.) If $\mu \in L^2(C^n, l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$ is given, we can associate a hyperfunction on $|x| < \varepsilon$ with it in the following way: we fix a finite number of open convex cones $\Gamma_j \subset \dot{R}^n = R^n \setminus \{0\}$, $j = 1, \ldots, s$ so that $\cup_{j=1}^s \Gamma_j = \dot{R}^n$ and consider functions $\mu_j : C^n \to C$, $j = 0, 1, \ldots, s$ so that

a) $\sum_{j=0}^{s} \mu_j = \mu$,

b) supp $\mu_j \subset \{\zeta \in C^n; Re\,\zeta \in \Gamma_j\}$, $j = 1, \dots, s$,

c) supp $\mu_0 \subset \{\zeta \in C^n; |Re\,\zeta| \leq 1\}$,

d) $|\mu_j(\zeta)| \leq |\mu(\zeta)|$, $\forall \zeta \in C^n$.

Next, we define functions $z \to h_j(z)$ for $j = 0, 1, \dots, s$, by

$$h_j(z) = (2\pi)^{-n} \int_{C^n} \exp[i\langle z, \zeta\rangle]\mu_j(\zeta)\, d\lambda, \tag{1.1}$$

where $d\lambda$ is the Lebesgue measure in C^n. It is then immediate that the h_j are analytic functions defined on the sets $\{z = x+iy \in C^n; |x| < \varepsilon, y \in \Gamma_j^{\perp}\}$ (x and y are assumed real, $\Gamma^{\perp}$ is the polar of Γ) for $j = 1, \dots, s$, whereas h_0 is analytic on $\{z \in C^n; |x| < \varepsilon, y \in R^n\}$. Also denote by $b(h)$ (if h is an analytic function defined on some wedge) the hyperfunctional boundary value of h. In particular, $u = \sum_{j=0}^{s} b(h_j)$, h_j as in (1.1), is then a hyperfunction in $\mathcal{B}(x; |x| < \varepsilon)$. It is easy to see that this hyperfunction does not depend on the particular way in which we have chosen the cones Γ_j, or splitted μ into the sum $\mu = \sum_{j=0}^{s} \mu_j$. We call u the Fourier-inverse transform of μ and shall write that $u = \mathcal{F}^{-1}(\mu)$. (This is similar to the way in which Fourier-inverse transforms in hyperfunctions are considered in [K], the main difference being that in [K]the functions μ and μ_j are defined on R^n, rather than on C^n. Indeed, we shall use the notation $\mathcal{F}^{-1}(\mu)$ also when $\mu \in L^2(R^n, l(\xi))$. In [M] a related construction is attributed to Carleman. The fact that we work on C^n and not on R^n is essential in our constructions and is related to the fact that our theory is local.) We recall the following results from [L2]:

Theorem 1.1. *a) Let $u \in \mathcal{B}(x; |x| < \varepsilon)$. Then there is $\mu \in L^2(C^n, l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$ so that $u = \mathcal{F}^{-1}(\mu)$.*

b) Assume that $\mu \in L^2(C^n, l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$ is given so that $\mathcal{F}^{-1}(\mu) = 0$ in $\mathcal{B}(x; |x| < \varepsilon)$. Also fix $\varepsilon' < \varepsilon$. Then there is $l' \in \mathcal{H}$ and $\chi_k \in L^2(C^n, l'(Re\,\zeta), -\varepsilon'|Im\,\zeta|)$, $k = 1, \dots, n$, so that

$$\mu = \sum_{j=1}^{n} (\partial/\partial\bar{\zeta}_k)\chi_k.$$

Let us also write the preceding theorem for germs of hyperfunctions at 0. It is then convenient to work with $(0,q)$-forms, identifying $(0,n)$-forms with functions. If we denote in fact by $X_{(0,q)}$ the space of $(0,q)$-forms with coefficients in X, then it follows from the preceding result that the inverse Fourier transform $\mathcal{F}^{-1}$ establishes an isomorphism

$$\frac{\cup_{\varepsilon>0, l\in\mathcal{H}} L^2_{(0,n)}(C^n, l(Re\,\zeta), -\varepsilon|Im\,\zeta|)}{\bar\partial[\cup_{\varepsilon'>0, l'\in\mathcal{H}} L^2_{(0,n-1)}(C^n, l'(Re\,\zeta), -\varepsilon'|Im\,\zeta|)]} \to \mathcal{B}.$$

Actually, "analysis" is much easier in the space $\mathcal{B}/\mathcal{A}$ than in $\mathcal{B}$. (We will give some arguments to justify this statement later on.) It is therefore convenient to see what results one can obtain from the above for $\mathcal{B}/\mathcal{A}$. Rather than working on all of C^n, we may then work on

$$U(c) = \{\zeta \in C^n; |Im\,\zeta| < c|Re\,\zeta|\},$$

or, actually, on what we call "germs" of these sets, $U(0)$. We shall in fact say that two functions f' defined on $U(c')$ and f'' defined at $U(c'')$ define the same germ at $U(0)$ if there is $c \le \min(c', c'')$ so that $f' = f''$ on $U(c)$. In this situation, we shall also write that $f' \sim f''$. Clearly $\bar\partial$ is well-defined on such germs, so we can consider $(0,q)$-forms with coefficients in $L^2(U(0), l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$. Moreover, if $f' \sim f''$, then $\mathcal{F}^{-1}(f - f'') \in \mathcal{A}(x; |x| < \varepsilon)$. This justifies the following notation:

$$L^2(U(0), l(Re\,\zeta), -\varepsilon|Im\,\zeta|) = \cup_{c>0} L^2(U(c), l(Re\,\zeta), -\varepsilon|Im\,\zeta|)/\sim.$$

We have then the following result:

Theorem 1.2. $\mathcal{F}^{-1}$ *defines by factorization an isomorphism:*

$$\frac{X_{(0,n)}}{X_{(0,n-1)}} \to \mathcal{B}/\mathcal{A}, \tag{1.2}$$

where $X = \cup_{\varepsilon>0, l\in\mathcal{H}} L^2(U(0), l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$.

2 A duality theory for $\mathcal{B}/\mathcal{A}$

1. The reason why it is often simpler to work in the space $\mathcal{B}/\mathcal{A}$ than in $\mathcal{B}$ is that for $\mathcal{B}/\mathcal{A}$ one can develop a duality theory which is much more

complete than what one can do for $\mathcal{B}$ itself. We shall explain at first some "positive" results in $\mathcal{B}/\mathcal{A}$ and then compare the situation with the one in $\mathcal{B}$. The spaces which we need for a duality theory related to $\mathcal{B}/\mathcal{A}$ are

$$\mathcal{A}(U(c); -l(Re\,\zeta), \varepsilon|Im\,\zeta|) = \{h \in \mathcal{A}(U(c)); \\ h\exp[l(Re\,\zeta) - \varepsilon|Im\,\zeta|] \in L^2(U(c))\}.$$

The topology is of course given by the $L^2(U(c))$-norm of $h\exp[l(Re\,\zeta) - \varepsilon|Im\,\zeta|]$.

The following two results imply that the spaces $\mathcal{A}(U(c); -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$ are "rich spaces", in the sense that they contain "many elements":

Proposition 2.1. *For any $c > 0$ there is $c' > 0$ and a domain of holomorphy $\Omega \subset C^n$ so that $U(c') \subset \Omega \subset U(c)$.*

(From the point of view of the applications we have in mind, the $U(c)$ behave thus as if they were domains of holomorphy.)

Proposition 2.2. *Consider $l \in \mathcal{H}$, $\delta > 0$ and $c < 1$. Then there is $l' \in \mathcal{H}$ and a plurisubharmonic function $\rho : U(c) \to R$ so that $-l(Re\,\zeta') \leq \rho(\zeta) \leq l(Re\,\zeta) + \delta|Im\,\zeta|$.*

Let us next consider $L \in \mathcal{A}'(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$. In view of the Hahn-Banach theorem, we can assume that L has the form

$$L(h) = \int_{U(c)} h(-\zeta)\mu(\zeta)\, d\lambda(\zeta),$$

for some $\mu \in L^2(U(c), l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$. Of course, μ is not unique with this property but we have the following result:

Proposition 2.3. *Fix $\varepsilon' < \varepsilon$. Assume $\nu \in L^2(U(c), l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$ is so that*

$$\int_{U(c)} h(-\zeta)\nu(\zeta)\, d\lambda(\zeta) = 0,$$

for all $h \in \mathcal{A}(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$, then it follows from the results in [H] that $\nu = \bar{\partial}\chi$ for some $\chi \in L^2(U(c'), l'(Re\,\zeta), -\varepsilon'|Im\,\zeta|)$, provided $c' > 0$, $l' \in \mathcal{H}$ are suitable.

(It is in this sense that we say that the spaces $\mathcal{A}'(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$ are very rich. Also cf. proposition 2.6 below.)

Let us also give the following definition:

Definition 2.4. *Let* $L, L' \in \mathcal{A}'(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$. *We say that* $L \sim L'$ *if we can find* $\varepsilon' > 0, d > 0, c > 0$ *so that* $|(L-L')(h)| \leq c$ *for any* $h \in \mathcal{A}(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$ *so that* $|h(\zeta)| \leq \exp[d|Re\,\zeta| + \varepsilon'|Im\,\zeta|]$.

It follows then from above that we have

Theorem 2.5. $\mathcal{B}/\mathcal{A}$ *is isomorphic to* $\cup_{\varepsilon,l}\mathcal{A}'(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)/\sim$.

We conclude that we can represent $\mathcal{B}/\mathcal{A}$ in the form of a factor space of analytic functionals. One can use this result to prove theorems on propagation of singularities as in [L1].

2. Let us compare the situation in $\mathcal{B}/\mathcal{A}$ with that in $\mathcal{B}$. In fact, it is standard to observe that for particular choices of $l \in \mathcal{H}$ the spaces $\mathcal{A}(C^n, -l(Re\,\zeta), -\varepsilon|Im\,\zeta|)$ are not "rich" at all. In fact, in the case $n = 1$ the following result is classical (in the theory of non-quasianalytic classes):

Proposition 2.6. *Assume that* $l : R^n \to R_+$, $l(\xi) \leq 1 + |\xi|$, *and* $h \in \mathcal{A}(C)$, $h \neq 0$, *are given with* $|h(\zeta)| \leq \exp[-l(Re\,\zeta) + A|Im\,\zeta|]$. *Then it follows that*

$$\int_{-\infty}^{\infty} \frac{l(\xi)}{1+\xi^2}\, d\xi < \infty. \tag{2.1}$$

On the other hand, it is very easy to construct sub-linear positive functions $l \in \mathcal{H}$, $n = 1$, for which (2.1) does not hold and for such l therefore $\mathcal{A}(C, -l(Re\,\zeta), \varepsilon|Im\,\zeta|) = \{0\}$. Of course this also shows that the spaces $\mathcal{A}(C^n, -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$ are often trivial even when $n \geq 1$, so that a theory of duality similar to the above for $\mathcal{B}/\mathcal{A}$ cannot exist for $\mathcal{B}$.

3 Fourier inverses of elements in $\mathcal{A}(U(c), l, -\varepsilon|Im\,\zeta|)$

1. The results in the following sections are for the moment not directly related to our ongoing work on second hyperfunctions. They are however a direct continuation of the results in the preceding two sections and should shed a new light on the relation between ultradistributions and hyperfunctions. We start with the following

Remark 3.1. *If* $h \in \mathcal{A}(C^n, -l(Re\,\zeta), \varepsilon|Im\,\zeta)$, *then the restriction of* h *to* R^n *is an element in* $L^2(R^n)$ *and as such has a Fourier inverse* $\mathcal{F}^{-1}(h)$ *in* $L^2(R^n)$. *(Which is of course equal to its Fourier-inverse transform calculated in terms of hyperfunctions in the sense of [K] or [M].) When*

$l \in \mathcal{H}$ is rapidly increasing at infinity, then $\mathcal{F}^{-1}(h)$ has to be very regular in terms of differentiability properties. Thus e.g., if $l(\xi) = (1 + |\xi|)^{1/2}$, then $\mathcal{F}^{-1}(h)$ is in the Gevrey class $\mathcal{G}^2(R^n)$ of functions on R^n for which there exists $c > 0$ so that $|(\partial/\partial x)^\alpha \mathcal{F}^{-1}h(x)| \le c^{|\alpha|+1}|\alpha|^{2|\alpha|}$, $\forall\alpha$, (c should not depend on x as long as x stays in a fixed compact set in R^n.) The dual of $\mathcal{A}(C^n, -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$ corresponds then in this case roughly speaking to Gevrey-ultradistributions of order two. This gives an interpretation of theorem 2.5: it shows that in some sense, the elements from $\mathcal{B}/\mathcal{A}$ are just ultradistributions with weight l factorized by $\mathcal{A}$. We shall add additional evidence in this same direction later on.

In view of the preceding remark it seems interesting to analyze Fourier-inverses of functions from $\mathcal{A}(C^n, -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$ in more detail. Actually we shall start our analysis with a study of Fourier-inverses of elements in the larger space $\mathcal{A}(C^n, l(Re\,\zeta), \varepsilon|Im\,\zeta|)$ or, rather of the restriction of such functions to R^n. Indeed, such functions lie in $L^2(R^n, l(\xi))$, so again we can define $\mathcal{F}^{-1}(h_{|R^n})$ in the sense of hyperfunctions. We have the following result:

Theorem 3.2. *Consider $h \in \mathcal{A}(C^n, l(Re\,\zeta), \varepsilon|Im\,\zeta|)$. $\mathcal{F}^{-1}(h)$ is real-analytic for $|x| > \varepsilon$ and is rapidly decreasing at infinity. In addition, if x^0 is fixed with $|x^0| > \varepsilon$, then there are constants c_1, c', c'' so that*

$$|\partial_x^\alpha(\mathcal{F}^{-1}h)(x)| \le c_1 c'^{|\alpha|}|\alpha|^{|\alpha|} \text{ for } |x - x^0| < c''. \tag{3.1}$$

Here c', c'' depend only on c, ε, x^0 and c_1 depends in addition on the norm of h in $\mathcal{A}(U(c),\ l(Re\,\zeta),\ \varepsilon|Im\,\zeta|)$.

4 The local Fourier transform

1. We describe here a calculus for a Fourier transform which is defined only locally and characterizes functions or hyperfunctions only locally. The inverse of this transform will be the Fourier-inverse transform in hyperfunctions which we have considered above. (Recall that this Fourier inverse transform referred also to hyperfunctions which were defined on subsets in R^n.) The local Fourier transform will be defined only modulo exponentially decreasing functions, whereas for the inverse transform, it should be calculated only modulo real-analytic functions. We also notice right away that our Fourier transform is closely related to the

FBI-transform of Sjöstrand. In particular part of the results which we consider are quite close to what happens in Sjöstrand's theory.

To make the definitions easier to understand, let us assume at first that f is a L^1-function defined in a neighborhood of $|x| \leq A$. We define its FBI transform then by

$$(4.1) \qquad (FBI_A f)(\xi, y) = \int_{|x|\leq A} \exp[-i\langle x, \xi\rangle - |\xi||x-y|^2/2]\, f(x)\, dx.$$

Part of the usefulness of the FBI-transform comes from the fact that it is quite easy to recover f on $|x| \leq A$ from its FBI-transform. This comes from the remark that

$$\int_{R^n} |\xi|^{n/2} \exp[-|\xi||x-y|^2/2]\, dy = (2\pi)^{n/2}.$$

It follows in fact from this that

$$\int_{R^n} (FBI\ f)(\xi, y)|\xi|^{n/2}\, dy = (2\pi)^{n/2} \hat{f}(\xi),$$

where $\hat{f}$ is the standard Fourier transform of f. One of the basic classical remarks is now that if f is real-analytic in a neighborhood of some point x^0 which satisfies $|x^0| < A$, then $FBI_A f(\xi, y)$ will be exponentially decreasing for y in a neighborhood of x^0. We shall use this later on in the following form:

Remark 4.1. *Assume that f is real-analytic for $|x| \geq \varepsilon$. Also fix B with $\varepsilon < B < A$. Then there are $c > 0$, $d > 0$, so that*

$$|FBI_A f(\xi, y)| \leq c \exp[-d|\xi|], \text{ for } B \leq |y| \leq A.$$

2. In (4.1) we assumed f to be a function in $L^1(x; |x| \leq A)$, but we can consider the FBI-transform also in more general situations by slightly changing definitions. Let us in fact assume that f is a hyperfunction supported in $|x| \leq A$, say. f defines then a real analytic functional concentrated in $|x| \leq A$. In particular it follows that if we fix some complex neighborhood U of $\{x; |x| \leq A\}$, then there will be some constant c so that

$$|f(h)| \leq c \sup_{z\in U} |h(z)|, \forall h \in \mathcal{A}(C^n).$$

(Here we have already assumed that we have associated with f an analytic functional on $\mathcal{A}(C^n)$.) Then we define $(FBI_A f)(\xi, y) = f(\exp[-i\langle x, \xi\rangle - |\xi||x-y|^2/2])$. It is useful to note that one can extend $\xi \to (FBI_A f)(\xi, y)$ to an analytic function $\zeta \to (FBI_A f)(\zeta, y)$ for $\zeta \in U(1)$. In fact, let us for this purpose denote by $|\zeta\lceil$ the expression $|\zeta\lceil = (\zeta_1^2 + \cdots + \zeta^2)^{1/2}$, where the square root $t^{1/2}$ is defined for $t \in \{t' \in C; Im\, t' > 0\}$ by the condition Re $t^{1/2} \geq 0$. Then the function

$$(\zeta, y) \to (FBI_A f)(\zeta, y) = f(\exp[-i\langle x, \zeta\rangle - |\zeta\lceil|x-y|^2/2])$$

defines the desired analytic extension to $|Im\,\zeta| < |Re\,\zeta|$. The following remark is trivial:

Remark 4.2. *If $f \in \mathcal{B}$ and supp $f \subset \{x; |x| \leq \varepsilon\}$, then there is $l \in \mathcal{H}$ and c so that*

$$|(FBI_A f)(\zeta, y)| \leq c \exp[l(Re\,\zeta) + l(Im\,\zeta) + \varepsilon|Im\,\zeta|], \text{ if } |Im\,\zeta| < |Re\,\zeta|.$$

3. Together with the FBI-transform, we shall now consider the "local Fourier transform"

$$(\mathcal{F}_{loc,A,B} f)(\zeta) = \int_{|y|\leq B} (FBI_A f)(\zeta, y)\, dy,$$

where $B < A$: the parameter y has "disappeared" in view of the integration. From remark 4.2 we obtain

Remark 4.3. *Let supp $f \subset \{x; |x| \leq \varepsilon\}$, $\varepsilon < A$. Then there is c, $l \in \mathcal{H}$ so that*

$$|(\mathcal{F}_{loc,A,B} f)(\zeta)| \leq c \exp[l(Re\,\zeta) + l(Im\,\zeta) + \varepsilon|Im\,\zeta|], \text{ if } |Im\,\zeta| < |Re\,\zeta|.$$

Also the following result, in which we compare the local Fourier transform with the standard Fourier transform in a particular case, is quite easy.

Proposition 4.4. *Let $B_1 < B < A$ be fixed and assume that supp $f \subset \{x; |x| \leq B_1\}$. Then there are c, d, so that*

$$|(\hat{f} - \mathcal{F}_{loc,A,B} f)(\zeta)| \leq c \exp[-d|\zeta|], \text{ if } |Im\,\zeta| < |Re\,\zeta|/2.$$

4. The assumption that the support of f lies in a rather small set is too restrictive in what follows. In fact, we are mainly interested in the case when f is real-analytic in some region of form $\varepsilon \leq |x| \leq A'$ for some $A' > A$. The first remark is here the following:

Remark 4.5. *Let $0 < \varepsilon < B < A < A'$ be given and assume that $f \in \mathcal{B}(|x| < A')$ is real-analytic for $|x| > \varepsilon$. Define a hyperfunction f_A as follows:*

$$f_A(x) = f(x) \text{ for } |x| \leq A, f_A(x) = 0, \text{ for } |x| > A, \tag{4.2}$$

and consider $\mathcal{F}_{loc,A,B} f_A(\zeta)$. Then $\mathcal{F}_{loc,A,B} f_A(\zeta)$ does not depend modulo an exponentially decreasing term on the choice of A, B. This follows indeed from remark 4.1.

Since we are interested only in a calculus modulo exponentially decaying terms, we shall write henceforth sometimes "$\mathcal{F}_{loc,A,B} f$" rather than "$\mathcal{F}_{loc,A,B} f_A$".

5. We now consider an extension of remark 4.2.

Theorem 4.6. *Consider $\varepsilon < \varepsilon' < B < A < A'$ and let $f \in \mathcal{B}(x; |x| < A')$ be real-analytic for $\varepsilon < |x| < A'$. Then there are constants c, c_1 and $l \in \mathcal{H}$ so that the function $R^n \ni \xi \to (\mathcal{F}_{loc,A,B} f)(\xi)$ can be extended to an analytic function $F : \{\zeta \in C^n; |Im\, \zeta| < c|Re\, \zeta|\} \to C$ satisfying*

$$|F(\zeta)| \leq c_1 \exp[l(Re\, \zeta) + \varepsilon' |Im\, \zeta|], \text{ if } |Im\, \zeta| < c|Re\, \zeta|. \tag{4.3}$$

Remark 4.7. *Here the interesting part is not so much the fact that $\xi \to (\mathcal{F}_{loc,A,B} f)(\xi)$ admits an extension (which fact is trivial), but the estimate we can obtain for it.*

The following two results now show how the local Fourier transform is related to the Fourier-inverse transform considered above.

Theorem 4.8. *Let $f \in C^\infty(x; |x| < A)$ be real-analytic for $|x| > \varepsilon$ and fix B with $\varepsilon < B < A$. Then $f - \mathcal{F}^{-1} \mathcal{F}_{loc,A,B} f$ is real-analytic for $|x| < B$.*

Theorem 4.9. *Assume $h \in \mathcal{A}(U(c), 0, \varepsilon |Im\, \zeta|)$ and fix A, B with $A > B > \varepsilon$. Then there are constants c, d such that*

$$|(h - \mathcal{F}_{loc,A,B} \mathcal{F}^{-1} h)(\zeta)| \leq c \exp[-d |Re\, \zeta|], \text{ if } |Im\, \zeta| < |Re\, \zeta|/2.$$

5 The spaces $\mathcal{E}_{-l,\varepsilon,c}$ and $\mathcal{E}_{-l,\varepsilon,c}/\mathcal{A}$

1. Let ε, A, B be fixed with $0 < \varepsilon < B < A$. Also consider $c > 0$. We denote by $\mathcal{E}_{-l,\varepsilon,c}$ the space of functions f so that $\mathcal{F}_{loc,A,B}f$ is analytic on $|Im\,\zeta| < c|Re\,\zeta|$ and satisfies for some constant $\tilde{c}$

$$(5.1) \qquad |(\mathcal{F}_{loc,A,B}f)(\zeta)| \le \tilde{c}\exp[-l(Re\,\zeta) + \varepsilon|Im\,\zeta|], |Im\,\zeta| < c|Re\,\zeta|.$$

The smallest constant $\tilde{c}$ for which (5.1) is valid is called the norm of f in $\mathcal{E}_{-l,\varepsilon,c}$. Note that for suitable l the assumption implies that $\mathcal{F}^{-1}\mathcal{F}_{loc,A,B}f$ is C^∞ and real-analytic for $|x| > \varepsilon$. Since we also know that $f - \mathcal{F}^{-1}\mathcal{F}_{loc,A,B}f$ is real-analytic for $|x| < B$, we can conclude that we have:

Proposition 5.1. *If $l \in \mathcal{H}$ increases sufficiently rapidly at infinity, it follows that $\mathcal{E}_{-l,\varepsilon,c} \subset C^\infty(x; |x| < B)$ and the elements $f \in \mathcal{E}_{-l,\varepsilon,c}$ are real-analytic for $\varepsilon < |x| < B$.*

Remark 5.2. *If $h \in \mathcal{A}(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$, then $f = \mathcal{F}^{-1}h \in \mathcal{E}_{-l,\varepsilon,c'}$ for some c'. Indeed, $(\mathcal{F}_{loc,A,B}\mathcal{F}^{-1}h - h)(\zeta) = O(\exp[-d|\zeta|])$ so that $\mathcal{F}_{loc,A,B}f$ satisfies (5.1).*

2. Let us now consider $u \in \mathcal{B}/\mathcal{A}$. Then we can find $\varepsilon, l \in \mathcal{H}$ and $\mu \in L^2(C^n, l, -\varepsilon|Im\,\zeta|)$ with $u = \mathcal{F}^{-1}(\mu)$ in $\mathcal{B}/\mathcal{A}$. We can then at first define a functional on $\mathcal{A}(U(c), -l, \varepsilon|Im\,\zeta|)$ associated with u by

$$(5.2) \qquad T(h) = \int_{U(c)} h(-\zeta)\mu(\zeta)d\lambda(\zeta).$$

We also write $u(\mathcal{F}^{-1}h) = T(h)$ (this is thus the definition of $u(\mathcal{F}^{-1}h)$) in this situation, although the values of $T(h)$ depend on the choice of μ. However, different values of μ will lead to equivalent analytic functionals. Conversely, u is determined uniquely in $\mathcal{B}/\mathcal{A}$ by this duality. Indeed, if $u(\mathcal{F}^{-1}h) = 0$ for any $h \in \mathcal{A}(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$, then $\mu = \sum_{k=1}^n \bar{\partial}_k\chi_k$, $\chi_j \in L^2(U(c), l'(Re\,\zeta), -\varepsilon'|Im\,\zeta|)$, so $u = \mathcal{F}^{-1}(\mu)$ is real analytic near zero.

The functional (5.2) leads to a useful test of regularity:

Proposition 5.3. *Let $u \in \mathcal{B}/\mathcal{A}$ be given and let T be associated with a representant of u as in (5.2). Then u vanishes (i.e. some, and then for that matter, any representant v of the class u in $\mathcal{B}/\mathcal{A}$ is real-analytic) if and only if there is $d > 0$, $\varepsilon' > 0$, $c_1 > 0$ so that $h \in \mathcal{A}(U(c), -l, \varepsilon|Im\,\zeta|)$, $|h(\zeta)| \le \exp[d|Re\,\zeta| + \varepsilon'|Im\,\zeta|]$ implies $|T(h)| \le c$. (This must happen for any T which gives u. We say in this situations also that $T \sim 0$.)*

Remark 5.3. *It is part of the statement that the condition regarding T does not depend on the choice of μ in (5.2), provided of course that μ is a representation function for u.*

We also note that if in (5.2) we replace integration on $|Im\,\zeta| < c|Re\,\zeta|$ by integration on $|Im\,\zeta| < c'|Re\,\zeta|$ for some $c' < c$, $c' > 0$, then the functional $T'(h)$ so obtained is equivalent with $T(h)$.

3. We want to give an interpretation in x-space to the duality $u(\mathcal{F}^{-1}h)$, and more generally, to $u(f)$ for $f \in \mathcal{E}_{-l,\varepsilon,c}$. The first step is that if u is fixed, then we can define for suitable l, ε, c a functional $S : \mathcal{E}_{-l,\varepsilon,c} \to C$ by

$$S(f) = \int_{U(c)} (\mathcal{F}_{loc,A,B}f)(-\zeta)\mu(\zeta)\,d\lambda(\zeta),$$

where μ is chosen again with $u = \mathcal{F}^{-1}(\mu)$. This is thus in close analogy with (5.2) Also here u is determined in $\mathcal{B}/\mathcal{A}$ by its action on $\mathcal{E}_{-l,\varepsilon,c}$. Assume in fact that $S(f) = 0$ for any $f \in \mathcal{E}_{-l,\varepsilon,c}$. Since $\mathcal{F}^{-1}h \in \mathcal{E}_{-l,\varepsilon,c}$ (by remark (5.2) if h lies in $\mathcal{A}(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$, it follows that $S(\mathcal{F}^{-1}h) = 0$ for any $h \in \mathcal{A}(U(c)), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$. It also follows then in fact that $h - \mathcal{F}_{loc,A,B}\mathcal{F}^{-1}h = 0(\exp[-d|\zeta|])$, with uniform "constants" if we are given an estimate $|h(\zeta)| \le \exp[d|Re\,\zeta| + \varepsilon|Im\,\zeta|]$. We next write

$$\int_{U(c)} h(-\zeta)\mu(\zeta)\,d\lambda(\zeta)$$
$$= \int_{U(c)} [(h - \mathcal{F}_{loc,A,B}\mathcal{F}^{-1}h)(-\zeta) + \mathcal{F}_{loc,A,B}\mathcal{F}^{-1}h(-\zeta)]\mu(\zeta)\,d\lambda(\zeta).$$

We conclude that $h \in \mathcal{A}(U(c), -l', \varepsilon'|Im\,\zeta|)$ and $|h(\zeta)| \le \exp[d|Re\,\zeta| + \varepsilon|Im\,\zeta|]$ together imply that $|\int_{U(c)} h(-\zeta)\mu(\zeta)\,d\lambda(\zeta)| \le c$. This implies of course that $\mathcal{F}^{-1}(\mu)$ is real-analytic on $|x| < \varepsilon$.
We have thus associated a functional on $\mathcal{E}_{-l,\varepsilon,c}$ with μ. Conversely, if a continuous functional is given on $\mathcal{E}_{-l,\varepsilon,c}$ then we can use that $\mathcal{F}^{-1}[\mathcal{A}(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)] \subset \mathcal{E}_{-l,\varepsilon,c}$ and obtain a functional on $\mathcal{A}(U(c), -l(Re\,\zeta), \varepsilon|Im\,\zeta|)$. This functional will then define an element in $\mathcal{B}/\mathcal{A}$ so we can associate hyperfunctions with functionals on $\mathcal{E}_{-l,\varepsilon,c}$. One can check that if one starts from some hyperfunction u, associates with this a functional on $\mathcal{E}_{-l,\varepsilon,c}$ and then associates with this functional

an hyperfunction as above, then one reobtains, modulo a real analytic function, the hyperfunction from which one started in the first place.

4. We want next to transform the integral

$$\int (\mathcal{F}_{loc,A,B} f)(-\zeta)\mu(\zeta)\, d\lambda(\zeta)$$

into an expression calculated in the space of the z-variables (which is the complexification of the base-space). We may assume for this purpose that $u = b(H)$ for some $H \in \mathcal{A}(U \times iG_d)$, where $U = \{x \in R^n; |x| < A', A' > A\}$, G is some open convex cone in $\dot{R}^n$ and $G_d = \{y; y \in G, |y| < d\}$. If $\mu : C^n \to C$ is a representation function of H in the sense that

$$H(z) = \int_{C^n} \exp[i\langle z, \zeta\rangle]\mu(\zeta)\, d\lambda, |x| < A', y \in G_d,$$

we may thus assume that it satisfies
(5.3)
$$|\mu(\zeta)| \leq c\exp[l(Re\,\zeta) - (A + c')|Im\,\zeta| - H_{\Gamma,d}(Re\,\zeta)] \text{ for some } c' > 0,$$

where $\Gamma = G^{\perp}$ and where $H_\Gamma(\xi) = \sup_{x\in G_d}\langle x, \xi\rangle$ is the supporting function of G_d. Denote $\Omega(\varepsilon) = \{x \in R^n; |x| < \varepsilon\}$. We also associate with $f \in \mathcal{E}_{-l,\varepsilon,c}$, $\delta > 0$, $t^0 \in G$, $\delta > 0$ small, the analytic functional f_{δ,t^0} on $\mathcal{A}(z; |x| \leq A, y \in G_d)$:

$$h \to \int_{|x|\leq A} f(x + it^0 \text{ dist}(x, \Omega(\varepsilon)))h(x + it^0 \text{ dist}(x, \Omega(\varepsilon)) + i\delta t^0)d\,z$$
$$= f_{\delta,t^0}(h),$$

where we assume that $\delta > 0$ and the norm of t^0 are small enough in order to have that t^0 dist $(x, \Omega(\varepsilon)) + \delta t^0 \in G_d$ for $|x| \leq A$.

Also note that f_{δ,t^0} can be regarded as a density on the set $x \to x + it^0 \text{ dist}(x, \Omega(\varepsilon)) + i\delta t^0$. Actually, f_{δ,t^0} is just the translation by $i\delta t^0$ of the analytic functional f_{0,t^0}, where explicitly

$$(5.4)\quad f_{0,t^0}(h) = \int_{|x|\leq A} f(x+it^0 \text{ dist}(x, \Omega(\varepsilon)))h(x+it^0 \text{ dist}(x, \Omega(\varepsilon)))d\,z.$$

(Here h is asumed real-analytic in a neighborhood of $\{x; |x| \leq A\}$. Note that for $|x| \leq \varepsilon$ we integrate on a real contour.) At the level of Fourier-Borel transforms of analytic functionals, it follows thus that

$$(\mathcal{F} f_{\delta,t^0})(\zeta) = (\mathcal{F} f_{0,t^0})(\zeta) \exp[-\delta \langle t^0, \zeta \rangle]. \tag{5.5}$$

As for f_{0,t^0}, it is carried by the set $\{x+it^0 \text{ dist } (x, \Omega(\varepsilon)); |x| \leq A\}$. (Recall that analytic functionals have a "carrier" rather than a "support".) One can now show that if t^0 is suitably small, then the functional

$$f \to \lim_{\delta \to 0} f_{\delta,t^0}(H) = S'(f), \tag{5.6}$$

exists for any $f \in \mathcal{E}_{-l,\varepsilon,c}$ and is essentially equivalent with $S(f)$, in the sense that the two functionals S and S' both determine $u = b(H)$ in $\mathcal{B}/\mathcal{A}$.

The upshot of this discussion is roughly speaking that there are natural spaces of functions, namely the spaces $\mathcal{E}_{-l,\varepsilon,c}$, so that the elements in $\mathcal{B}/\mathcal{A}$ can be regarded as functionals on these spaces. We conclude this report with some remarks on the spaces $\mathcal{E}_{-l,\varepsilon,c}$. Our first remark is that:

Proposition 5.5. *$\mathcal{E}_{-l,\varepsilon,c}$-classes are stable under multiplication.*

This is a consequence of the following proposition.

Proposition 5.6. *Let $l \in \mathcal{H}$ be given. Then there is $l' \in \mathcal{H}$ with the following properties: a) if $f, g \in \mathcal{A}(U(c), -l', \varepsilon|Im\,\zeta|)$, then h defined for $\xi \in R^n$ by*

$$h(\xi) = \int_{R^n} f(\xi - \theta) g(\theta)\, d\theta, \tag{5.7}$$

lies in $L^2(R^n, -l)$ and admits an analytic estension to the set $|Im\,\zeta| < c'|Re\,\zeta|$ which stays in $\mathcal{A}(U(c'), -l, \varepsilon|Im\,\zeta|)$.

b) A similar statement is valid if $f \in \mathcal{A}(U(c), l, \varepsilon|Im\,\zeta|)$, and $g \in \mathcal{A}(U(c)$, $-l, \varepsilon|Im\,\zeta|)$. In that case, h defined by (5.7) stays in $L^2(R^n, l')$ and admits an analytic extension to an element in $\mathcal{A}(U(c'), l', \varepsilon|Im\,\zeta|)$.

Finally, we want to discuss the relation between the spaces $\mathcal{E}_{-l,\varepsilon,c}$ and inhomogeneous Gevrey classes introduced in [LR].We recall at first the following definition from [LR]:

Definition 5.7. *Consider $l \in \mathcal{H}$ and let f be a C^∞ function defined on $|x| < A'$. We write that $f \in \mathcal{G}_l(x; |x| < A')$ if we can find for every $A < A'$ a sequence $k \to f_k$ of $C_0^\infty(x; |x| \leq (A + A'')/2)$ functions so that $f(x) = f_k(x)$ for $|x| < A$, so that*

$$|\hat{f}_j(\xi)| \leq c(cj/l(\xi))^j, \forall j, \forall \xi \in R^n,$$

and so that the sequence $j \to f_j$ is bounded in $\mathcal{E}'(R^n)$. We say that such functions f belong to the "inhomogeneous Gevrey class" $\mathcal{G}_l$.

We have then the following result:

Proposition 5.8. *Let $l' \in \mathcal{H}$ be given. Then we can find $l \in \mathcal{H}$ with the following property: assume that $f \in G_l$ on $|x| < A + c$ and that it is real analytic for $\varepsilon < |x| < A + c$. Then $f \in \mathcal{E}_{-l',\varepsilon',c'}$ if $\varepsilon' > \varepsilon$.*

The proofs of the results in this report will appear elsewhere.

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