数理解析研究所講究録
1090 巻 1999 年 174-179



# On the uniqueness theorem for nonlinear singular partial differential equations

田原 秀敏 (Hidetoshi TAHARA)
(Dept. of Math., Sophia University, Tokyo)

In this note, I will discuss the uniqueness of the solution of nonlinear singular partial differential equations

$$(t\partial/\partial t)^m u = F(t,x,\{(t\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j+|\alpha|\le m, j<m}).$$

**Notations.** $t\in \boldsymbol{C}$, $x=(x_1,\ldots,x_n)\in \boldsymbol{C}^n$, $\boldsymbol{N}=\{0,1,2,\ldots\}$, $\boldsymbol{N}^*=\{1,2,\ldots\}$, $m\in \boldsymbol{N}^*$, $j\in \boldsymbol{N}$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in \boldsymbol{N}^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$,

$$\Big(\frac{\partial}{\partial x}\Big)^\alpha=\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n},$$

and $N=\#\{(j,\alpha)\in \boldsymbol{N}\times \boldsymbol{N}^n; j+|\alpha|\le m \text{ and } j<m\}$. We denote by $\mathcal{R}(\boldsymbol{C}\setminus\{0\})$ the universal covering space of $\boldsymbol{C}\setminus\{0\}$.

## §1. Equations and assumptions.

Let

$$t\in \boldsymbol{C},\quad x=(x_1,\ldots,x_n)\in \boldsymbol{C}^n,\quad Z=\{Z_{j,\alpha}\}_{j+|\alpha|\le m, j<m}\in \boldsymbol{C}^N,$$

and let $F(t,x,Z)$ be a function in $(t,x,Z)$. In this note I will discuss the uniqueness of the solution of the following equation

$$\text{(E)}\qquad \Big(t\frac{\partial}{\partial t}\Big)^m u=F\Big(t,x,\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big\}_{j+|\alpha|\le m, j<m}\Big)$$

with an unknown function $u=u(t,x)$.

The main assumptions are as follows:

(A$_1$) $F(t,x,Z)$ is holomorphic in a neighborhood of $(t,x,Z)=(0,0,0)$;

(A$_2$) $F(0,x,0)\equiv 0$ near $x=0$;

(A$_3$) $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0,x,0)\equiv 0$ near $x=0$, if $|\alpha|>0$.

We denote by $\lambda_1(x),\dots,\lambda_m(x)$ the roots of the equation in $\lambda$:

$$\lambda^m-\sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\lambda^j=0$$

and call them the characteristic exponents of (E).

**Examples.** The followings are typical examples of our equation:

(1) $t\dfrac{\partial u}{\partial t}=\lambda u+u\Big(\dfrac{\partial u}{\partial x}\Big)$,

(2) $\Big(t\dfrac{\partial}{\partial t}\Big)^2u=3u\Big(\dfrac{\partial^2 u}{\partial x^2}\Big)$,

(3) $\Big(t\dfrac{\partial}{\partial t}\Big)^2u+\Big(t\dfrac{\partial}{\partial t}\Big)u=(2u+x+1)\Big(\dfrac{\partial u}{\partial x}\Big)^2$.

## §2. Some results by Gérard-Tahara (1993).

Gérard-Tahara [2] proved the following result on the existence of holomorphic solutions.

**Theorem 1 (holomorphic solutions).** *Assume* (A$_1$), (A$_2$) *and* (A$_3$). *If* $\lambda_i(0)\notin\{1,2,3,\cdots\}$ *for* $i=1,\cdots,m$, *the equation* (E) *has a unique holomorphic solution* $u(t,x)$ *near the origin of* $\boldsymbol{C}\times\boldsymbol{C}^n$ *satisfying* $u(0,x)\equiv 0$.

Moreover, about the uniqueness of the solution of (E), Gérard-Tahara [2] has proved Theorem 2 below.

**Definition 1.** We denote by $\mathcal{S}_+$ the set of functions $u(t,x)$ satisfying the following: $u(t,x)$ is a holomorphic function on $\{(t,x) \in \mathcal{R}(\boldsymbol{C}\backslash\{0\}) \times \boldsymbol{C}^n\,;\, 0 < |t| < \varepsilon, |\arg t| < \theta$ and $|x| \leq \delta\}$ for some $\varepsilon > 0$, $\theta > 0$, $\delta > 0$ and satisfies

$$\max_{|x|\leq\delta} |u(t,x)| = O(|t|^a) \ \ (\text{as } t \longrightarrow 0)$$

for some $a > 0$.

**Theorem 2 (Uniqueness of the solution).** *Assume* $(\mathrm{A}_1)$, $(\mathrm{A}_2)$ *and* $(\mathrm{A}_3)$. *If*

$$\operatorname{Re}\lambda_i(0) \leq 0 \quad (i = 1, \ldots, m)$$

*holds, the uniqueness of the solution of* (E) is valid in $\mathcal{S}_+$.

Since in this case the equation (E) has a unique holomorphic solution, the above uniqueness theorem yields .

**Corollary.** *Assume* $(\mathrm{A}_1)$,$(\mathrm{A}_2)$,$(\mathrm{A}_3)$ *and*

$$\operatorname{Re}\lambda_i(0) \leq 0 \quad (i = 1, \ldots, m).$$

*Then, if $u(t,x)$ is a solution of (E) belonging to $\mathcal{S}_+$, $u(t,x)$ is holomorphic in a neighborhood of the origin.*

Thus, from the uniqueness theorem we can get the result on removable singularities of the solution of (E).

## §3. New uniqueness theorem.

### 3.1. Class of solutions.

A function $\mu(t)$ on $(0,T)$ is called *a weight function* if it satisfies the following conditions $\mu_1) \sim \mu_4)$:

$\mu_1)$ $\quad \mu(t) \in C^0((0,T))$,

$\mu_2)$ $\quad \mu(t) > 0$ on $(0,T)$ and $\mu(t)$ is increasing in $t$,

$\mu_3)$ $\quad \displaystyle\int_0^T \frac{\mu(s)}{s} ds < \infty$,

$\mu_4)$ $\quad \mu(t + ct) = O(\mu(t))$ (as $t \longrightarrow +0$) for some $c > 0$.

By $\mu_2$) and $\mu_3$) the condition $\mu(t) \longrightarrow 0$ (as $t \longrightarrow +0$) is clear. The following functions are typical examples:

$$\mu(t) = t^a, \quad \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $a > 0$, $b > 1$, $c > 1$.

**Definition 2.** For $a > 0$, we denote by $\mathcal{S}_a(\mu(t))$ the set of functions $u(t,x)$ satisfying the following: $u(t,x)$ is a holomorphic function on $\{(t,x) \in \mathcal{R}(\boldsymbol{C} \setminus \{0\}) \times \boldsymbol{C}^n\,;\, 0 < |t| < \varepsilon, |\arg t| < \theta \text{ and } |x| \leq \delta\}$ for some $\varepsilon > 0$, $\theta > 0$, $\delta > 0$ and satisfies

$$\max_{|x|\leq\delta} |u(t,x)| = O(\mu(|t|)^a) \quad (\text{as } t \longrightarrow 0).$$

**Remark 1.**

$$\mathcal{S}_+ = \bigcup_{a>0} \mathcal{S}_a(\mu(t) \equiv t).$$

Note that $\mu(t) \equiv t$ is a weight function.

### 3.2. A conjecture.

About the uniqueness of the solution of (E) in $\mathcal{S}_a(\mu(t))$, I have one conjecture:

**Conjecture.** *Assume* ($A_1$), ($A_2$) *and* ($A_3$). *Let* $\mu(t)$ *be a weight function. If*

$$\operatorname{Re}\lambda_i(0) \leq 0 \quad (i = 1, \ldots, m)$$

*holds, the uniqueness of the solution of* (E) is valid in $\mathcal{S}_m(\mu(t))$.

In the case $m = 1$ this is already proved (see [1]). But in the case $m \geq 2$ this is still open. In the next section I will report a weaker result.

### 3.3. A weaker result.

**Theorem A ([5]).** *Assume* ($A_1$), ($A_2$) *and* ($A_3$). *Let* $\mu(t)$ *be a weight function. If*

$$\operatorname{Re}\lambda_i(x) \leq 0 \quad (i = 1, \ldots, m)$$

*holds in a neighborhood of* $x = 0$, *the uniqueness of the solution of* (E) is valid in $\mathcal{S}_m(\mu(t))$.

**Example 1.** Let us consider

$$(e_1) \qquad \Big(t\frac{\partial}{\partial t}\Big)^2 u = 3u\Big(\frac{\partial^2 u}{\partial x^2}\Big)$$

where $(t, x) \in \boldsymbol{C}^2$. The characteristic exponents are $\lambda_1 = 0$ and $\lambda_2 = 0$. In this case we have:

1) $u(t, x) \equiv 0$ is the unique holomorphic solution of $(e_1)$ under the condition $u(0, x) \equiv 0$.

2) By Theorem A we see that the uniqueness of the solution of $(e_1)$ is valid in $\mathcal{S}_2(\mu(t))$ for any weight function $\mu(t)$.

3) Since $\mathcal{S}_a(\mu(t)) \subset \mathcal{S}_2(\mu(t))$ holds for $a \geq 2$, the uniqueness of the solution of $(e_1)$ is valid in $\mathcal{S}_a(\mu(t))$ for any $a \geq 2$ and any weight function $\mu(t)$.

4) Note that $(e_1)$ has a family of non-trivial solutions

$$u(t, x) = \frac{x^2 + \alpha x + \beta}{(C - \log t)^2} \qquad (\alpha, \beta, C \in \boldsymbol{C}).$$

This implies that if $0 < a < 2$ the uniqueness is not valid in $\mathcal{S}_a(\mu(t))$ for $\mu(t) = 1/(-\log t)^c$ with $1 < c \leq 2/a$.

### 3.4. Another uniqueness theorem.

In case $\mathcal{S}_a(\mu(t))$ with $a < m$, what happen ? About this we have:

**Theorem B ([4],[5]).** *If for some* $p$ *with* $0 \leq p \leq m - 1$ *the characteristic exponents of* (E) *satisfy*

$$\begin{cases} \operatorname{Re} \lambda_i(x) \leq 0 & \text{for } i = 1, \ldots, p, \\ \operatorname{Re} \lambda_i(0) < 0 & \text{for } i = p + 1, \ldots, m \end{cases}$$

*in a neighborhood of* $x = 0$ *and if* $a > p$, *then the uniqueness of the solution of* (E) *is valid in* $\mathcal{S}_a(\mu(t))$.

**Example 2.** Let us consider

$$(e_2) \qquad \Big(t\frac{\partial}{\partial t}\Big)^2 u + \Big(t\frac{\partial}{\partial t}\Big)u = (2u + x + 1)\Big(\frac{\partial u}{\partial x}\Big)^2$$

where $(t, x) \in \boldsymbol{C}^2$. The characteristic exponents are $\lambda_1 = 0$ and $\lambda_2 = -1$. In this case we have:

1) $u(t, x) \equiv 0$ is the unique holomorphic solution of $(e_2)$ under the condition $u(0, x) \equiv 0$.

2) By Theorem B we see that if $a > 1$ the uniqueness of the solution of $(e_2)$ is valid in $\mathcal{S}_a(\mu(t))$ for any weight function $\mu(t)$.

3) Note that $(e_2)$ has a family of non-trivial solutions

$$u(t, x) = \frac{x + 1}{(C - \log t)} \qquad (C \in \boldsymbol{C}).$$

This implies that if $0 < a < 1$ the uniqueness is not valid in $\mathcal{S}_a(\mu(t))$ for $\mu(t) = 1/(-\log t)^c$ with $1 < c \leq 1/a$.

4) In the case $a = 1$ it is still unknown whether the uniqueness of the solution of $(e_2)$ is valid in $\mathcal{S}_1(\mu(t))$ for any weight function or not.

# References


[1] R. Gérard and H. Tahara : *Holomorphic and singular solutions of non-linear singular first order partial differential equations,* Publ. RIMS, Kyoto Univ. 26 (1990), 979-1000.

[2] ——— : *Solutions holomorphes et singulières d'équations aux dérivées partielles singulières non linéaires,* Publ. RIMS, Kyoto Univ. 29 (1993), 121-151.

[3] ——— : *Singular nonlinear partial differential equations,* Aspects of Mathematics, E 28, Vieweg, 1996.

[4] H. Tahara : *Uniqueness of the solution of non-linear singular partial differential equations,* J. Math. Soc. Japan, 48 (1996), 729-744.

[5] ——— : *On the uniqueness theorem for nonlinear singular partial differential equations,* preprint.