数理解析研究所講究録
1090 巻 1999 年 33-45

Lagrangian properties for the diffraction in the complex domain

Pascal Laubin
University of Liège, Belgium

1 Introduction

Let M be a real manifold with boundary and P a second order differential operator with smooth coefficients and real principal symbol p. We assume that p is of real principal type and not characteristic on the boundary. Let us consider the classical Dirichlet problem

$$Pu=0 \quad \text{in} \quad M, \quad u_{|\partial M}=0.$$

If the equation of the boundary is $f=0$ with $f>0$ in M, the diffractive region is defined by

$$\mathcal{G}_{+}=\{\rho\in\dot{T}^{*}\partial M:p(\rho)=0, \quad \{p,f\}=0, \quad \frac{\{p,\{p,f\}\}_{\rho}}{\{\{p,f\},f\}_{\rho}}>0\}$$

and corresponds to rays tangent to the boundary. The propagation of singularities of C^{∞}, Gevrey and analytic singularities is known in this setting, see [12], [7], [8]. However, very few lagrangian properties are preserved along diffractive rays. In [9], Lebeau proves that, far away from the data, the operator mapping the Dirichlet data to the normal derivative of the solution belongs to a class of lagrangian Gevrey 3 distributions with weight.

We review a result on the lagrangian properties of the solution at the transition from the shadow to the illuminated region in the C^{∞} framework. Using the canonical invariance, we prove that the solution belongs to a class of lagrangian distributions associated to a pair of lagrangian submanifolds. As a consequence, we see that, for a conormal data, the second wave front lies in a lagrangian submanifold.

We next investigate the same problem in the analytic category. Here we use the geometry of complex canonical transforms and the H_{φ} spaces of Sjöstrand. We generalize the definition of bilagrangian distributions in this framework and describe the FBI transform of the solution of the boundary value problem.

2 Pairs of lagrangian submanifolds

2.1 Microlocal phase

Let X be a C^{∞} manifold of real dimension n and with local coordinates $x_1,\ldots,x_n$. On the cotangent bundle T^*X, we consider the canonical 2-form

$$\sigma=\sum_{j=1}^{n}d\xi_{j}\wedge dx_{j}$$

where the dual coordinates are defined by $d\xi_j(D_{x_k}) = \delta_{jk}$. This manifold is conic for the multiplication $M_t : (x,\xi) \mapsto (x, t\xi)$. We denote by $\dot{T}^*X = T^*X \setminus \{0\}$ the cotangent bundle with the zero section removed.

A submanifold Λ of $\dot{T}^*X$ of dimension n is lagrangian if $\sigma_{|\Lambda} = 0$. It is said conic if it is invariant through T_t for every $t > 0$.

The classical definition of a phase function for a conic lagrangian submanifold is the following, [1]. For simplicity, we restrict ourself to the case of a real non-degenerate phase function.

Definition 1 *Let X be a C^∞ manifold and φ be a C^∞ real valued function in an open conic subset Γ of $X \times \mathbb{R}^N \setminus \{0\}$ which is homogeneous of degree 1. The function φ is called a local phase function of X if $d\varphi \neq 0$ in Γ and $\mathrm{rg}(\varphi''_{\theta x}, \varphi''_{\theta\theta}) = N$ in the set*

$$C_\varphi = \{(x,\theta) \in \Gamma : \varphi'_\theta(x,\theta) = 0\}.$$

If φ is a local phase function then the differential of the map

$$j_\varphi : C_\varphi \to \dot{T}^*X : (x,\theta) \mapsto (x, \varphi'_x(x,\theta))$$

is of rank n. If it is an embedding then φ is called a *phase function*. Since

$$j_\varphi^*\sigma = j_\varphi^* d(\xi dx) = d(\varphi'_x dx) = d(d\varphi_{|C_\varphi}) = 0,$$

its image $\Lambda_\varphi = j_\varphi(C_\varphi)$ is a lagrangian submanifold of $\dot{T}^*X$.

2.2 2-microlocal phase

The second wave front set along a lagrangian submanifold Λ is defined as a subset of the cotangent bundle of Λ. To define lagrangian distributions associated to this geometric setting, we introduce new phase functions.

If Λ is a conic lagrangian submanifold of $\dot{T}^*X$, then we have the identification

$$\dot{T}^*\Lambda \sim T_\Lambda \dot{T}^*X$$

where the right hand side is the normal bundle of Λ. Indeed, if k is a normal to Λ at a point ρ then $T_\rho\Lambda \ni h \mapsto \sigma(h,k)$ is a well-defined 1-form.

Moreover this manifold has two homogeneities: one inherited from Λ and another one as a cotangent bundle. A lagrangian submanifold of $\dot{T}^*\Lambda$ is said *conic bilagrangian* if it is conic for both homogeneities. We introduce phase functions that parameterize such a manifold.

Let Γ_0 be an open subset of $X \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^M \setminus \{0\}$ such that $(x,\theta,\eta) \in \Gamma_0$ and $s,t > 0$ imply $(x, t\theta, st\eta) \in \Gamma_0$. Such an open set is called a *profile*. An open subset Γ of $X \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^M \setminus \{0\}$ is said *biconic with profile* Γ_0 if

- $(x,\theta,\eta) \in \Gamma$ and $t > 0$ imply $(x, t\theta, t\eta) \in \Gamma$,
- for each compact subset K of Γ_0, there is $\epsilon > 0$ such that $(x,\theta,s\eta) \in \Gamma$ if $(x,\theta,\eta) \in K$ and $0 < s < \epsilon$.

If Γ is biconic with respect to a family of profiles, it is also biconic with respect to their union. The *profile* of Γ is the largest profile Γ_0 such that the last condition is satisfied.

We also introduce

$$\Gamma_1 = \{(x,\theta) : \exists \eta \text{ such that } (x,\theta,\eta) \in \Gamma\}.$$

This is an open conic subset of $X \times \mathbb{R}^N \setminus \{0\}$.

Let $p, q \in \mathbb{R}$ and $r \in \mathbb{N}_0$. A C^∞ function $f : \Gamma \to \mathbb{R}^m$ is said *bihomogeneous of degree* $(p, q; r)$ if

- $f(x, t\theta, t\eta) = t^p f(x, \theta, \eta)$ if $(x, \theta, \eta) \in \Gamma$, $t > 0$,
- for every $(x_0, \theta_0, \eta_0) \in \Gamma_0$, there is a neighborhood V of (x_0, θ_0, η_0) and a C^∞ function F in $V \times]-\epsilon, \epsilon[$ satisfying

$$f(x, \theta, s\eta) = s^q F(x, \theta, \eta, s^{1/r})$$

  if $(x, \theta, \eta, s) \in V \times]0, \epsilon[$.

The integer r is inserted here essentially for technical reasons. In the application, it does not affect the 2-microlocal geometry but has some effects on the microlocal lagrangian submanifolds involved. We say that f has the regularity r.

Definition 2 *Let*

- Λ *be a conic lagrangian submanifold of* $\dot{T}^*X$,
- φ *be a* C^∞ *real valued function which is homogeneous of degree* 1 *in* Γ_1,
- ψ *be a* C^∞ *real valued function which is bihomogeneous of degree* $(1, 1; r)$ *in* Γ

and

$$C_{\varphi,\psi} = \{(x,\theta,\eta) \in \Gamma_0 : \varphi'_\theta(x,\theta) = 0, \psi'_{1,\eta}(x,\theta,\eta) = 0\}.$$

The pair (φ, ψ) *is a local 2-phase function of* Λ *(with regularity* r*) if*

- φ *is a local phase function that parameterizes* Λ,
- *at each point of* $C_{\varphi,\psi}$, *the vector* $(\psi'_{1,x}, \psi'_{1,\theta})$ *is different from* 0 *and*

$$\operatorname{rk}\begin{pmatrix} \psi''_{1,\eta x} & \psi''_{1,\eta\theta} & \psi''_{1,\eta\eta} \\ \varphi''_{\theta x} & \varphi''_{\theta\theta} & 0 \end{pmatrix} = N + M.$$

If φ is a phase function, the last condition means that the map $(\rho, \eta) \mapsto \psi_1(j_\varphi^{-1}(\rho), \eta)$ is a local phase function of Λ. This definition has the following consequences.

a) *The map*

$$j_{\varphi,\psi} : C_{\varphi,\psi} \to \dot{T}^*\Lambda : (x,\theta,\eta) \mapsto ((x,\varphi'_x), j_{\varphi *}((\psi'_{1,x}, \psi'_{1,\theta})_{|TC_\varphi})).$$

is a lagrangian immersion.

Following the identification $\dot{T}^*\Lambda \sim T_\Lambda \dot{T}^*X$, *the map* $j_{\varphi,\psi}$ *can be identified with*

$$C_{\varphi,\psi} \to \dot{T}_\Lambda T^*X : (x,\theta,\eta) \mapsto ((x,\varphi'_x),(h,\tilde{\psi}'_{1,x}+\varphi''_{xx}.h+\varphi''_{x\theta}.k))$$

where h, k *satisfy*

$$\varphi''_{\theta x}.h+\varphi''_{\theta\theta}.k+\tilde{\psi}'_{1,\theta}=0.$$

b) Let (φ,ψ) be a local 2-phase function (with regularity r) in a biconic set Γ and $(x_0,\theta_0,\eta_0) \in C_{\varphi,\psi}$. By the definition, φ is a local phase function in Γ_1 and there is a biconic open subset $\tilde{\Gamma}$ of Γ whose profile contains (x_0,θ_0,η_0) such that $(x,(\theta,\eta)) \mapsto \varphi(x,\theta)+\psi(x,\theta,\eta)$ is a local phase function in $\tilde{\Gamma}$. A local 2-phase function (φ,ψ) is called a *2-phase function* if j_φ, $j_{\varphi+\psi}$ and $j_{\varphi,\psi}$ are embeddings.

One can verify that *if* (φ,ψ) *is a local 2-phase function in* Γ *and* $(x_0,\theta_0,\eta_0) \in C_{\varphi,\psi}$ *then there is a biconic open set* $\tilde{\Gamma}$ *whose profile contains* (x_0,θ_0,η_0) *such that* (φ,ψ) *is a 2-phase function in* $\tilde{\Gamma}$.

Hence, *if* (φ,ψ) *is a 2-phase function then*

$$\{((x,\varphi'_x),(h,\psi'_{1,x}+\varphi''_{xx}.h+\varphi''_{x\theta}.k)) : (x,\theta) \in C_{\varphi,\psi}, \psi'_{1,\theta}+\varphi''_{\theta x}.h+\varphi''_{\theta\theta}.k=0\}$$

is a conic bilagrangian submanifold of $\dot{T}^*\Lambda_\varphi$. It is denoted $\Lambda_{\varphi,\psi}$.

c) *If* (φ,ψ) *is a 2-phase function, then*

$$n-\mathrm{rg}(\pi_{\Lambda_\varphi,X}) = N-\mathrm{rg}(\varphi''_{\theta\theta}) \quad , \quad n-\mathrm{rg}(\pi_{\Lambda_{\varphi,\psi},\Lambda_\varphi}) = M-\mathrm{rg}(\psi''_{1,\eta\eta}),$$

and

$$n-\mathrm{rg}(\pi_{\Lambda_{\varphi,\psi},X}) = N+M-\mathrm{rk}\begin{pmatrix} \psi''_{1,\eta\eta} & \psi''_{1,\eta\theta} \\ 0 & \varphi''_{\theta\theta} \end{pmatrix}.$$

2.3 Pairs of lagrangian submanifolds

We now describe the geometric setting associated to a 2-phase. If Y is a submanifold of a C^∞ manifold X, the blowup of X along Y is

$$\hat{X}_Y = (X \setminus Y) \cup \dot{T}_Y X.$$

The sets

$$\bigcap_{1\le j\le p} \Big(\{x \in \omega : f_j(x)>0\} \cup \{(x,h) \in \dot{T}_Y X : x \in \omega, df_j(x).h>0\}\Big)$$

where ω is an open subset of X and $f_j \in C^\infty(\omega)$, $f_{j|Y\cap\omega}=0$ for all j, form a basis of topology of $\hat{X}_Y$. For this topology, the projection $\pi : \hat{X}_Y \to X$ is continuous.

Definition 3 A pair (Λ_0,Λ_1) is a *2-microlocal pair* of lagrangian submanifolds of $\dot{T}^*X$ if

- Λ_0 is a conic lagrangian submanifolds of $\dot{T}^*X$, $\Lambda_1 \subset (\dot{T}^*X)^\wedge_{\Lambda_0}$,
- $\Lambda_1 \cap (\dot{T}^*X \setminus \Lambda_0)$ is a conic lagrangian submanifold of $\dot{T}^*X$,

- for each $(\rho, h) \in \Lambda_1 \cap \dot{T}_{\Lambda_0} T^*X$, there is an open neighborhood V of (ρ, h) in $(\dot{T}^*X)^{\wedge}_{\Lambda_0}$ and a 2-phase function (φ, ψ) such that

$$\Lambda_0 \cap \pi(V) = \Lambda_\varphi \quad \text{and} \quad \Lambda_1 \cap V = \Lambda_{\varphi+\psi} \cup \Lambda_{\varphi,\psi}.$$

In this situation, we say that the 2-phase function (φ, ψ) defines (Λ_0, Λ_1). Let $T_{\Lambda_0}\Lambda_1 = \Lambda_1 \cap \dot{T}_{\Lambda_0}(T^*X)$. This is a conic bilagrangian submanifold of $\dot{T}^*\Lambda_0$.

Example 4 *In $\dot{T}^*\mathbb{R}^n$, consider*

$$\varphi(x,\xi) = x.\xi \quad , \quad \psi(x,\xi,\eta') = \frac{\eta'.\xi'}{\xi_n} - H(\eta', \xi_n).$$

where $\xi = (\xi', \xi_n)$ and H is bihomogeneous of degree $(1,1;r)$. We have

$$\Lambda_\varphi = \{(0,\xi) : \xi_n \neq 0\}$$

and

$$\Lambda_{\varphi+\psi} = \{((-\frac{\eta'}{\xi_n}, \frac{\eta'.H'_{\eta'}}{\xi_n} + H'_{\xi_n}), (\xi_n H'_{\eta'}, \xi_n)) : \xi_n \neq 0\}.$$

If $H(\eta', \xi_n) = \eta_1^3/\eta_2^2$ in $\mathbb{R}^3$, the projection of $T_{\Lambda_\varphi}\Lambda_{\varphi+\psi}$ on Λ_φ is the cusp

$$\{(0,\xi) : (\frac{\xi_1}{3})^3 = (\frac{\xi_2}{2})^2 \xi_3 : \xi_3 \neq 0\}.$$

It can be shown, see [4], that the property of being a microlocal pair of lagrangian submanifolds is preserved by an homogeneous canonical transformation.

Let us describe the equivalence of 2-phase functions.

Two 2-phase functions (φ, ψ) and $(\tilde{\varphi}, \tilde{\psi})$ defined in biconic open subsets Γ and $\tilde{\Gamma}$ of $X \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^M \setminus \{0\}$ are said *equivalent* if there is a C^∞ diffeomorphism $\Gamma \to \tilde{\Gamma}$: $(x,\theta,\eta) \mapsto (x, f(x,\theta,\eta), g(x,\theta,\eta))$ such that

- $\varphi(x, f(x,\theta,\eta)) + \psi(x, f(x,\theta,\eta), g(x,\theta,\eta)) = \tilde{\varphi}(x,\theta) + \tilde{\psi}(x,\theta,\eta)$,
- f is strictly bihomogeneous of degree $(1,0;r)$ and g is bihomogeneous of degree $(1,1;r)$,
- $D_\theta f_0$ and $D_\eta g_1$ are invertible in Γ_0.

These two pairs define the same 2-microlocal pair.

If Δ is a diagonal real invertible matrix, the pair of phases

$$\varphi(x,\theta) = \tilde{\varphi}(x,\theta'') + \frac{\langle \Delta\theta', \theta' \rangle}{2|\theta''|} \quad , \quad \psi(x,\theta'',\eta) = \tilde{\psi}(x,\theta'',\eta)$$

defines the same lagrangian submanifolds as $\tilde{\varphi}$ and $\tilde{\psi}$. In the same way,

$$\varphi(x,\theta) = \tilde{\varphi}(x,\theta) \quad , \quad \psi(x,\theta,\eta) = \tilde{\psi}(x,\theta,\eta'') + \frac{\langle \Delta\eta', \eta' \rangle}{2|\eta''|}$$

defines the same lagrangian submanifolds as $\tilde{\varphi}$ and $\tilde{\psi}$.

It can be shown that the transition between two 2-phase functions defining the same 2-microlocal pair of lagrangian submanifolds can be obtained by a composition of the previous reductions.

3 Bilagrangian distributions

3.1 Symbols

We use only classical symbols. This is enough for the applications that we consider here.

Definition 5 *If $m, p \in \mathbb{R}$ and X is an open subset of $\mathbb{R}^n$, we denote by $S^{m,p}(X, \mathbb{R}^N, \mathbb{R}^M)$ the set of all $a \in C^\infty(X \times \mathbb{R}^N \times \mathbb{R}^M)$ such that for every compact subset K of X and all multiorders α, β, γ there is a $C > 0$ satisfying*

$$|D_x^\alpha D_\theta^\beta D_\eta^\gamma a(x, \theta, \eta)| \leq C(1 + |\theta| + |\eta|)^{m-|\beta|}(1 + |\eta|)^{p-|\gamma|}$$

for all $(x, \theta, \eta) \in K \times \mathbb{R}^N \times \mathbb{R}^M$.

Write

$$S_2^\infty = \bigcup_{m,p\in\mathbb{R}} S^{m,p} \quad , \quad S^{m,-\infty} = \bigcap_{p\in\mathbb{R}} S^{m,p}.$$

It is clear that $S^{m,p}$ is a Fréchet space with semi-norms given by the smallest constants which can be used in the definition.

Oscillatory integrals can be defined using symbols in $S^{m,p}$ and 2-phase functions.

Theorem 6 *Let (φ, ψ) be a 2-phase function in an open biconic set Γ and let F be a closed conic subset of Γ such that $F \ll \Gamma$. For every $u \in C_0^\infty(X)$, the linear form*

$$a \mapsto \iiint e^{i(\varphi(x,\theta)+\psi(x,\theta,\eta))} a(x, \theta, \eta) u(x)\, dx d\theta d\eta$$

defined in the set of all $a \in S^{-\infty}(X; \mathbb{R}^N \times \mathbb{R}^M)$ satisfying $\mathrm{supp}(a) \subset F$, *can be extended on S_2^∞ in a unique way such that it is continuous on the set of $a \in S^{m,p}(X, \mathbb{R}^N, \mathbb{R}^M)$ satisfying* $\mathrm{supp}(a) \subset F$ *for every m, p.*

3.2 Distribution class

Let X be a C^∞ manifold of dimension n and let (Λ_0, Λ_1) be a 2-microlocal pair of lagrangian submanifolds of $\dot{T}^*X$.

Definition 7 *The space $I^{m,p}(X, \Lambda_0, \Lambda_1)$ is the set of all locally finite sums of an element of $I^m(X, \Lambda_0)$, an element of $I^{m+p}(X, \Lambda_1 \cap T^*X)$ and distributions of the form*

$$I_{\varphi,\psi,a}(u) = (2\pi)^{-(n+2(N+M))/4} \iiint e^{i(\varphi(x,\theta)+\psi(x,\theta,\eta))} a(x, \theta, \eta) u(x)\, dx d\theta d\eta$$

where (U, χ) is a chart of X, $u \in C_0^\infty(X)$, (φ, ψ) is a 2-phase function of (Λ_0, Λ_1) defined in an open biconic subset Γ of $\chi(U) \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^M \setminus \{0\}$ and

$$a \in S^{m+(n-2N)/4, p-M/2}(\chi(U), \mathbb{R}^N, \mathbb{R}^M)$$

satisfies $\mathrm{supp}(a) \ll \Gamma$.

It can be shown that this space is invariant by composition with a Fourier integral operators. Moreover, any 2-phase function defining the pair (Λ_0, Λ_1) near a point $\rho_0 \in \Lambda_0$ can be used to define any element of $I^{m,p}(X, \Lambda_0, \Lambda_1)$ near ρ_0.

The singularities of an element of $I^{m,p}(X, \Lambda_0, \Lambda_1)$ are included in the lagrangian submanifolds involved, [4].

Theorem 8 *If* $u \in I^{m,p}(X, \Lambda_0, \Lambda_1)$ *then*

$$WF(u) \subset \Lambda_0 \cup \Lambda_1 \ , \quad WF^{(2)}_{\Lambda_0}(u) \subset T_{\Lambda_0}\Lambda_1.$$

4 Application to diffraction

Let us consider the boundary value problem

$$\begin{cases} (-\Delta + (1+x_n)\partial_t^2)u = 0 \\ u_{|x_n=0} = \delta_0 \quad , \quad u_{|t<0} = 0 \end{cases}$$

where we use the decomposition $(t, x', x_n) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$. This is a model for the strictly diffractive problems in the C^∞ category, see [11].

Let

$$p(x_n, \tau, \xi) = |\xi|^2 - (1+x_n)\tau^2$$

be the principal symbol of the operator and $r(\tau, \xi') = |\xi'|^2 - \tau^2$ be the boundary hamiltonian. Two lagrangian submanifolds are involved here. On one hand, we consider the flowout $\Lambda_0 = \Lambda_{0,+} \cup \Lambda_{0,-}$ of

$$\{((0,0),(\tau,\xi)) : \tau = \pm|\xi'| \neq 0, \ \xi_n = 0\}$$

through H_r on the boundary and followed by H_p intersected with $t > 0$ and $x_n > 0$. On the other hand, the flowout $\Lambda_1 = \Lambda_{1,+} \cup \Lambda_{1,-}$ of

$$\{((0,0),(\tau,\xi)) : \tau = \pm|\xi|, \xi_n \neq 0\}$$

through H_p intersected with $t > 0$ and $x_n > 0$. These two manifolds are smooth but are tangent at their intersection.

It can be checked that $(\Lambda_{0,\pm}, \Lambda_{1,\pm})$ is a 2-microlocal pair of lagrangian submanifolds with

$$\begin{aligned} T_{\Lambda_{0,\pm}}\Lambda_{1,\pm} \ = \ & \{(((\frac{2}{3}x_n^{3/2} + 2\sqrt{x_n}, x', x_n), (\pm|\xi'|, \xi', \mp|\xi'|\sqrt{x_n})), \\ & ((0,0,0),(\pm\frac{1}{2}\sigma, 0, \mp\frac{1}{2}\sigma(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}))) : \sigma, x_n > 0, \xi' \neq 0\}. \end{aligned}$$

A 2-phase function $(\varphi_\pm ; \psi_\pm)$ of $(\Lambda_{0,\pm}, \Lambda_{1,\pm} \cup T_{\Lambda_{0,\pm}}\Lambda_{1,\pm})$ is given by

$$\varphi_\pm(t, x, \xi') = x'.\xi' \pm |\xi'|(t - \frac{2}{3}x_n^{3/2})$$

and

$$(\varphi_\pm + \psi_\pm)(t, x, \sigma, \xi') = x'.\xi' \pm |\xi'|(1 - \frac{\sigma}{|\xi'|})^{-1/2}(t - \frac{2}{3}((x_n + \frac{\sigma}{|\xi'|})^{3/2} - (\frac{\sigma}{|\xi'|})^{3/2})).$$

This 2-phase function has the regularity 2.

We denote by $I^m_\rho(X, \Lambda_0)$ the set of all lagrangian distributions on Λ_0 with symbol in S^m_ρ. This means that the symbol satisfies the following inequalities

$$|D^\alpha_x D^\beta_\theta a(x,\theta)| \leq C_{\alpha,\beta}(1+|\theta|)^{m-|\beta|+(1-\rho)(|\alpha|+|\beta|)}.$$

An analysis of the solution of the initial boundary value problem given in [2] leads to the following result.

Theorem 9 *The solution u of the previous boundary value problem belongs to*

$$\begin{aligned} & I^{\frac{n}{4}-1,\frac{3}{4}}(\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}_+, \Lambda_0, \Lambda_1 \cup T_{\Lambda_0}\Lambda_1)) \\ + \quad & I_{2/3}^{\frac{n}{4}-\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}_+, \Lambda_0). \end{aligned}$$

5 The geometry in the complex domain

Our purpose is to define the phase functions used to characterize the bilagrangian distributions in the formalism of the Fourier-Bros-Iagolnitzer transform. In the microlocal case, we closely follow [6] and collect some material from [9], see also [13].

As usual, we identify

- $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ and write $z = x + iy$,
- $\zeta \in T^*_z\mathbb{C}^n$ with $(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ using $\zeta(h) = \sum_j \zeta_j h_j$,
- $T^*_z\mathbb{C}^n$ with $T^*_{(x,y)}\mathbb{R}^{2n}$ by mapping the $\mathbb{C}$-linear form $\zeta \in T^*_z\mathbb{C}^n$ to the $\mathbb{R}$-linear form $h \mapsto -\Im\zeta(h)$.

This map is symplectic if $T^*\mathbb{R}^{2n}$ is endowed with the usual canonical 2-form and $T^*\mathbb{C}^n$ with the 2-form $-\Im\sigma$ defined below.

It follows that if f is a holomorphic function, $\partial f \in T^*_z\mathbb{C}^n$ is identified with $d(-\Im f) \in T^*_{(x,y)}\mathbb{R}^{2n}$ since $d(-\Im f) = -\Im(df) = -\Im(\partial f)$.

In the same way, if φ is a real function then $d\varphi \in T^*_{(x,y)}\mathbb{R}^{2n}$ is identified with $\frac{2}{i}D_z\varphi \in \mathbb{C}^n$.

All the constructions described in this section are local even this is not stated explicitly.

5.1 FBI transform

Writing $z = x + iy$ and $\zeta = \xi + i\eta$, the canonical 2-form on $T^*\mathbb{C}^n$ is

$$\sigma = \sum_j d\zeta_j \wedge dz_j.$$

Its real and imaginary parts

$$\Re\sigma = \sum_j (d\xi_j \wedge dx_j - d\eta_j \wedge dy_j), \quad \Im\sigma = \sum_j (d\eta_j \wedge dx_j + d\xi_j \wedge dy_j)$$

are symplectic forms on $\mathbb{R}^{2n}$.

Let φ be a real C_1 function defined in a neighborhood of $z_0 \in \mathbb{C}^n$ and

$$\Lambda_\varphi = \{(z, \frac{2}{i} D_z \varphi(z)) : z \in \mathbb{C}^n\}.$$

This manifold is $\Im$-lagrangian since it is identified with

$$\{(z, d\varphi(z)) : z \in \mathbb{C}^n\} \subset T^*\mathbb{R}^{2n}.$$

If j_φ denotes the immersion $z \mapsto (z, \frac{2}{i} D_z \varphi(z))$ then

$$j_\varphi^*(\Re\sigma) = j_\varphi^*(\sigma) = j_\varphi^*(d(\zeta dz)) = d(\frac{2}{i}\partial\varphi) = \frac{2}{i}\overline{\partial}\partial\varphi.$$

It follows that, if $\overline{\partial}\partial\varphi$ is non degenerate, j_φ is a symplectic map from $(\mathbb{C}^n, \frac{2}{i}\overline{\partial}\partial\varphi)$ onto $(\Lambda_\varphi, \Re\sigma)$. Its inverse is the projection.

The following result is proven in [7], see also [3].

Theorem 10 *Let φ be a strictly plurisubharmonic function near $z_0 \in \mathbb{C}^n$ and $\chi : \dot{T}^*\mathbb{R}^n \to \Lambda_\varphi$ a canonical transform defined near (y_0, η_0) such that $\chi(y_0, \eta_0) = (z_0, \frac{2}{i} D_z \varphi(z_0))$. Here Λ_φ is endowed with the 2-form $\Re\sigma$. There is a unique holomorphic function $g(z, y)$ near (z_0, y_0), such that*

- *the complexification of χ is*

$$\chi^{\mathbb{C}} : T^*\mathbb{C}^n \to T^*\mathbb{C}^n : (y, -D_y g(z, y)) \mapsto (z, D_z g(z, y)),$$

- $ig(z_0, y_0) = \varphi(z_0)$, $-D_y g(z_0, y_0) = \eta_0$,
- *the function $y \mapsto -\Im g(z, y)$ has a non degenerate critical point $y(z)$ with signature $(0, n)$ and critical value $\varphi(z)$. Moreover, we have*

$$(y(z), -D_y g(z, y(z))) = \chi^{-1}(z, \frac{2}{i} D_z \varphi(z)).$$

For example, if $\chi : (x, \xi) \mapsto (x - i\xi, \xi)$ and $\varphi(z) = \frac{1}{2}|\Im z|^2$, then $g(z, y) = \frac{i}{2}(z - y)^2$. The FBI transform associated to φ, χ near the points (y_0, η_0), z_0 is

$$T_\chi u(z, \lambda) = \int e^{i\lambda g(z,y)} a(z, y, \lambda) u(y)\, dy$$

where a is a classical symbol.

5.2 Lagrangian submanifolds

In this setting, lagrangian submanifold can be parameterized by a holomorphic function.

Proposition 11 *Let Λ be a lagrangian submanifold of $\dot{T}^*\mathbb{R}^n$, h be a phase function of Λ near ρ_0 and χ be a local canonical map from $\dot{T}^*\mathbb{R}^n$ to Λ_φ mapping ρ_0 to z_0. If g the FBI phase defined in theorem 10 and*

$$\phi_\Lambda(z) = \mathrm{cv}_{(x,\theta)}(g(z,x) + h(x,\theta))$$

then $\varphi_\Lambda = -\Im\phi_\Lambda$. The critical points are given by

$$(x,\theta) = j_{\mathbb{C}}^{-1} \circ \chi_{\mathbb{C}}^{-1}(z, D_z\phi_\Lambda(z)).$$

Here j is the immersion $(x,\theta) \mapsto (x, h'_x)$ and $j_{\mathbb{C}}$ is its complexification.

We have

$$\chi^{\mathbb{C}}(\Lambda^{\mathbb{C}}) = \{(z, D_z\phi_\Lambda(z)) : z \in \mathbb{C}^n\}$$

and

$$\varphi_\Lambda(z) \leq \varphi(z).$$

The equality holds if and only if $(z, \frac{2}{i}D_z\varphi(z)) \in \chi(\Lambda)$.

In this formalism, the lagrangian distributions are defined in the following way.

Definition 12 *Let u be a distribution in an open subset Ω of $\mathbb{R}^n$, Λ a lagrangian submanifold of $\dot{T}^*\Omega$. With the notations of proposition 11, u is said lagrangian at ρ_0 if, in a neighborhood of z_0, we have*

$$(T_\chi u)(z,\lambda) = e^{i\lambda\phi_\Lambda(z)}b(z,\lambda)$$

where b is a classical analytic symbol.

This is equivalent to the fact that u can be written $u = u_1 + u_2$ with $\rho_0 = j_h(x_0,\theta_0)$ not in the singular spectrum of u_2 and

$$u_1(x) = \int_\Gamma e^{ih(x,\theta)}a(x,\theta)\,d\theta$$

where Γ is a conic neighborhood of θ_0 and a is a classical analytic symbol near (x_0,θ_0).

5.3 Pairs of lagrangian submanifolds

Let us consider the FBI transform of a 2-phase function. For simplicity, we restrict ourself to the case of one 2-microlocal parameter.

Proposition 13 *Let (Λ_0,Λ_1) be a 2-microlocal pair of lagrangian submanifolds and (h,ψ) be a 2-phase function for the pair (Λ_0,Λ_1) near a point $\rho_0 \in \Lambda_0$. We assume that h is analytic and that ψ is an analytic function of $(x,\theta,\sigma^{1/2})$,*

$$\psi(x,\theta,\sigma) = \psi_1(x,\theta)\sigma + \psi_{3/2}(x,\theta)\sigma^{3/2} + \psi_2(x,\theta)\sigma^2 + \mathcal{O}(\sigma^{5/2}).$$

If g is an FBI phase function associated to a local canonical map χ such that $\chi(\rho_0) = z_0 \in \mathbb{C}^n$, we have

$$\begin{aligned}\phi(z,\sigma) &= \mathrm{cv}_{(x,\theta)}\big(g(z,x) + h(x,\theta) + \psi(x,\theta,\sigma)\big)\\ &= \Phi_{\Lambda_0}(z) + \Phi_1(z)\sigma + \Phi_{3/2}(z)\sigma^{3/2} + \Phi_2(z)\sigma^2 + \mathcal{O}(\sigma^{5/2}).\end{aligned}$$

Here Φ_1 and $\Phi_{3/2}$ are real on $\pi \circ \chi(\Lambda_0)$, $\Phi_1(z_0) = 0$, $D_z\Phi_1(z_0) \neq 0$ and $\Im\Phi_2(z_0) > 0$.

With the notations of the proposition 13, a distribution u is said *analytic bilagrangian* at ρ_0 with respect to (Λ_0, Λ_1) if, in a neighborhood of z_0, we have

$$(T_\chi u)(z,\lambda) = \int_0^\delta e^{i\phi(z,\sigma)} a(z,\sigma,\lambda)\, d\sigma$$

where a is holomorphic in an open set of the form

$$\{(z,\sigma) \in \mathbb{C}^n \times \mathbb{C} : |z - z_0| < \epsilon, |\Im\sigma| < c\Re\sigma\}$$

and is bounded by $C\lambda^m$ for $\lambda > 1$.

Since $\Im\Phi_2(z_0) > 0$ and $\Phi_1(z_0)$, $\Phi_{3/2}(z_0)$ are real, we can choose $\delta > 0$ small such that

$$-\Im\phi(z_0,\delta) < -\Im\varphi_{\Lambda_0}(z_0).$$

For example, if

$$\Lambda_0 = \{((0,x_n),(\xi',0))\}, \quad \Lambda_1 = \{((0,0),(\xi',\xi_n))\}$$

and $g(z,y) = i(z-y)^2/2$, we have

$$\Phi_{\Lambda_0}(z) = \frac{i}{2} z'^2, \quad \Phi_{\Lambda_1}(z) = \frac{i}{2} z^2$$

and

$$\phi(z,\sigma) = \frac{iz'^2}{2} + \sigma z_n + \frac{i\sigma^2}{2}.$$

6 Bilagrangian structure of the parametrix

Let us show how, at the transition of the shadow and the illuminated region, the parametrix defines a bilagrangian distribution if the boundary data is conormal.

Using [11], we may assume that the operator can be written

$$P(x,D) = D_{x_n}^2 + R(x, D_{x'})$$

in the half space $\{x_n > 0\}$. Its principal symbol is

$$p(x,\xi) = \xi_n^2 + r(x,\xi').$$

Let $r_0(x',\xi') = r(x',0,\xi')$. We assume that the point (x_0',ξ_0') is diffractive. This means that $r_0(x_0',\xi_0') = 0$ and $dr_0 \neq 0$, $\partial_{x_n} r < 0$.

Following [7], we first perform a complex canonical transform. We choose the weight function $\varphi_0(z') = |\Im z'|^2/2$ and a canonical map

$$\chi_0 : \dot{T}^*\mathbb{R}^{n-1} \to (\Lambda_{\varphi_0}, \Re\sigma)$$

mapping (x_0',ξ_0') to $(0,0)$ and the glancing region $\{r_0 = 0\}$ to $\{\Im z_1 = 0\}$. To this canonical map is associated a FBI transform.

After this transform, we obtain a pseudodifferential operator

$$P(x, \tilde{D}, \lambda) = \tilde{D}_{x_n}^2 + R(x, \tilde{D}_{x'}, \lambda)$$

near $(0,0)$ on Λ_{φ_0}. Its principal symbol $p(x,\xi) = \xi_n^2 + r(x,\xi')$ is real on Λ_{φ_0} and $p(x,\xi) = 0$ is equivalent to $x_n + q(x',\xi) = 0$ with

$$q(x',\xi) = \xi_1 - e(x',\xi')\xi_n^2 + \mathcal{O}(\xi_n^4), \quad e(0,0) > 0.$$

In the H_φ space, the problem is reduced to find an outgoing solution to

$$P(x, \tilde{D}, \lambda)u(x,\lambda) = 0, \quad u_{|x_n=0} = g. \tag{1}$$

Define, as above, Λ_0 as the flowout of the set of diffractive points through the boundary hamiltonian H_r followed by H_p and Λ_1 as the flowout of all the characteristic points at $x = 0$ through H_p.

In the boundary value problem (1), we consider the boundary data $g(x', \lambda) = \exp(i\lambda z'^2)$ corresponding to a Dirac mass. Using the Lebeau construction of the parametrix, we obtain the following estimation.

Theorem 14 *The function*

$$\begin{aligned} \varphi(z,\sigma) &= \mathrm{cv}_{(x,\eta'')}\Big(\frac{i}{2}(z_n - x_n)^2 + H(z', \sigma, \eta'', \sqrt{x_n + \sigma}) \\ &\qquad - x_1\sigma - x''.\eta'' + F(x', \sqrt{\sigma}, \eta'') + \frac{ix'^2}{2}\Big) \end{aligned}$$

satisfies the conditions of proposition 13. Moreover, the solution u of the boundary value problem (1) can be written $u_1 + u_2$ where u_1 is analytic bilagrangian and

$$|u_2(z,\lambda)| \leq C_\epsilon e^{\lambda(\varphi_{\Lambda_0}(z) + Cd(z, \pi\circ\chi(\Lambda_0))^3) + \epsilon\lambda}$$

near 0 for every $\epsilon > 0$.

References

[1] Hörmander, L., *The analysis of linear partial differential operators I-IV*, Springer-Verlag, 1983-85.

[2] Friedlander, F.G. and Melrose, R.B., *The wave front set of the solution of a simple initial-boundary value problem with glancing rays II*, Math. Proc. Camb. Phil. Soc., 87, 1977, 97-120.

[3] Laubin, P., *Etude 2-microlocale de la diffraction*, Bull. Soc. Roy. Sc. Liège, 4, 1987, 295-416.

[4] Laubin P., Willems B., *Distributions associated to a 2-microlocal pair of lagrangian manifolds*, Comm. in Part. Diff. Eq., 19, 1994, 1581-1610.

[5] Laubin P., *Conormality and lagrangian properties in diffractive boundary value problems*, Microlocal analysis and spectral theory, 91-113, Kluwer Academic Press, 1997.

[6] Lebeau, G., *Deuxième microlocalisation sur les sous-variétés isotropes*, Ann. Inst. Fourier, Grenoble, 35, 1985, 145-216.

[7] Lebeau, G., *Régularité Gevrey 3 pour la diffraction*, Comm. in Part. Diff. Eq., 9(15), 1984, 1437-1494.

[8] Lebeau, G., *Propagation des singularités Gevrey pour le problème de Dirichlet*, Advances in microlocal analysis, Nato ASI, C168, 1986, 203-223.

[9] Lebeau, G., *Scattering frequencies and Gevrey* 3 *singularities*, Invent. math., 90, 1987, 77-114.

[10] Melrose, R. B., *Local Fourier-Airy integral operators*, Duke Math. J., 42, 1975, 583-604.

[11] Melrose, R. B., *Transformation of boundary value problems*, Acta Math. J., 147, 1981, 149-236.

[12] Sjöstrand, J., *Propagation of analytic singularities for second order Dirichlet problems, I-III*, Comm. Part. Diff. Eq., 5(1), 1980, 41-94; 5(2), 1980, 187-207; 6(5), 1981, 499-567.

[13] Sjöstrand, J., *Singularités analytiques microlocales*, Astérisque, 95, 1982, 1-166.