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CAUCHY PROBLEMS FOR MIXED-TYPE OPERATORS

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ABSTRACT. We study a general theory of mixed-type operators containing the Tricomi operators, degenerate hyperbolic operators, and elliptic operators. We will give a necessary and sufficient condition for the Cauchy problems to be well-posed.

Let $P(x, D)$ be a microdifferential operator defined at $x^* = (0; 0, \cdots, 0, \sqrt{-1}) \in \sqrt{-1}\mathbf{T}^*\mathbf{R}^n$ of order $m \geq 2$, written in the form

$$(1) \qquad \begin{cases} P(x,D) = D_1^m + \sum_{0\leq j\leq m-1} P_j(x,D')D_1^j, \\ \operatorname{ord} P_j \leq m-j. \end{cases}$$

Here we have written $D' = (D_2, \cdots, D_n)$. We also write as $D'' = (D_1, \cdots, D_{n-1})$, $D''' = (D_2, \cdots, D_{n-1})$. Let $\sigma_m(P)(x, \xi)$ be the principal symbol of $P(x, D)$. We assume that

$$(2) \qquad \begin{cases} \text{if } x_1 = 0, \text{ then } \sigma_m(P) = \xi_n^m; \\ \text{if } x_1 \neq 0, \text{ then the equation } \sigma_m(P) = 0 \text{ has } m \text{ distinct roots} \\ \xi_1 = \varphi_1(x, \xi'), \cdots, \varphi_m(x, \xi'). \end{cases}$$

We denote by $\mathcal{O}$ (resp. $\mathcal{O}_{(j)}$) the sheaf of holomorphic functions (resp. the sheaf of functions $f(x_1^{1/j}, x')$ such that $f(x)$ are holomorphic). Without loss of generality, we may assume that $\varphi_j(x, \xi') \in \mathcal{O}_{(m'),x^*}$ for some $m' \in \mathbf{N}$, that they are homogeneous in ξ' of degree 1, and that they vanish when $x_1 = 0$. From now on, we denote $\bar{\mathcal{O}} = \mathcal{O}_{(m')}$. It follows that

$$\begin{cases} \text{for some } q_j \in \mathbf{N}/m' \text{ and some } a_j(x,\xi') \in \bar{\mathcal{O}}_{x^*} \text{ we have} \\ \varphi_j(x,\xi') = x_1^{q_j} a_j(x,\xi'),\ a_j(x^*) \neq 0\ (1 \leq j \leq m). \end{cases}$$

We also assume that

$$(3) \qquad i \neq j \quad \Longrightarrow \quad (q_i, a_i(x^*)) \neq (q_j, a_j(x^*)).$$

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We denote by $\mathcal{C}$ (resp. $\mathcal{E}$) the sheaf of microfunctions (resp. microdifferential operators). Let us consider the Cauchy problem

$$\left\{\begin{array}{l} Pu=0, \\ D_1^{j-1}u(0,x')=v_j(x'), \quad 1\le j\le m, \end{array}\right. \tag{4}$$

where $u\in\mathcal{C}_{\mathbf{R}^n,x^*}$ and $v_j\in\mathcal{C}_{\mathbf{R}^{n-1},x^{*\prime}}$ ($x^{*\prime}=(0;0,\cdots,0,\sqrt{-1})\in\sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1}$). If $P(x,D)$ is microhyperbolic, (4) is well-posed for arbitrary initial values, as is well-known (See [3]). Otherwise (4) may be solvable for some initial values (e.g., for $v_1=\cdots=v_m=0$), but may be unsolvable for other initial values. Therefore there arises a problem to know for which initial values (4) becomes solvable.

To give the main theorem we need to prepare some preliminaries. Let $A(x',D')$ be an both-side invertible $m\times m$ matrix whose components $A_{(\mu,\nu)}(x',D')\in\mathcal{E}^{\mathbf{R}}_{x^*}$ are independent of (x_1,D_1). Here we denote by $\mathcal{E}^{\mathbf{R}}$ the sheaf of holomorphic microlocal operators (c.f. [1,6]). We choose r rows of this matrix in an arbitrary way. To be clear, let $1\le j_1<j_2<\cdots<j_r\le m$ and choose the $j_1,\cdots,j_r$-th rows of A. Then we obtain an $r\times m$ matrix $A'(x',D')$ of holomorphic microlocal operators. We say that $v_1(x'),\cdots,v_m(x')\in\mathcal{C}_{\mathbf{R}^{n-1},x^{*\prime}}$ satisfy an r-relation if choosing some r rows of some $A(x',D')$ we have $A'(x',D')\vec{v}(x')=\vec{0}$. Here $\vec{v}$ denotes ${}^t(v_1,\cdots,v_m)$. Note that even if $v_1(x'),\cdots,v_m(x')$ satisfy an r-relation and another s-relation, it does not necessarily mean an $(r+s)$-relation.

We next define a classification of the characteristic roots. Let $\theta\in\{0,\pi\}$. Let

$$(x,\xi')\in\mathbf{R}^n\times\mathbf{R}^{n-1},\ x_1\neq 0,\ \arg x_1=\theta. \tag{5}$$

We define

$$\begin{aligned} M&=\{1,2,\cdots,m\},\\ M_{0,\theta}&=\{\lambda\in M;\ \mathrm{Re}(x_1\varphi_\lambda(x,\xi'))=0,\ \text{if } (x,\xi') \text{ satisfies (5)}\},\\ M_{\pm,\theta}&=\{\lambda\in M;\ \pm\mathrm{Re}(x_1\varphi_\lambda(x,\xi'))>0,\ \text{if } (x,\xi') \text{ satisfies (5)}\},\\ M'_\theta&=M\setminus M_{0,\theta}\setminus M_{+,\theta}\setminus M_{-,\theta}\ . \end{aligned}$$

It is easy to see that $M_{0,\theta}\cup M_{+,\theta}\cup M_{-,\theta}\cup M'_\theta=M$ is a disjoint union.

Let $m_{0,\theta}$, $m_{\pm,\theta}$ be the number of the elements belonging to $M_{0,\theta}$, $M_{\pm.\theta}$, respectively. We assume that

$$M'_\theta=\emptyset,\ \forall\theta\in\{0,\pi\}. \tag{6}$$

We also need a condition for the microfumctions. Let

$$\begin{aligned} \omega(r)&=\{(x,\xi)\in\sqrt{-1}\mathbf{T}^*\mathbf{R}^n;\ |x|<r,\ |\xi''|<r\,\mathrm{Im}\,\xi_n\},\\ \omega'(r)&=\{(x',\xi')\in\sqrt{-1}\mathbf{T}^*\mathbf{R}^{n-1};\ |x'|<r,\ |\xi'''|<r\,\mathrm{Im}\,\xi_n\}, \end{aligned}$$

and

$$\begin{aligned} \omega_0(r)&=\{(x,\xi)\in\omega(r);\ |x'|\le r^{-1}|x_1|,\ |\xi''|\le r^{-1}|x_1|\,\mathrm{Im}\,\xi_n\},\\ \omega'_0(r)&=\{tx'^*;\ t>0\}. \end{aligned}$$

We define

$$\mathcal{C}_0 = \varinjlim_{r>0} \Gamma_{\omega_0(r)}(\mathcal{C}_{\mathbf{R}^n},\ \omega_0(r)),$$
$$\mathcal{C}'_0 = \varinjlim_{r>0} \Gamma_{\omega'_0(r)}(\mathcal{C}_{\mathbf{R}^{n-1}},\ \omega'_0(r)).$$

Then we have the following

Theorem. *We assume* (1) – (3) *and* (6). *Let* $v_1(x'), \cdots, v_m(x') \in \mathcal{C}'_0$. *Then there exists an* $m_{+,0}$*-relation and an* $m_{+,\pi}$*-relation such that the Cauchy problem* (4) *has a solution* $u \in \mathcal{C}_0$ *if, and only if,* $v_1(x'), \cdots, v_m(x')$ *satisfy these relations.*

We give some examples. At first we remind the reader of the well-known result for the operators of principal type.

Example 0 (*Lewy-Mizohata operators*). If $P_\pm = D_1 \pm \sqrt{-1}x_1 D_n$, then we have $M_{\pm,\theta} = \{1\}(= M)$, $M_{\mp,\theta} = \emptyset$. The above theorem means that $P_- u = 0$, $u(0, x') = v(x')$ is solvable for any $v \in \mathcal{C}'_0$ without any relations. In fact using the defining function we only need to let $u(x) = v(x''', x_n + \sqrt{-1}x_1^2/2)$. On the other hand, $P_+ u = 0$, $u(0, x') = v(x')$ is solvable only for the case when $v(x')$ satisfies a one-relation. This means $v = 0$, and $u = 0$. It follows that $P_+ u = 0 \Rightarrow u = 0$, i.e., P_+ is hypo-elliptic in $\mathcal{C}_0$ (See [6]).

Lewy-Mizohata operators are the simplest case of our theory, and our theorem gives a similar result even for more complicated operators. The characteristic roots belonging to $M_{+,\theta}$ cause obstruction, and correspondingly the Cauchy data must satisfy so many relations. Let us see the case $m = 2$.

Example 1 (*microhyperbolic operators*). Let $P(x, D) = D_1^2 - x_1^2 D_n^2 + P'(x, D)$, $\operatorname{ord} P' \leq 1$. Without loss of generality, we may assume that P' is a polynomial in D_1 of degree 1. Since $\varphi_1(x, \xi') = x_1\xi_n, \varphi_2(x, \xi') = -x_1\xi_n$, and $\arg \xi_n = \pi/2$, it is easy to see that $M_{0,\theta} = \{1, 2\}$, $M_{\pm,\theta} = \emptyset$ for $\theta \in \{0, \pi\}$. It follows that that (4) is solvable for arbitrary $v_1(x'), v_2(x') \in \mathcal{C}'_0$ without any relations (See [3]).

Example 2 (*Tricomi operators*). Let $P(x, D) = D_1^2 - x_1 D_n^2 + P'(x, D)$, $\operatorname{ord} P' \leq 1$. We have $\varphi_1(x, \xi') = \sqrt{x_1}\xi_n$, $\varphi_2(x, \xi') = -\sqrt{x_1}\xi_n$. It follows that $M_{0,0} = \{1, 2\}$, $m_{+,0} = 0$, and that $M_{+,\pi} = \{1\}$, $M_{-,\pi} = \{2\}$, $m_{+,0} = 1$. It follows that there exists a 1-relation, and (4) is solvable if, and only if, the Cauchy data satisfy this relation. We can understand this phenomenon as follows. Let $\omega \subset \sqrt{-1}\mathbf{T}^*\mathbf{R}^n$ be a small neighborhood of x^*, and let $\omega^\theta = \{(x, \xi) \in \omega;\ x_1 \neq 0,\ \arg x_1 = \theta\}$, $\theta \in \{0, \pi\}$. At first we consider an elliptic boundary value problem in ω_π, giving one boundary datum on $\{x_1 = 0\}$. Then we can always extend this solution to the hyperbolic region ω_0. This case was considered also by [4]

Example 3 (hypoelliptic operators). Let $P(x, D) = D_1^2 + x_1^2 D_n^2 + P'(x, D)$, $\operatorname{ord} P' \leq 1$. Since $\varphi_1(x, \xi') = \sqrt{-1}x_1\xi_n, \varphi_2(x, \xi') = -\sqrt{-1}x_1\xi_n$, it is easy to see that $M_{-,\theta} = \{1\}$, $M_{+,\theta} = \{2\}$, $m_{+,\theta} = 1$ for $\theta \in \{0, \pi\}$. There exist an $m_{+,0}$-relation and an $m_{+,\pi}$-relation such that the Cauchy problem (4) uniquely has a solution $u \in \mathcal{C}_0$ if, and only if, $v_1(x'), \cdots, v_m(x') \in \mathcal{C}'_0$ satisfy both of these relations. In most cases two 1-relations mean a 2-relation, but this is not always true. If this is true (4) is solvable only in the case $v_1 = v_2 = 0$, and $u = 0$. In other words, $Pu = 0$ does not have any non-trivial solutions. It is well-known that this is true if the principal symbol $\sigma_1(P')$ of the lower order term satisfies $\xi_n^{-1}\sigma_1(P') \notin \{\sqrt{-1},\ \sqrt{-13},\ \sqrt{-15},\cdots\}$(See [2,5]). Of course our result applies for higher order operators, too.

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