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A Survey of Removable Singularities in the Theory of Linear Differential Equations

MOTOO UCHIDA

Department of Mathematics, Graduate School of Science, Osaka University (Japan)

内田素夫 (大阪大学理学研究科)

This is a survey of some classical and recent results on local continuation of solutions of differential equations in the real domain. In particular, we are interested in the problem of removal of point singularity. We do not try to make it exhaustive here. We will not prove any results stated. Instead we give references to the literature.

Let M be a paracompact real analytic manifold, $n = \dim M$. Let X be a complex neighborhood of M, T^*X its cotangent bundle. (Through this paper, we keep this notation.) Let $\mathcal{D}_X$ be the sheaf of rings of differential operators with holomorphic coefficients on X. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, $\mathrm{Ch}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$, which is a closed analytic subset of T^*X. For $x \in X$, we set

$$\mathrm{Ch}_x(\mathcal{M}) = \mathrm{Ch}(\mathcal{M}) \cap T^*_x X.$$

0. A First Remark

Let M be an open ball in $\mathbf{R}^n$ centred at 0, and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Let $\mathcal{B}_M$ denote the sheaf of hyperfunctions on M.

Recall that $\mathcal{M}$ is hyperbolic in direction dx_1 at 0 if

$$(0, dx_1) \notin C_{T^*_M X}(\mathrm{Ch}(\mathcal{M})).$$

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The following is then an immediate consequence of the theory of micro-supports (Cf. Kashiwara and Schapira [KS]).

For a closed subset K of M, let $N^*(K)$ denote the conormal cone to K in M.

Theorem 0.1. *1) Let K be a closed subset of M, $0 \in K$. Assume $N_0^*(K) \neq T_0^*M$. If $\mathcal{M}$ is hyperbolic in every direction belonging to $N_0^*(K)$,*

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}\Gamma_K \mathcal{B}_M)_0 = 0.$$

2) Let N be a real analytic submanifold germ of M at 0. Let $p \in (T_N^*M)_0$. If $\mathcal{M}$ is hyperbolic in direction p, we have

$$\mu_N(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))_p = 0.$$

In view of this, if we assume hyperbolicity of the system of differential equations, the continuation problem of its solutions becomes trivial.

By combining the above theorem and the formula of Kashiwara and Kawai [KK] for elliptic boundary value problems, we immediately obtain

Theorem 0.2. *Let N be a real analytic submanifold germ of M at 0 of codimension d. Let $p \in (T_N^*M)_0$. Assume*

$$p \notin C_{T_M^*X}(V)$$

for every irreducible component V of $\mathrm{Ch}(\mathcal{M})$ of codimension $< d$, and

$$T_M^*X \cap W \subset T_X^*X \quad \textit{and} \quad T_Y^*X \cap W \subset T_X^*X$$

for every irreducible component W of $\mathrm{Ch}(\mathcal{M})$ of codimension $\geq d$, where Y is the complexification of N in X. Then we have

$$H^j \mu_N(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))_p = 0 \quad \textit{for } j < d.$$

The above results are still true if we replace $\mathcal{B}_M$ by $\mathcal{A}_M$, the sheaf of real analytic functions. In this article, we do not mention any generalization in this direction. For this, see the expose of Takeuchi in this volume.

1. Extension of Solutions of Overdetermined Systems of Differential Equations

Let M be a paracompact real analytic manifold, and X its complexification. Let $\mathcal{B}_M$ denote the sheaf of hyperfunctions on M.

Let K be a closed subset of M. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.

Hypothesis 1.1. There exists a complex submanifold Z of X of codimension d, and its real submanifold L such that

(1) Z is the complexification of L in X,
(2) $K \subset L$, and
(3) Z is non-characteristic for $\mathcal{M}$.

Theorem 1.2. *Assume Hypothesis 1.1. Then we have*

$$H^j \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}\Gamma_K \mathcal{B}_M) = 0 \quad \textit{for } j < d.$$

This implies in particular the extension of all sections of $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{B}_M)$ defined outside K to the whole M if the system of differential equations $\mathcal{M}$ satisfies the conditions in Hypothesis 1.1 for $d = 2$. Moreover, in this case, if

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^{n_0} \xleftarrow{P} \mathcal{D}_X^{n_1} \xleftarrow{Q} \mathcal{D}_X^{n_2}$$

is a resolution of $\mathcal{M}$ on M, with P, Q matrices of differential operators, we see that the mapping

$$\{u \in \mathcal{B}(M)^{n_0} \mid Pu = f\} \to \{u \in \mathcal{B}(M \setminus K)^{n_0} \mid Pu = f\}$$

is one-to-one and onto for any hyperfunction n_1-vector f which satisfies the compatibility condition $Qf = 0$.

In the case where K is a one-point subset, we can state Hypothesis 1.1 in the following form. Hence the extendability to a point of all hyperfunction solutions defined outside follows from the overdetermined character of the system of differential equations.

Lemma 1.3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and $x \in M$. If $\mathrm{Ch}_x(\mathcal{M})$ is of codimension $\geq d$ in $T_x^* X$, the conditions in Hypothesis 1.1 are satisfied for $K = \{x\}$.*

Proof. Immediate. Moreover we can take L as a real submanifold of M.

The proof of Theorem 1.2 is given by Kawai [Kw]. The key is the following two formulas.

Formula 1.4. Structure theorem of hyperfunctions : Let L be a real analytic submanifold of X, and Z the complexification of L in X. If $K \subset L \cap M$, we have locally (up to orientation sheaf factors)

$$\mathrm{R}\Gamma_K \mathcal{B}_M \cong \mathcal{D}^{\infty}_{X \leftarrow Z} \otimes_{\mathcal{D}^{\infty}_Z} \mathrm{R}\Gamma_K \mathcal{B}_L$$

as $\mathcal{D}^{\infty}_X$-module. Here $\mathcal{D}^{\infty}_{X \leftarrow Z}$ is a $(\mathcal{D}^{\infty}_X, \mathcal{D}^{\infty}_Z)$-bimodule (see [SKK]), and $\mathcal{B}_L$ denotes the sheaf of hyperfunctions on L.

Formula 1.5. Let $\varphi : Z \to X$. Assume φ is non characteristic for $\mathcal{M}$. Let $\mathcal{M}_Z$ denote the induced $\mathcal{D}_Z$-module of $\mathcal{M}$: $\mathcal{M}_Z = \mathcal{D}_{Z \to X} \otimes_{\mathcal{D}_X} \mathcal{M}$. Then $\mathcal{M}_Z$ is a coherent $\mathcal{D}_Z$-module, and we have

$$\varphi^{-1}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}^{\infty}_{X \leftarrow Z}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{M}_Z, \mathcal{D}^{\infty}_Z)[-\operatorname{codim} Z].$$

Let $\mathcal{D}b_M$ be the sheaf of Schwartz distributions on M.

Since we have also the structure formula for distributions, by the same proof, we have an analogue of Theorem 1.2 for the sheaf of distributions.

Theorem 1.6. *Assume Hypothesis 1.1. Then we have*

$$H^j \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_K \mathcal{D}b_M) = 0 \quad \textit{for } j < d.$$

2. Continuation of Regular Solutions of Single Differential Equations

By the results of Section 1, we have general theorems on local extension of solutions of overdetermined systems of differential equations. Hence we are now interested mainly in the case of determined systems of differential equations, in particular, of single differential equations. In this case, we have to consider regular solutions of the equation or solutions with a growth condition near the singular locus. In this section, we collect some results for regular (analytic in most cases) solutions. We restrict ourselves to the case of point singularities. The interesting results early obtained are the following.

Theorem 2.1 [G]. *Let P be a differential operator with constant coefficients in n variables, $P \neq 0$. Assume that the polynomial $P(\zeta)$ is irreducible and*

$$P_m(1, 0, \ldots, 0) = 0 \textit{ and } dP_m(1, 0, \ldots, 0) \neq 0,$$

where P_m denotes the principal part of P. Then any smooth solution of the differential equation $Pu = 0$ on $U \setminus \{0\}$ is continued to the whole U, where U is a neighborhood of 0 in $\mathbf{R}^n$.

Theorem 2.2 [Kn1]. *Let P be a differential operator with constant coefficients in n variables, $P \neq 0$. Assume that the algebraic variety $P(\zeta) = 0$ in $\mathbf{C}^n$ has no elliptic irreducible components. Then any real analytic solution of the differential equation $Pu = 0$ on $U \setminus \{0\}$ is continued to the whole U, where U is a neighborhood of 0 in $\mathbf{R}^n$.*

In fact, Kaneko [Kn1] proved the extendability of analytic solutions to compact convex subsets of $\mathbf{R}^n$.

In what follows, we state a corresponding result in the case of equations with variable coefficients.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_M$-module. Let $x \in M$.

Hypothesis 2.3. (0) $\mathrm{Ch}(\mathcal{M}) \subset V_1 \cup \cdots \cup V_r \cup W$ in a neighborhood of x, where V_j is a homogeneous analytic subset given by $P_j = 0$ $(j = 1, \ldots, r)$ and W is a homogeneous analytic subset of codimension ≥ 2.
(1) P_j is a homogeneous holomorphic function which is real valued on $T^*_M X$ $(j = 1, \ldots, r)$.
(2) For any j and any irreducible component V' of $V_j \cap T^*_x X$, there exists $q \in T^*_M X \cap V'$ where $d_\xi P_j(q) \neq 0$.
(3) $W \cap T^*_x X$ is of codimension ≥ 2 in $T^*_x X$.

Theorem 2.4. *Suppose $\mathcal{M}$ satisfies the conditions of Hypothesis 2.3. Let U be a neighborhood of x in M. Any real analytic solution of $\mathcal{M}$ on $U \setminus \{x\}$ is extendable to the whole U as a hyperfunction solution of $\mathcal{M}$, namely as a section of $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{B}_M)$. If we assume moreover that all the P_j's are of simple real characteristics (i.e., $d_\xi P_j \neq 0$ on $T^*_M X \setminus M$) and that $W \cap T^*_M X \subset M$, the extension is analytic.*

In particular, we have :

Hypothesis 2.5. (0) $\mathrm{Ch}(\mathcal{M}) \subset V_1 \cup \cdots \cup V_r$ in a neighborhood of x, where V_j is a homogeneous analytic subset given by $P_j = 0$ $(j = 1, \ldots, r)$.
(1) P_j is a homogeneous holomorphic function which is real valued on $T^*_M X$ and $d_\xi P_j(q) \neq 0$ on $T^*_M X \setminus M$ $(j = 1, \ldots, r)$.
(2) $V_j \cap T^*_x X$ has no elliptic irreducible components $(j = 1, \ldots, r)$.

Theorem 2.6. *Suppose $\mathcal{M}$ satisfies the conditions of Hypothesis 2.5. Let U be a neighborhood of x in M. Any analytic solution of $\mathcal{M}$ on $U \setminus \{x\}$ is analytically extendable to the whole U.*

Note that all hyperfunction solutions are not necessarily extendable for the equation $\mathcal{M}$ above.

The proof of Theorem 2.6 is contained in [U1], where the proof for $r = 1$ is given. There is no difficulty in proving Theorem 2.6, and also Theorem 2.4, in the same way.

As a particular case, we have the following. Let P be a differential operator (with analytic coefficients) on M, $P \neq 0$. We denote simply by $\mathrm{Ch}(P)$ the characteristic variety of the $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{D}_X P$, and $\mathrm{Ch}_x(P) = \mathrm{Ch}(P) \cap T_x^* X$.

Corollary 2.7. *Let P be a differential operator (with analytic coefficients) on M of real principal type (i.e., with real principal part and of simple real characteristics). Assume that $\mathrm{Ch}_x(P)$ has no elliptic irreducible components. Then any analytic solution of $Pu = 0$ on $U \setminus \{x\}$ is analytically continued to the whole U.*

Example. $P = D_1^3 D_3 + (D_2^2 + D_3^2)^2 - D_4^4 + Q(x, D)$ on $\mathbf{R}^4$, where $Q(x, D)$ is a differential operator of order 4 with real principal part which vanishes at x. (Note that this P is not hyperbolic in any direction.)

Note that Kaneko [Kn3] has earlier obtained the essentially same result as Corollary 2.7 in the case where the analytic subset $\mathrm{Ch}_x(P)$ is irreducible ([Kn3, Theorem 21]). On the other hand, in [Kn2], a removable singularity result is proved by assuming

(1) P is of real principal type,
(2) $x_1 = 0$ is non-characteristic for P, and
(3) the roots of $P_m(x, \zeta_1, \xi')$ in ζ_1 are all real and simple if (x, ξ') is in a neighborhood of $(0, dx_2)$ in $M \times \mathbf{R}^{n-1}$.

This is also a particular case to which Corollary 2.7 is applied.

The method of the proof of Theorem 2.4 given in [U1] is the same as Kaneko presented in Part III of [Kn3] (there, with always fixing the hypersurface $x_1 = 0$, the argument is left partially completed). That is a kind of Fundamental Principle in the conormal sphere bundle [Kn3]. We believe that Theorem 2.4 or Corollary 2.7 is a complete result obtained in this direction which Kaneko aimed to arrive at by the method of loc.cit. (see Introduction of [Kn3]).

Theorem 2.4 is generalized to the case where the singular locus of u is contained in a real analytic submanifold of codimension ≥ 2. We do not give the details. (The results are parallel with suitable modification. See also [Kn3], Part III.)

3. Continuation of Analytic Solutions to the Vertex of a Convex Proper Closed Cone

Kaneko [RIMS Kōkyūroku 592 (1986), pp.149–172] conjectured that all real analytic solutions of the wave equation or the ultra-hyperbolic equation

$$(D_1^2 + \cdots + D_k^2 - D_{k+1}^2 - \cdots - D_n^2)u = 0$$

defined outside

$$K = \{(x_1, x') \in \mathbf{R}^n \mid x_1 \leq -C\|x'\|\},$$

with $C > 0$, are analytically continued to a neighborhood of $x = 0$.

As to this problem, we have :

Theorem 3.1. *Let P be a second order differential operator with analytic coefficients on M. Assume that P is of real principal type and is not elliptic. Let K be a closed subset of M, and $x \in K$. Assume that K is convex in local coordinates of class $\mathcal{C}^1$ in a neighborhood of x and the tangent cone $C_x(K)$ is proper. Then any real analytic solution of the equation $Pu = 0$ defined outside K is analytically continued to a neighborhood of x in M.*

The proof is in [U1]. We can apply this theorem to the example of the conjecture above for any $C > 0$ (i.e., without taking C large).

4. Extension of Solutions of Differential Equations with Growth Restriction

For the result of this section, it is sufficient that M is a smooth real manifold. See the work of Bochner [B], Littman [L], Harvey and Polking [HP], Eells and Polking [EP] and the survey report [P] of Polking for the details in this direction. (The author would like to thank A. Kaneko who kindly let the author know the theorem of Bochner and the survey report of Polking.)

The following basic theorem of extension of solutions with a growth restriction is due to Bochner [B].

Theorem 4.1. *Let U be an open subset of $\mathbf{R}^n$, F a closed subset of U. Let $P(x, D)$ be an $n_1 \times n_0$ matrix of differential operators of order m with smooth coefficients on U. Let u be an n_0 vector of $L_{1,\mathrm{loc}}(U)$ which satisfies*

$$P(x, D)u = 0 \quad \textit{in } \mathcal{D}b(U \setminus F).$$

If

$$\varlimsup_{\varepsilon\to+0} \varepsilon^{-m} \int_{K(\varepsilon)} |u|dx = 0,$$

for any compact subset K of F, where $K(\varepsilon) = \{x \in U \mid d(x,K) < \varepsilon\}$, we have

$$P(x,D)u = 0 \quad \textit{in } \mathcal{D}b(U).$$

As an immediate corollary of this theorem, we have the following interesting two results (which are also due to Bochner [B]). (Bochner stated them in a more general setting.)

Let M be a real smooth paracompact manifold. Let $P(x,D)$ be an $n_1 \times n_0$ matrix of differential operators of order m with smooth coefficients on M. Let F be a locally finite union of closed submanifolds of M of class $\mathcal{C}^1$ of codimension $\geq d$.

Theorem 4.2. *Let u be an n_0 vector of $L_{p,\mathrm{loc}}(M)$, $p \geq 1$, which satisfies $P(x,D)u = 0$ in $\mathcal{D}b(M \setminus F)$. If $m \leq d(1-1/p)$, we have $P(x,D)u = 0$ on M.*

Theorem 4.3. *Let u be an n_0 vector of $L_{1,\mathrm{loc}}(M)$ which satisfies $P(x,D)u = 0$ in $\mathcal{D}b(M \setminus F)$. If $|u(x)| = o(d(x,F)^{-\gamma})$, locally uniformly on M, with $\gamma \leq d - m$, we have $P(x,D)u = 0$ on M.*

The first result is generalized by Littman [L] in terms of (s,q)-polar closed subsets ($s > 0$, $q > 1$). The fundamental properties of (s,q)-polar subsets are :

Theorem 4.4. (1) *If A is contained in a closed submanifold of M of class $\mathcal{C}^\infty$ of codimension d, A is (s,q)-polar if $sq \leq d$.*
(2) *A countable union of (s,q)-polar subsets is (s,q)-polar.*

We have :

Theorem 4.5 [L]. *Let $q > 1$, and let F be an (m,q)-polar closed subset of M. Let u be an n_0 vector of $L_{p,\mathrm{loc}}(M)$, with $p = q/(q-1)$, which satisfies $P(x,D)u = 0$ in $\mathcal{D}b(M \setminus F)$. Then we have $P(x,D)u = 0$ on M.*

4a. Appendix to Section 4

Bochner's results are immediately extended to a class of semi-linear differential equations. (Such results are not found in the literature in a general form, but the proof seems well known as a standard argument.)

Let M be a real smooth paracompact manifold. Let $P(x, D)$ be an $n_1 \times n_0$ matrix of differential operators of order m with smooth coefficients on M. Let $Q(x, u)$ be an n_1 vector of continuous functions, satisfying

$$\|Q(x,u)\| \leq A\|u\|^{\rho} + B, \quad \text{with } \rho \geq 0 \text{ and } A,\ B > 0.$$

Consider the semi-linear differential equation

$$(\#) \qquad P(x, D)u = Q(x, u).$$

Let F be a locally finite union of closed submanifolds of M of class $\mathcal{C}^1$ of codimension $\geq d$.

Theorem 4.6. *Let u be an n_0 vector of $L_{p,\mathrm{loc}}(M)$, $p \geq \max\{1, \rho\}$, which satisfies (#) in $\mathcal{D}b(M \setminus F)$. If $m \leq d(1 - 1/p)$, u satisfies (#) in $\mathcal{D}b(M)$.*

Theorem 4.7. *Let u be an n_0 vector of $L_{\max\{1,\rho\},\mathrm{loc}}(M)$ which satisfies (#) in $\mathcal{D}b(M \setminus F)$. If $|u(x)| = o(d(x, F)^{-\gamma})$, locally uniformly on M, with $\gamma \leq d - m$, u satisfies (#) in $\mathcal{D}b(M)$.*

It is possible to give a similar result to the differential equation

$$P(x, D)u = Q(x, D^{\alpha}u).$$

Eells and Polking [EP] applied the argument of Bochner and Littman to the equation of harmonic maps.

5. Continuation of Analytic Solutions of Single Differential Equations II

We stated in Section 2 a few removable point singurality theorems for single differential equations. It was then essential to assume the equation to have no elliptic factors. In this section, we consider equations of which the characteristic variety possibly has elliptic irreducible components.

Let P be a differential operator with analytic coefficients on M. We denote simply by $\mathrm{Ch}(P)$ the characteristic variety of $\mathcal{D}_X/\mathcal{D}_X P$. We assume $\mathrm{Ch}(P) \neq T^*X$.

In what follows, we assume that the principal symbol of P is real. (Or, more generally, we have only to assume that $\mathrm{Ch}(P)$ is real in a neighborhood of $T^*_M X \setminus M$.)

Let $x \in M$. We set $\mathrm{Ch}_x(P) = \mathrm{Ch}(P) \cap T^*_x X$.

Hypothesis 5.1. (0) $\mathrm{Ch}_x(P) \neq T_x^*X$.
(1) Let $\sigma(P)$ denote the principal symbol of P. Then $\sigma(P)(x,\xi) = P_1(\xi)\cdots P_r(\xi)Q(\xi)$ for the above fixed x, where P_j is an irreducible homogeneous polynomial ($j = 1,\ldots,r$), with $P_j \neq P_k$ ($j \neq k$), and Q is a homogeneous polynomial of degree q.
(2) P_j is real : $P_j = \overline{P_j}$ ($j = 1,\ldots,r$).
(3) $P_j^{-1}(0) \cap \mathbf{R}^n \neq \{0\}$, for any $j = 1,\ldots,r$.
(4) $Q \neq 0$ on $\mathbf{R}^n \setminus \{0\}$.

We then have :

Theorem 5.2. *Let P be a differential operator with real principal part. Let $x \in M$ (and take a local coordinate z with $z = 0$ at x). Assume Hypothesis 5.1 at x. Moreover we assume $P_j^{-1}(0)^{\mathrm{reg}} \cap \mathbf{R}^n \neq \{0\}$ for any $j = 1$, $\ldots, r$. Let U be a neighborhood of x in M. Let u be a real analytic solution of $Pu = 0$ on $U \setminus \{x\}$. If*

(a) *u is in $L_p(U \setminus \{x\})$ for $p \geq 1$ and $q \leq n(1 - 1/p)$,*

or if

(b) *$|u(z)| = o(d(z)^{-\gamma})$ for $\gamma \leq n - q$,*

where $d(z)$ is the distance of z and 0 in the local chart, u is then extendable to the whole U as a hyperfunction which satisfies $Pu = 0$.

Corollary 5.3. *Let P be a differential operator with real principal part P_m and of simple real characteristics (i.e., $d_\xi P_m \neq 0$ on $T_M^*X \setminus M$). Let $x \in M$, and assume Hypothesis 5.1. Then any analytic solution of $Pu = 0$ on $U \setminus \{x\}$ is analytically continued to the whole U if u satisfies one of the growth condition* (a) *or* (b).

If $q = 0$, by Theorem 2.4, the growth restriction on u (a) nor (b) is not needed. The proof will be in [U2].

6. Extension of Solutions of Differential Equations with Growth Restriction II

In this section, we consider the removal of point singularity of solutions of systems of differential equations again.

Let $\mathrm{Gr}(\mathcal{D}_X)$ denote the graded ring of $\mathcal{D}_X$ for the filtration by the order.

Let M be an open ball of $\mathbf{R}^n$ centred at 0. Let P_1 and P_2 be differential operators with analytic coefficients on M. Letting $\sigma(P_\nu) = \varphi_\nu Q$ be a factorization in $\mathrm{Gr}(\mathcal{D}_X)_0$ ($\nu = 1, 2$), we assume

Hypothesis 6.1. (0) $Q(x,\xi)$ is a homogeneous polynomial in ξ of degree q, and $Q(0,\xi) \not\equiv 0$.
(1) $\{\varphi_1(0,\xi) = \varphi_2(0,\xi) = 0\}$ is an algebraic subvariety of T_0^*X of codimension ≥ 2.

We consider the system of differential equations

$$(\#) \qquad P_1 u = P_2 u = 0.$$

Then we have

Theorem 6.2. *Suppose Hypothesis 6.1. Let* u *be in* $L_{1,\mathrm{loc}}(M)$ *and satisfy the equation* $(\#)$ *in* $\mathcal{D}b(M \setminus \{0\})$. *If*

(a) u *is in* $L_{p,\mathrm{loc}}(M)$ *for* $p \geq 1$ *and* $q \leq n(1-1/p)$,

or if

(b) $|u(x)| = o(d(x)^{-\gamma})$ *for* $\gamma \leq n - q$,

where $d(x)$ *is the distance of* x *and* 0, u *then satisfies* $(\#)$ *in* $\mathcal{D}b(M)$.

This is a refinement of the results of Bochner in the point singularity case for systems of differential equations

$$P_1 u = \cdots = P_r u = 0.$$

In the above theorem, we restricted ourselves to the case $r = 2$ by a certain technical reason for the proof. We think however that the result itself is true for any $r \geq 2$ and the proof works with a minor modification. (If $P_1, \ldots, P_r$ are differential operators with constant coefficients, the proof works for any $r \geq 2$ as it is.)

The proof (hopefully for any $r \geq 2$) will be in [U3].

References

[B] Bochner, S., *Weak solutions of linear partial differential equations*, J. Math. Pures Appl. **35** (1956), 193–202.

[EP] Eells, J. and Polking, J. C., *Removable singularities of harmonic maps*, Indiana Univ. Math. J. **33** (1984), 859–871.

[G] Grusin, V. V., *On solutions with isolated singularities for partial differential equations with constant coefficients*, Trans. Moscow Math. Soc. **15** (1966), 295–315.

[HP] Harvey, R. and Polking, J. C., *Removable singularities of solutions of linear partial differential equations*, Acta Math. **125** (1970), 39–56.

[KK] Kashiwara, M. and Kawai, T., *On the boundary value problem for elliptic system of linear partial differential equations, I and II*, Proc. Japan Acad., Ser. A, **48** (1972), 712–715; *ibid.* **49** (1973), 164–168.

[KS] Kashiwara, M. and Schapira, P., *Sheaves on Manifolds*, Springer-Verlag, 1990.

[Kn1] Kaneko, A., *On continuation of regular solutions of partial differential equations to compact convex sets*, J. Fac. Sci. Univ. Tokyo **17** (1970), 567–580.

[Kn2] Kaneko, A., *On continuation of regular solutions of linear partial differential equations with real analytic coefficients*, Proc. Japan Acad. **51** (1975), 798–801.

[Kn3] Kaneko, A., *On continuation of real analytic solutions of linear partial differential equations*, Astérisque **89-90** (1981), S.M.F., 11–44.

[Kw] Kawai, T., *Extension of solutions of systems of linear differential equations*, Publ. RIMS, Kyoto Univ., **12** (1976), 215–227.

[L] Littman, W., *Polar sets and removable singularities of partial differential equations*, Ark. Math. **7** (1967), 1–9.

[P] Polking, J. C., *A survey of removable singularities*, Seminar on Nonlinear Partial Differential Equations (S. S. Chern, ed.), MSRI Publ., vol. 2, Springer, 1984, pp. 261–292.

[U1] Uchida, M., *Continuation of analytic solutions of linear differential equations up to convex conical singularities*, Bull. Soc. Math. France **121** (1993), 133–152.

[U2] Uchida, M., in preparation.

[U3] Uchida, M., in preparation.