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Simple algorithm for recognizing lake-free 4-map graphs

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1 Introduction

Suppose that we are given a set S of countries on the sphere. Traditional planarity says that two countries are adjacent iff they share a border line. In [1], Chen et al. suggest a modified notion of planarity which says that two countries are adjacent iff they share at least one point. The graph G abstracting this adjacency is called a *map graph*. If the countries of S together cover the sphere completely, G is called a *lake-free map graph*. If G is a map graph (or lake-free map graph, resp.) and no k countries of S meet at a single point for some natural number k, then G is called a *k-map graph* (resp., *lake-free k-map graph*).

The problem of recognizing map graphs and its extensions have been studied for geographic information systems. Chen et al. [1] proved that the problem of recognizing map graphs is in NP and gave a very complicated $O(n^3)$-time algorithm for recognizing 4-map graphs. Subsequently, Thorup [2] came up with a complicated $\Omega(n^{125})$-but polynomial-time algorithm for recognizing map graphs.

This paper was motivated by the necessity of simplifying the algorithms mentioned above. It shows that there is a relatively simple $O(n^4)$ algorithm for recognizing lake-free 4-map graphs.

2 Basics

A *marked* graph is a graph in which zero or more edges are colored and the rest are not. Let G be a marked graph. $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. For a vertex v in G, $N_G(v)$ denotes the set of vertices adjacent to v in G. Let U be a subset of E. $N_G(U)$ denotes $\cup_{u \in U} N_G(u)$. Let F be a subset of E. $G-U-F$ denotes the marked graph obtained from G by deleting the edges in F and the vertices in U together with the edges incident to them. When U or F is empty, we drop it from the notation $G-U-F$. $G[U]$ denotes $G-(V(G)-U)$.

Throughout the rest of this paper, fix a marked graph G with vertex set V and edge set E. Let U be a subset of V. A *layout* $\mathcal{L}$ of $G[U]$ is a drawing of the vertices of U on the sphere satisfying the following three conditions:

1. Each $u \in U$ is drawn as a disc homeomorph $\mathcal{R}(u)$ on the sphere; for every pair of distinct vertices u and v in U, the interiors of $\mathcal{R}(u)$ and $\mathcal{R}(v)$ are disjoint.

2. Two vertices u and v are adjacent in $G[U]$ iff the boundaries of $\mathcal{R}(u)$ and $\mathcal{R}(v)$ have a nonempty intersection. In addition, if $\{u,v\}$ is a colored edge in G, then the boundaries of $\mathcal{R}(u)$ and $\mathcal{R}(v)$ share a curve segment (not a single point).

3. No point in the drawing is shared by at least five $\mathcal{R}(u)$'s.

If we remove all $\mathcal{R}(u)$ with $u \in U$ from the sphere, we may be left with a number of connected regions. The closure of each of these regions is called a *lake* in $\mathcal{L}$. Note that each $\mathcal{R}(u)$ is a closed set and its removal from the sphere includes the removal of its boundary. A vertex $u \in U$ *touches* a lake $\mathcal{H}$ if the boundaries of $\mathcal{R}(u)$ and $\mathcal{H}$ have a nonempty intersection; u *strongly touches* $\mathcal{H}$ if the boundaries of $\mathcal{R}(u)$ and $\mathcal{H}$ share a curve segment. A *2-lake* is a lake strongly touched by exactly two vertices. *Erasing a 2-lake* $\mathcal{H}$ in $\mathcal{L}$ is the operation of modifying $\mathcal{L}$ by extending $\mathcal{R}(u)$ to occupy $\mathcal{H}$, where u is a vertex strongly touching $\mathcal{H}$. $\mathcal{L}$ is a *map* of G if $U=V$ and there is no lake in $\mathcal{L}$. When $U \neq V$, $\mathcal{L}$ is *extensible* if we can obtain a map of G by somehow drawing the vertices of $V-U$ as disc homeomorphs in the lakes of $\mathcal{L}$. $\mathcal{L}$ is *transformable* to another layout $\mathcal{L}'$ of $G[U]$ if whenever $\mathcal{L}$ is extensible, so is $\mathcal{L}'$. $\mathcal{L}$ is *well-formed* if for every edge $\{u,v\}$ in $G[U]$, $\mathcal{R}(u)$ and $\mathcal{R}(v)$ share either exactly one point or exactly one curve segment (but not both) of their boundaries. If $\mathcal{M}$ is a map of G and W is a subset of V, then $\mathcal{M}|_W$ denotes the extensible layout of $G[W]$ inherited from $\mathcal{M}$ in an obvious way.

Our goal is to design an algorithm which given G, constructs a map of G if one exists, and reports "failure" otherwise. Since checking the correctness of a map of G can be done in linear time, we can *assume* that G has a map and only need to show how to find one. So, in the rest discussion of this paper, *we assume that G has a map.* We also call the vertices in G *nations.* Throughout the rest of this paper, we will use lower-case letters to denote nations.

Let $\mathcal{M}$ be a map of G. A *k-point* in $\mathcal{M}$ is a point shared by exactly k nations. Let u and v be two nations. A *(u,v)-point* in $\mathcal{M}$ is a 4-point p at which u and v together with two other nations x and y meet cyclically in the order u, x, v, y. *Erasing the (u,v)-point* p in $\mathcal{M}$ is the operation of modifying $\mathcal{M}$ by extending nation x to occupy a disc that is centered at p and touches no nation other than u, v, x, and y. A *(u,v)-segment* in $\mathcal{M}$ is a curve segment S shared by the boundaries of u and v such that each endpoint of S is a 3- or 4-point. Note that two (u,v)-segments must be disjoint. An edge $\{u,v\}$ of G is *good* in $\mathcal{M}$ if either (i) there is exactly one (u,v)-segment but no (u,v)-point in $\mathcal{M}$ or (ii) there is exactly one (u,v)-point but no (u,v)-segment in $\mathcal{M}$. An edge that is not good in $\mathcal{M}$ is *bad* in $\mathcal{M}$. Note that $\mathcal{M}$ is well-formed iff every edge of G is good in $\mathcal{M}$.

For every $v \in V$, since nation v is a disc homeomorph in $\mathcal{M}$, removing v from $\mathcal{M}$ leaves exactly one connected region. So, G must be biconnected.

Fact 1 Let u and v be two distinct vertices of G. Then, the following statements hold:

1. $G-\{u,v\}$ is disconnected iff there are at least two (u,v)-segments in $\mathcal{M}$.
2. Suppose that $G-\{u,v\}$ is disconnected and its connected components are G_1, ..., G_k. Then for each $i \in \{1,\ldots,k\}$, the marked graph G'_i obtained from $G[V(G_i) \cup \{u,v\}]$ by coloring edge $\{u,v\}$ has a map. Moreover, given a map $\mathcal{M}_i$ for each G'_i, we can easily construct a map of G.

In light of Fact 1, we hereafter assume that G is 3-connected.

Fact 2 G is 3-connected iff G has a well-formed map.

Throughout the rest of this paper, unless stated otherwise, $\mathcal{M}$ denotes a well-formed map of G. Two nations u and v *strongly touch* in $\mathcal{M}$ if there is a (u,v)-segment in $\mathcal{M}$; they *weakly touch* in $\mathcal{M}$ if there is a (u,v)-point in $\mathcal{M}$. To simplify the discussions in the sequel, we assume that $|V| \geq 9$; the problem is easily solved when $|V| < 9$.

Fact 3 Let $C = \{a,b,c\}$ be a set of three distinct vertices in G. Then, the following statements hold:

1. When C is not a clique in G, $G-C$ is connected.
2. When C is a clique in G, $G-C$ is disconnected iff (i) the nations in C do not meet at a point in $\mathcal{M}$ and (ii) each pair of nations in C strongly touch in $\mathcal{M}$.
3. Suppose that $G-C$ is disconnected. Then, (i) $G-C$ has exactly two connected components G_1 and G_2, and (ii) both G'_1 and G'_2 have a well-formed map, where G'_1 (respectively, G'_2) is the marked graph obtained from $G[V(G_1) \cup C]$ (respectively, $G[V(G_2) \cup C]$) by coloring the edges in $E(G[C])$. Moreover, given a well-formed map of G'_1 and one of G'_2, we can easily construct one of G.

A clique consisting of k vertices is called a *k-clique*. A clique C in G is *maximal* if no clique in G properly contains C. A maximal k-clique is denoted by MC_k. Let k be a positive integer. Two maximal cliques C_1 and C_2 are *k-sharing* if $|C_1 \cap C_2| = k$. It is easy to see that G has no 7-clique.

Fact 4 Suppose that G is 4-connected. Then, G has no 6-clique.

A *correct 4-pizza* is a list $\langle a,b,c,d \rangle$ of four nations in G such that G has a well-formed map in which nations a, b, c, d meet at a point cyclically in this order. *Removing a correct 4-pizza* $\langle a,b,c,d \rangle$ from G is the operation of modifying G as follows: Delete the edge $\{a,c\}$ from G and color the edges $\{b,d\}$, $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, and $\{d,a\}$.

Lemma 2.1 Let G' be the marked graph obtained from G by removing a correct 4-pizza $\langle a,b,c,d \rangle$. Then, G' has a well-formed map. Moreover, given a well-formed map of G', we can easily construct a well-formed map of G.

A simple inspection shows that every extensible layout of an MC_5 in G must be a "pizza with crust". Thus, in every extensible layout of an MC_5 C, there is a point at which exactly four nations of C meet. This motivated the following definition. A *correct center* of an MC_5 C in G is a list $\langle a,b,c,d \rangle$ of four nations in C such that C has a well-formed extensible layout in which nations a, b, c, d meet at a point cyclically in this order. The unique nation in $C-\{a,b,c,d\}$ is called a *correct crust* of C.

Fact 5 Let C be an MC_5 in G. Then, every correct center of C is a correct 4-pizza in G.

3 Advanced reductions

Let U be a subset of V. A *figure* of $G[U]$ is a list $\mathcal{D} = \langle \mathcal{L}, \mathcal{S}, L_1, \ldots, L_k \rangle$, where $\mathcal{L}$ is a layout of $G[U]$, $\mathcal{S}$ is a set of curve segments in $\mathcal{L}$ whose internal points are disjoint, and L_1, ..., L_k are disjoint lists of vertices in U. We call $\mathcal{L}$ the *layout* in $\mathcal{D}$, call the curve segments in $\mathcal{S}$ the *contractible segments* in $\mathcal{D}$, and call L_1, ..., L_k the *permutable lists* in $\mathcal{D}$. Associated with $\mathcal{D}$ is the set $\mathcal{X}$ of all layouts of $G[U]$ that can be obtained from $\mathcal{L}$ by (i) contracting some contractible segments each to a single point while erasing all resulting 2-lakes and (ii) for each permutable list L_i, selecting a permutation π of L_i and renaming each nation $u \in L_i$ as $\pi(u)$. $\mathcal{D}$ *displays* $\mathcal{L}'$ if each $\mathcal{L}' \in \mathcal{X}$. $\mathcal{D}$ *displays* $G[U]$ if $\mathcal{D}$ displays an extensible layout of $G[U]$. We will frequently illustrate $\mathcal{D}$ by first drawing $\mathcal{L}$ and then modifying it by (i) interrupting each contractible segment while emphasizing its endpoints and (ii) for each permutable list L_i, renaming each nation $u \in L_i$ as u^i. For example, when $U = \{a, b, c, d, e\}$ is an MC_5 in G, Figure 2.1 displays $\mathcal{M}|_U$. Actually, by contracting a set of contractible segments in the figure, we can obtain Figure 2.2(1) through (5); only they can possibly display $\mathcal{M}|_U$, as can be easily checked. We note that Figure 2.1 has a unique permutable set, namely, U itself. $\mathcal{D}$ is *transformable* to another figure $\mathcal{D}'$ of $G[U]$ if whenever $\mathcal{D}$ displays $\mathcal{M}|_U$, so does $\mathcal{D}'$.

For an edge $\{a, b\}$ in G, $\mathcal{E}[a, b]$ denotes the set of uncolored edges $\{x, y\} \in E$ such that $\{x, y\} \cap \{a, b\} = \emptyset$ and $\{x, y, a, b\}$ is an MC_4 in G. A *separating edge* of G is an edge $\{a, b\} \in E$ such that $G - \{a, b\} - \mathcal{E}[a, b]$ is disconnected. A *shrinkable segment* S in $\mathcal{M}$ is a (u, v)-segment in $\mathcal{M}$ such that (i) $\{u, v\}$ is an uncolored edge in G, (ii) both endpoints of S are 3-points, and (iii) one endpoint of S is touched by a nation $a \notin \{u, v\}$ and the other is touched by a nation $b \notin \{u, v, a\}$. Nations a and b are called the *ending nations* of S.

Lemma 3. 1 Suppose that G is 4-connected. Assume that G has a separating edge $\{a, b\}$. Let $G' = G - \{a, b\} - \mathcal{E}[a, b]$. Then, for every $\{x, y\} \in E$ such that x and y belong to different connected components of G', $\langle a, x, b, y\rangle$ is a correct 4-pizza in G.

Corollary 3. 2 Suppose that G is 4-connected. Assume that G has no 5-clique. Then, G has a separating edge iff there is a shrinkable segment in $\mathcal{M}$ whose ending nations are adjacent in G.

An *induced 4-cycle* in G is a set C of four vertices in G such that $G[C]$ is a cycle. A *separating 4-cycle* of G is an induced 4-cycle C in G such that $G - C$ is disconnected.

Lemma 3. 3 Suppose that G has a separating 4-cycle C. Then, $G - C$ has exactly two connected components G_1 and G_2. Moreover, we can easily construct two marked graphs G'_1 and G'_2 such that (i) each of G'_1 and G'_2 has a well-formed map and (ii) given a well-formed map of G'_1 and one of G'_2, we can easily construct one of G.

A *separating triple* of G is a list $\langle a, b, c\rangle$ of three vertices in G such that (i) $C = \{a, b, c\}$ is a clique in G and (ii) $G - C - \mathcal{E}[a, b]$ is disconnected.

Lemma 3. 4 Suppose that G is 4-connected and has no separating edge but has a separating triple $\langle a, b, c\rangle$. Then, $G - \{a, b, c\} - \mathcal{E}[a, b]$ has exactly two connected components G_1 and G_2. Moreover, $|V(G_1) \cap N_G(V(G_2))| = |V(G_2) \cap N_G(V(G_1))| = 1$ and $\langle a, u, b, v\rangle$ is a correct 4-pizza, where $\{u\} = V(G_1) \cap N_G(V(G_2))$ and $\{v\} = V(G_2) \cap N_G(V(G_1))$.

A *separating quadruple* is a list $\langle a, b, c, d\rangle$ of four vertices in G such that (i) $G[\{a, b, c, d\}]$ is a cycle and (ii) $G - \{a, b, c, d\} - \mathcal{E}[a, b]$ is disconnected. Using Lemma 3. 3 , we can modify the proof of Lemma 3. 4 to prove the following:

Lemma 3. 5 Suppose that G has neither separating edge nor separating 4-cycle, but has a separating quadruple $\langle a, b, c, d\rangle$. Then, $G - \{a, b, c, d\} - \mathcal{E}[a, b]$ has exactly two connected components G_1 and G_2. Moreover, $|V(G_1) \cap N_G(V(G_2))| = |V(G_2) \cap N_G(V(G_1))| = 1$ and $\langle a, u, b, v\rangle$ is a correct 4-pizza, where $\{u\} = V(G_1) \cap N_G(V(G_2))$ and $v = V(G_2) \cap N_G(V(G_1))$.

Fact 6 Suppose that G does not have an MC_5 or a separating edge. Then, G has a separating quadruple iff for some induced 4-cycle C in G, at most one pair of adjacent nations of C weakly touch in $\mathcal{M}$.

A *separating triangle* of G is a list $\langle a, b, c\rangle$ of three vertices in G such that (i) $C = \{a, b, c\}$ is a clique in G and (ii) $G' = G - C - (\mathcal{E}[a, b] \cup \mathcal{E}[a, c])$ is disconnected. If in addition, G' has a connected component consisting of a single vertex, then $\langle a, b, c\rangle$ is a *strongly separating triangle* of G.

Throughout the remainder of this section, we assume that G does not have a separating edge, triple or quadruple, but has a separating triangle $\langle a, b, c\rangle$. Let C and G' be as described in the last paragraph. Our goal is to show that using C and G', we can easily find two correct 4-pizzas.

Claim 1 Let Z be a subset of $V - C$. Suppose that $\{u, v\}$ is an edge of G such that u and v belong to different connected components of $G'[Z]$. Then, $a \in N_G(u) \cap N_G(v)$. Consequently, nations u, v, b, and c cannot meet at a 4-point in $\mathcal{M}$.

Claim 2 Let Z be a subset of $V - C$. Suppose that a subset $\{u, v, w\}$ of $V - C$ is a 3-clique of G such that u and v belong to different connected components of $G'[Z]$. Then, either (i) $C \subseteq N_G(u)$ and $\{C \cap N_G(v), C \cap N_G(w)\} = \{\{a,b\},\{a,c\}\}$ or (ii) $C \subseteq N_G(v)$ and $\{C \cap N_G(u), C \cap N_G(w)\} = \{\{a,b\},\{a,c\}\}$. Moreover, there is no $x \in Z - \{u,v,w\}$ with $\{u,v,w\} \subseteq N_G(x)$.

Note that $\mathcal{M}|_C$ can have at most two lakes. If $\mathcal{M}|_C$ has only one lake, then Figure 3.5(1), (2), or (3) displays it; otherwise, Figure 3.5(4) displays it.

Lemma 3. 6 Figure 3.5(1) does not display $\mathcal{M}|_C$.

Lemma 3. 7 Figure 3.5(2) does not display $\mathcal{M}|_C$.

Lemma 3. 8 Figure 3.5(3) does not display $\mathcal{M}|_C$.

By Lemma 3. 6 , 3. 7 and 3. 8 , only Figure 3.5(4) can display $\mathcal{M}|_C$.

Lemma 3. 9 Suppose that C is a strongly separating triangle of G. Then, we can easily find two correct 4-pizzas.

Lemma 3. 10 Suppose that there is no strongly separating triangle of G. Further assume that C is a separating triangle of G. Then, we can easily find two correct 4-pizzas.

Fact 7 Suppose that G does not have an MC_5, a separating edge, or a separating quadruple. Then, G has a separating triangle iff for some 3-clique C of G, (i) the nations of C do not meet at a point in $\mathcal{M}$ and (ii) at least one pair of nations of C strongly touch in $\mathcal{M}$.

4 Removing cliques of size 5

Throughout this section, we assume that G does not have a separating edge, quadruple, or triangle. This implies that G is 4-connected and has no separating triple. We further assume that G has an MC_5; our goal of this section is to show how to remove MC_5's from G. The idea behind the removal of an MC_5 C from G is to try to find and remove a correct center P of C. By Fact 5 , we make progress after removing P. After the removal of P, the resulting G may be not 4-connected and may have a separating 4-cycle, edge, triple, quadruple, or triangle. To maintain the assumption that G does not have a separating edge, quadruple, or triangle, we just apply the reductions in the last section to the resulting G. Also, not unexpectedly, our search of a correct center of C may fail. In this case, we will be able to decompose G into smaller graphs to make progress.

From Figure 2.2, it is easy to see that every MC_5 C of G is 4-sharing with at most two other MC_5's of G. We claim that at least one MC_5 of G is 4-sharing with two other MC_5's of G. Towards a contradiction, assume that the claim does not hold. Let $C = \{a,b,c,d,e\}$ be an MC_5 of G. Figure 2.2(1) does not display $\mathcal{M}|_C$; otherwise, either V would equal C or at least one of $\langle e^1, a^1, b^1\rangle$, $\langle e^1, b^1, c^1\rangle$, $\langle e^1, c^1, d^1\rangle$, and $\langle e^1, a^1, d^1\rangle$ would be a separating triangle of G, a contradiction. For similar reasons, when C is 4-sharing with no MC_5 of G, none of Figure 2.2(2) through (5) displays $\mathcal{M}|_C$. So, consider the case where C is 4-sharing with exactly one MC_5, say $C_1 = \{a^1, b^1, c^1, e^1, f\}$, of G. In this case, by the assumption that G has no separating triangle, Figure 2.2(2), (3), and (5) are transformable to Figure 4.1(1) and Figure 2.2(4) is transformable to Figure 4.1(2). By Figure 4.1(1) and (2), only Figure 4.2(1) or (2) can possibly display $\mathcal{M}|_{\{a,\dots,f\}}$. Actually, Figure 4.2(2) does not display $\mathcal{M}|_{\{a,\dots,f\}}$; otherwise, since C_1 is 4-sharing with no MC_5 of G other than C, there is no $g \in V - \{a,\dots,f\}$ with $\{a^1, b^1, e^1, f\} \subseteq N_G(g)$ and $\langle a^1, f, e^1\rangle$ would be a separating triangle of G, a contradiction. Similarly, Figure 4.2(1) does not display $\mathcal{M}|_{\{a,\dots,f\}}$; otherwise, since $|V| \geq 9$, $\langle a^1, f, b^1\rangle$ or $\langle a^1, f, e^1\rangle$ would be a separating triple of G, a contradiction. Therefore, the claim holds.

By the above claim, if G has an MC_5, then it has an MC_5 that is 4-sharing with two other MC_5's of G. By our assumption that G has an MC_5, G has an MC_5 $C = \{a,b,c,d,e\}$ that is 4-sharing with two other MC_5's, say $C_1 = \{a,c,d,e,f\}$ and $C_2 = \{a,b,c,e,g\}$, of G. Let $U = C \cup \{f,g\}$. We show how to find a correct center of C below. First, we make a simple but useful observation.

Fact 8 Let W be a subset of an MC_5 C' of G with $|W| \geq 3$. If all the edges in $E(G[W])$ are colored in G or $G - C'$ has a vertex x with $W = C' \cap N_G(x)$, then no nation in $C' - W$ is a correct crust of C'.

Vertices f and g are not adjacent in G; otherwise, only Figure 2.2(4) or (5) can display $\mathcal{M}|_C$, but after drawing nations f and g in the two figures, we see that the 4-connectedness of G would force V to equal U, a contradiction against the assumption that $|V| \geq 9$. So, only Figure 4.1(1) or Figure 4.1(2) can display $\mathcal{M}|_U$. By the figures, a correct center of C can be found from a correct crust immediately. So, it suffices to find out which one of a, c, and e is a correct crust of C. This is done by a case-analysis.

5 Removing cliques of size 4

Throughout this section, we assume that G does not have an MC_5, a separating edge, quadruple, or triangle. We further assume that G has an MC_4; our goal of this section is to show how to remove MC_4's from G. The idea behind the removal of an MC_4 C from G is to try to find and remove a correct 4-pizza via constructing an extensible layout of C. After the removal of a correct 4-pizza, the resulting G may be not 4-connected and may have a separating 4-cycle, edge, triple, quadruple, or triangle. To maintain the assumption that G does not have a separating edge, quadruple or triangle, we just apply the reductions in the last section to the resulting G.

Since $|V| > 8$ and G does not have a separating triangle, for every MC_4 $C = \{a, b, c, d\}$ of G, only Figure 5.1(1), (2) or (3) can possibly display $\mathcal{M}_C$, according to Fact 7.

5.1 Finding out rice-balls

Let $C = \{a, b, c, d\}$ be an MC_4 of G. For a subset W of C, let $\mathcal{E}[W]$ be the set of uncolored edges $\{u, v\} \in E$ such that $u \notin W$, $v \notin W$, and some MC_4 of G consists of u, v, and two vertices in W. Let $G' = G - C - \mathcal{E}[C]$. A *3-subset* of C is a subset S of C with $|S| = 3$. For each 3-subset S of C, let $V_S = \cup_K V(K)$, where K ranges over all connected components K of G' with $C \cap N_G(V(K)) = S$.

Lemma 5. 1 Figure 5.1(3) displays $\mathcal{M}|_C$ iff the following statements hold:

1. $V_{\{a,b,c\}}$, $V_{\{a,b,d\}}$, $V_{\{a,c,d\}}$, and $V_{\{b,c,d\}}$ each are nonempty and they together form a partition of $V - C$.
2. For every pair of two distinct 3-subsets S and T of C, $|V_S \cap N_G(V_T)| = 1$, $|V_T \cap N_G(V_S)| = 1$, and $(S \cap T) \cup (V_S \cap N_G(V_T)) \cup (V_T \cap N_G(V_S))$ is an MC_4 of G.

Since it is easy to check whether Statements 1 and 2 hold, we can easily decide whether C has an extensible "rice-ball" layout. Once we know that C has an extensible "rice-ball" layout, then by Figure 5.1(3) and Statement 2, we can easily find and then remove six correct 4-pizzas from G.

5.2 Distinguishing the remaining two

By the discussion in §5.1, we may assume that no MC_4 $C = \{a, b, c, d\}$ of G satisfies Statements 1 and 2 in Lemma 5. 1. Then, we have:

Corollary 5. 2 For every MC_4 C of G, the nations of C are related in map $\mathcal{M}$ in the same way as either Figure 5.1(1) or (2) shows.

Let $C = \{a, b, c, d\}$ be an MC_4 of G. Our goal is to find out which of Figure 5.1(1) and (2) displays $\mathcal{M}|_C$. This is achieved by a case-analysis.

5.3 Removing pizzas

By the discussions in the last two subsections, we may assume that for every MC_4 $C = \{a, b, c, d\}$ of G, only Figure 5.1(1) displays $\mathcal{M}|_C$. That is, the four nations of every MC_4 of G meet at a point in $\mathcal{M}$.

Fix an MC_4 $C = \{a, b, c, d\}$ of G. C is 3-sharing with no MC_4 C' of G because otherwise, C' would have a nonpizza layout. By Figure 5.1(1), there are distinct nations e, f, g and h in $V - C$ such that $C \cap N_G(e) = \{a^1, b^1\}$, $C \cap N_G(f) = \{b^1, c^1\}$, $C \cap N_G(g) = \{c^1, d^1\}$ and $C \cap N_G(h) = \{d^1, a^1\}$, because $\mathcal{M}$ has no lake. On the other hand, the existence of the nations e, f, g and h ensures that the nations of C have to meet at a point in $\mathcal{M}$ cyclically in the order a^1, b^1, c^1, d^1. Thus, by finding out nations e, f, g and h, we can find and remove a correct 4-pizza from G.

6 The case with no 4-clique

By the discussions in the previous sections, we may assume that G has a well-formed map but has no 4-cliques. Then, G is a 3-connected planar graph and hence has a unique plane embedding. The dual of the unique embedding is a well-formed map of G.

7 Time analysis

Let n be the number of vertices in the input graph G. We first claim that testing the existence of a separating triangle takes $O(n^2)$ time. We then claim that testing the existence of a separating quadruple takes $O(n^3)$ time.

Finally, we claim that the running time of the algorithm is $O(n^4)$. The proof is by induction. The claim is clearly true when $n \le 8$. Suppose $n > 8$. If some reduction in §2 or §3 applies, then we can perform such a reduction in $O(n^3)$ time as claimed above, and so the running time on G is $O(n^4)$ by the inductive hypothesis. If no reduction in §2 or §3 applies, we can either (i) find a correct 4-pizza in $O(n)$ time and remove it from G, or (ii) reduce the problem for G to that for a smaller graph in $O(1)$ time; so, the running time on G is $O(n^4)$ by the inductive hypothesis.

References

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[2] M. Thorup. Map Graphs in Polynomial Time. FOCS'98.

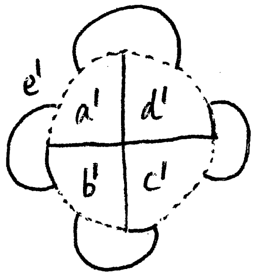
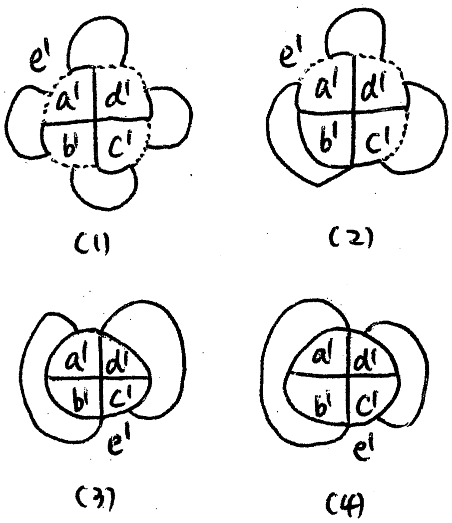


Figure 2.1



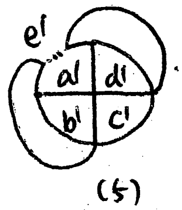


Figure 2.2

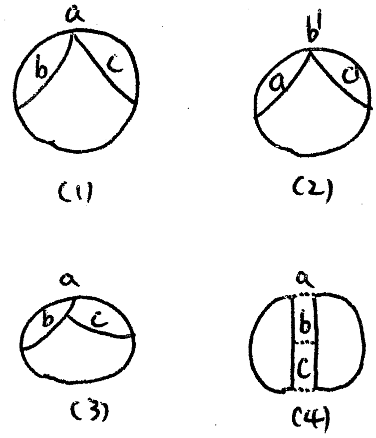


Figure 3.1

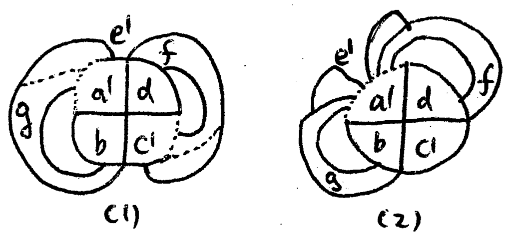


Figure 4.1

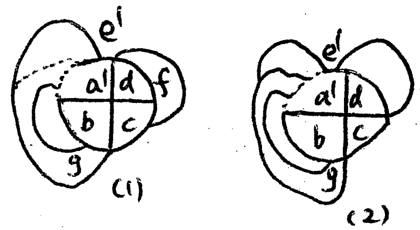


Figure 4.2

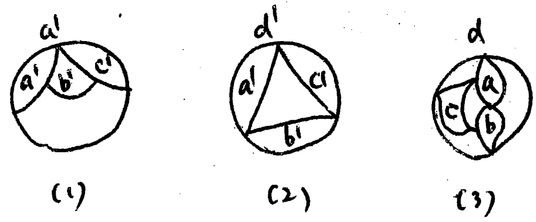


Figure 5.1