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POINCARÉ SERIES CONSTRUCTED FROM A WHITTAKER FUNCTION ON $Sp(2;\mathbb{R})$

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1. Whittaker function

1.1. Structure of Lie group and Lie algebra. Let G be the symplectic group $Sp(2;\mathbb{R})$ realized as

$$G=\{g\in SL_4(\mathbb{R})\mid {}^tgJg=J\},\quad \text{with } J=\begin{pmatrix}0 & 1_2\\ -1_2 & 0\end{pmatrix}\in M_4(\mathbb{R}),$$

where ${}^tg$ denotes the transpose of a matrix g and 1_2 denotes a unit matrix of size 2.

Let $O(4)$ be the orthogonal group of degree 2. Take a maximal compact subgroup $K=G\cap O(4)$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebra of G, K, respectively. Let $\theta(X)=-{}^tX$ be a Cartan involution and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$.

We set $\mathfrak{a}=\mathbb{R}H_1+\mathbb{R}H_2$ with $H_1=diag(1,0,-1,0),H_2=diag(0,1,0,-1)$. Then $\mathfrak{a}$ is a maximally Cartan subalgebra of $\mathfrak{g}$ and the restricted root system $\Delta=\Delta(\mathfrak{g};\mathfrak{a})$ is expressed as $\Delta=\Delta(\mathfrak{g};\mathfrak{a})=\{\pm\lambda_1\pm\lambda_2,\pm2\lambda_1,\pm2\lambda_2\}$, where λ_j is the dual of H_j. We choose a positive root system Δ^+ as $\Delta^+=\{\lambda_1\pm\lambda_2,2\lambda_1,2\lambda_2\}$.

We also denote the corresponding nilpotent subalgebra by $\mathfrak{n}=\sum_{\beta\in\Delta^+}\mathfrak{g}_\beta$. Here $\mathfrak{g}_\beta$ is the root subspace of $\mathfrak{g}$ corresponding to $\beta\in\Delta^+$. Then one obtains an Iwasawa decomposition of $\mathfrak{g}$ and G ; $\mathfrak{g}=\mathfrak{n}+\mathfrak{a}+\mathfrak{k}$, $G=NAK$ with $A=\exp\mathfrak{a}$, $N=\exp\mathfrak{n}$.

1.2. Representation of the maximal compact subgroup. Firstly, we review the parametrization of the finite-dimensional irreducible representations of $SL_2(\mathbb{C})$. Let $\{f_1,f_2\}$ be the standard basis of the vector space $V=V_1=\mathbb{C}\oplus\mathbb{C}$. Then $GL_2(\mathbb{C})$ acts on V by matrix multiplication. We denote the symmetric tensor space of 2 dimension by $V_d=S^d(V)$. Here $V_0=\mathbb{C}$. We consider V_d as a $SL_2(\mathbb{C})$-module by

$$sym^d(g)(v_1\otimes v_2\otimes\cdots\otimes v_d)=gv_1\otimes gv_2\otimes\cdots\otimes gv_d.$$

It is well known that all the finite-dimensional irreducible (polynomial) representations of $SL_2(\mathbb{C})$ can be obtained in this way. By Weyl's unitary trick, all irreducible unitary representations of $SU(2)$ are obtained by restriction of sym^d $(d\geq 0)$.

The maximal compact subgroup K is isomorphic to the unitary group $U(2)$ of degree 2 by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A+\sqrt{-1}B, \quad \text{for } \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K.$$

For $d, m \in \mathbb{Z}, d \geq 0$, we define a holomorphic representation $(\sigma_{d,m}, V_d)$ of $GL_2(\mathbb{C})$ by $\sigma_{d,m}(g) = sym^d(g) \otimes \det(g)^m$. Then we know $\hat{U(2)} = \{\sigma_{d,m}|_{U(2)} \mid d, m \in \mathbb{Z}, d \geq 0\}$. We set $\lambda = (\lambda_1, \lambda_2) = (m+d, m)$ and $\tau_\lambda = \sigma_{d,m}|_{U(2)}$. By the isomorphism between K and $U(2)$, we obtain

$\hat{K} = \{(\tau_\lambda, V_\lambda) \mid \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}, \lambda_1 \geq \lambda_2\}$. We choose the basis of V_λ as

$$V_\lambda = \left\{ v_k = \frac{n!}{k!(n-k)!} f_1^{\otimes k} \otimes f_2^{\otimes (n-k)} \text{ (symmetric tensor)} \mid 0 \leq k \leq n \right\}_{\mathbb{C}}.$$

1.3. Characters of the unipotent radical. The commutator subgroup $[N, N]$ of N is given by

$$[N, N] = \left\{ \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & 0 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \;\middle|\; n_1, n_2 \in \mathbb{R} \right\}.$$

Hence a unitary character η of N is written for some constant $\eta_0, \eta_3 \in \mathbb{R}$ as

$$\left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & n_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \mapsto \exp\{\sqrt{-1}(\eta_0 n_0 + \eta_3 n_3)\} \in \mathbb{C}^\times.$$

A unitary character η of N is said to be non-degenerate if $\eta_0\eta_3 \neq 0$.

1.4. Parametrization of the discrete series. Let us now parametrize the discrete series of $Sp(2;\mathbb{R})$. Take a compact Cartan subalgebra $\mathfrak{h}$ defined by $\mathfrak{h} = \mathbb{R}h_1 \oplus \mathbb{R}h_2$ with $h_1 = X_{13} - X_{31}$, $h_2 = X_{24} - X_{42}$, where the $X_{ij}'s$ are elementary matrices given by $X_{ij} = (\delta_{ip}\delta_{jq})_{1 \leq p,q \leq 4}$, with Kronecker's delta $\delta_{i,p}$, and let $\mathfrak{h}_\mathbb{C}$ be its complexification. Then the absolute root system is expressed as

$$\tilde{\Delta} = \Delta(\mathfrak{g}; \mathfrak{h}) = \{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\},$$

where by $\beta = (r, s)$, we mean $r = \beta(-\sqrt{-1}h_1), s = \beta(-\sqrt{-1}h_2)$. Let

$$\tilde{\Delta}^+ = \{(2,0), (0,2), (1,1)(1,-1)\}.$$

We write the set of compact positive roots by $\tilde{\Delta}_c^+ = \{(1,-1)\}$. Then there are 4 sets of positive roots $\tilde{\Delta}_J^+$ $(J = I, II, III, IV)$ of $(\mathfrak{g}, \mathfrak{h})$ containing $\Delta_c^+(\mathfrak{g}; \mathfrak{h})$ as follows:

$$\tilde{\Delta}_I^+ = \{(2,0), (1,1), (0,2), (1,-1)\}, \; \tilde{\Delta}_{II}^+ = \{(1,1), (2,0), (1,-1), (0,-2)\},$$

$$\tilde{\Delta}_{III}^+ = \{(2,0), (1,-1), (0,-2), (-1,-1)\}, \; \tilde{\Delta}_{IV}^+ = \{(1,-1), (0,-2), (-1,-1), (-2,0)\}.$$

We put $\delta_{G,J} = 2^{-1}\sum_{\beta\in\tilde{\Delta}_J^+}\beta$ (resp. $\delta_K = 2^{-1}\sum_{\beta\in\tilde{\Delta}_c^+}\beta$), the half sum of positive roots (resp. the half sum of compact positive roots). By definition, the space of Harish-Chandra parameters Ξ_c is given by

$$\Xi_c = \{\Lambda \in \mathfrak{h}_{\mathbb{C}}^* \mid \Lambda + \delta_{G,I} \text{ is analytically integral and } \Lambda \text{ is regular and } \tilde{\Delta}^+\text{-dominant}\}.$$

For each $J = I, II, III, IV$, we set $\Xi_J = \{\Lambda \in \Xi_c \mid \langle\Lambda, \alpha\rangle > 0 \ (\alpha \in \tilde{\Delta}_J^+)\}$. Then Ξ_c is written as a disjoint union $\Xi_c = \coprod_{J=I}^{IV} \Xi_J$.

It is well-known that there exists a bijection from Ξ_c to the set of equivalence classes of discrete series representations of G. Let π_Λ be the discrete series representation associated to Λ in Ξ_J , then τ_λ $(\lambda = \Lambda + \delta_{G,J} - 2\delta_K)$ is the unique minimal K-type of π_Λ. We note that for each Λ in Ξ_c, $\lambda = \Lambda + \delta_{G,J} - 2\delta_K$ is called the Blattner parameter. An easy computation implies

$$\Xi_c = \{(\Lambda_1, \Lambda_2) \in \mathbb{Z}\oplus\mathbb{Z} \mid \Lambda_1 \neq 0, \Lambda_2 \neq 0, \Lambda_2 < \Lambda_1, \Lambda_1 + \Lambda_2 \neq 0\}.$$

We note that Ξ_I (*resp.*Ξ_{IV}) corresponds to the holomorphic (resp. anti-holomorphic) discrete series, and Ξ_{II} and Ξ_{III} coresponds to the large discrete series in the sence of Vogan,[V].

1.5. Characterization of the minimal K-type of a discrete series representation. Let η be a unitary character of N. Then we set

$$C_\eta^\infty(N\setminus G) = \{\phi : G \to \mathbb{C},\ C^\infty\text{-}class \mid \phi(ng) = \eta(n)\phi(g),\ (n, g) \in N \times G\}.$$

By the right regular action of G, $C_\eta^\infty(N\setminus G)$ has a structure of smooth G-module. For any finite dimensional K-module (τ, V), we set

$$C_{\eta,\tau}^\infty(N\setminus G/K) = \{F : G \to V,\ C^\infty\text{-}class \mid F(ngk^{-1}) = \eta(n)\tau(k)F(g),\ (n, g, k) \in N \times G \times K\}.$$

Let (π_Λ, H) be the discrete series representation of G with Harish-Chandra parameter Λ in Ξ_J, $(J = I, II, III, IV)$, and denote its associated $(\mathfrak{g}_{\mathbb{C}}, K)$-module by the same symbol. For W in $Hom_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda^*, C_\eta^\infty(N\setminus G))$, we define F_W in $C_{\eta,\tau_\lambda}^\infty(N\setminus G/K)$ by

$$W(v^*)(g) = \langle v^*, F_W(g)\rangle, \quad (v^* \in V_\lambda^*, g \in G).$$

Here $(\tau_\lambda, V_\lambda)$ denotes the minimal K-type of π_Λ and $\langle *, *\rangle$ denotes the canonical pairing on $V_\lambda^* \times V_\lambda$.

Now let us recall the definition of the Schmid-operater. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and $Ad = Ad_{\mathfrak{p}_{\mathbb{C}}}$ be the adjoint representation of K on $\mathfrak{p}_{\mathbb{C}}$. Then we can define a differential operator $\nabla_{\eta,\lambda}$ from $C_{\eta,\tau_\lambda}^\infty(N\setminus G/K)$ to $C_{\eta,\tau_\lambda\otimes Ad}^\infty(N\setminus G/K)$

as $\nabla_{\eta,\lambda}F = \sum_i R_{X_i}F(\cdot) \otimes X_i$. Here the set $\{X_i\}_i$ is any fixed orthonormal basis of p with respect to the Klilling form on $\mathfrak{g}$ and $R_X F$ denotes the right differential of the function F by X in $\mathfrak{g}$ i.e. $R_X F(g) = \frac{d}{dt}F(g \cdot \exp tX)\Big|_{t=0}$. This operator $\nabla_{\eta,\lambda}$ is called the Schmid operator.

Let $(\tau_\lambda^-, V_\lambda^-)$ be the sum of irreducible K-submodules of $V_\lambda \otimes p_{\mathbb{C}}$ with heighest weight of the form $\lambda - \beta$ ($\beta \in \tilde{\Delta}^+_{J,n}$, $J = I, II, III, IV$). Let P_λ be the projection from $V_\lambda \otimes p_{\mathbb{C}}$ to V_λ^-. We define a differential operator from $C^\infty_{\eta,\tau_\lambda}(N \setminus G/K)$ to $C^\infty_{\eta,\tau_\lambda^-}(N \setminus G/K)$ by $\mathcal{D}_{\eta,\lambda}F(g) = P_\lambda(\nabla_{\eta,\lambda}F(g))$ for $F \in C^\infty_{\eta,\tau_\lambda}(N \setminus G/K)$, $g \in G$. Yamashita obtain the following result.

Proposition 1.1 ([Y1] H.Yamashita, Proposition(2.1)). *Let π_Λ be a representation of discrete series with Harish-Chandra parameter $\Lambda \in \Xi_J$ of $Sp(2;\mathbb{R})$. Set $\lambda = \Lambda + \delta_G - 2\delta_K$. Then the linear map*

$$W \in Hom_{\mathfrak{g}_{\mathbb{C}},K}(\pi^*_\Lambda, C^\infty_\eta(N \setminus G)) \to F_W \in Ker(\mathcal{D}_{\eta,\lambda})$$

is injective, and if Λ is far from the walls of the Wyel chambers, it is bijective.

1.6. A basis on the Whittaker space on $Sp(2;\mathbb{R})$. By the result of Kostant [Ko], and Vogan [V], if η is non-degenerate, we obtain the expression

$$dim_{\mathbb{C}} Hom_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, C^\infty_\eta(N\backslash G)) = \begin{cases} 4, & \text{if } \Lambda \in \Xi_{II} \cup \Xi_{III}, \\ 0, & \text{if } \Lambda \in \Xi_I \cup \Xi_{IV}. \end{cases}$$

Oda obtain the following result.

Theorem 1.1 ([O] Oda).

Let us assume that η is non-degenerate and $\Lambda \in \Xi_{II}$. We choose the basis $V_\lambda = \{v_k \mid 0 \le k \le d\}_{\mathbb{C}}$ defined in §4.2. Here $d = \lambda_1 - \lambda_2$. Then
(1) $F \in Ker\mathcal{D}_{\eta,\lambda}$ if and only if F satisfies the conditions

$$(\partial_1 - k)h_{d-k} + \sqrt{-1}\eta_0 h_{d-k-1} = 0, \qquad \text{for } 0 \le k \le d-1,$$

(1.1)
$$\{\partial_1\partial_2 + (a_1/a_2)^2\eta_0^2\}h_d = 0,$$

(1.2)
$$\{(\partial_1 + \partial_2)^2 + 2(\lambda_2 - 1)(\partial_1 + \partial_2) - 2\lambda_2 + 1 + 4\eta_3 a_2^2 \partial_2\}h_d = 0.$$

Here $\partial_i = \frac{\partial}{\partial a_i}$, $i = 1, 2$ and $\{h_k \mid 0 \le k \le d\}$ is determined by

$$F|_A(a) = \sum_{k=0}^{d} c^{(1)}_{i,k}(a)v_k,$$

$$c_k(a) = a_1^{\lambda_2+1}a_2^{\lambda_1}\left(\frac{a_1}{a_2}\right)^k \exp(\eta_3 a_2^2)h_k(a), \quad \text{for } a \in A,\ 0 \le k \le d.$$

(2) *If* $\eta_3 < 0$, $Ker\mathcal{D}_{\eta,\lambda}$ *contains the function* F *such that* $h_d(a)$ *has the integral representatio for* $a = diag(a_1, a_2, a_1^{-1}, a_2^{-1}) \in A$

$$h_d(a) = \int_0^\infty t^{-\lambda_2+\frac{1}{2}} W_{0,-\lambda_2}(t) \exp\left(\frac{t^2}{32\eta_3 a_2^2} + \frac{8\eta_0^2\eta_3 a_1^2}{t^2}\right) \frac{dt}{t}.$$

By Theorem 1.1, Oda showed that if $\Lambda \in \Xi_{II} \cup \Xi_{III}$ and η is non-degenerate,

$$Hom_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda^*, \mathcal{A}_\eta(N\backslash G)) \cong \begin{cases} \mathbb{C}, & \eta_3 < 0, \\ 0, & \eta_3 > 0. \end{cases}$$

Here we put

$$\mathcal{A}_\eta(N\backslash G) = \{F \in C_\eta^\infty(N\backslash G) \mid K\text{ -finite and for any } X \in U(\mathfrak{g}_{\mathbb{C}}) \text{ there exists a constant } C_X > 0 \text{ such that } |F(g)| \le C_X tr({}^t gg),\ g \in G\}$$

and $U(\mathfrak{g}_{\mathbb{C}})$ denotes the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

We set for $t \in \mathbb{C}$, $|\arg t| < \pi$,

$$k_{i,\nu}(t) = \begin{cases} I_\nu(\sqrt{t}/2), & \text{if } i = 1, 2, \\ K_\nu(\sqrt{t}/2), & \text{if } i = 3, 4. \end{cases}$$

and for $F \in C_\eta^\infty(N\backslash G/K)$, set $h_k, c_k \in C^\infty(A)$ $(0 \le k \le d)$ as in Theorem 1.1. Then we obtain the following results.

Theorem 1.2. *Let us assume that* η *is non-degenerate and* $\Lambda \in \Xi_{II}$.

(1) *Ker* $D_{\eta,\lambda}$ *has the basis* $\{F_i | 1 \le i \le 4\}$ *such that* $h_{i,d}(a)$ $(1 \le i \le 4)$ *have the integral representations for* $a = diag(a_1, a_2, a_1^{-1}; a_2^{-1}) \in A$

$$h_{i,d}(a) = \int_{\tilde{C}_i} t^{\frac{1}{2}(1-\lambda_2)} k_{i,-\lambda_2}(t) \exp\left(\frac{t}{32\eta_3 a_2^2} + \frac{8\eta_0^2\eta_3 a_1^2}{t}\right) \frac{dt}{t}.$$

Here we set the contours $\tilde{C}_i$ $(1 \le i \le 4)$

$$\int_{\tilde{C}_i} dt = \begin{cases} \int_C dt, & \textit{if } i = 1, 3, \\ \int_0^\infty dt, & \textit{if } i = 2, 4, \end{cases}$$

where $\int_C dt$ *is the contour integral on* C *given in Theorem* 2.1–(2) *and* $\int_0^\infty dt$ *is the usual integral on* $\mathbb{R}_{>0}$.

(2) *We set*

$$X = \left\{(k, j) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, 0 \le k \le \left[\frac{d-1}{2}\right],\ j = 1, 2,\ 0 \le 2k + j \le d\right\}$$

For $\eta_3 \in \mathbb{R}$ and for any $r, \epsilon_1, \epsilon_2 > 0$, there exist constants $b_{(3)}^{i,k} > 0$ $(1 \le i \le 4,\ (k,j) \in X)$ such that

$$|c_{i,d-(2k+j)}(a)| \le b_{(3)}^{i,k} a_1^{1+\lambda_1} a_2^{1-l_i\lambda_2} \left(\frac{a_1}{a_2}\right)^{\alpha^{(2)}_{i,j,k}} \times \exp\left\{((-1)^{i+1}|\eta_0| + \epsilon_1)\frac{a_1}{a_2} + (2|\eta_3| + \eta_3 + \epsilon_2)a_2^2\right\}, \quad \text{for } r \ge a_1 > 0,\ a_2 > 0,$$

where we set for $1 \le i \le 4$ and $(k,j) \in X$

$$\alpha^{(2)}_{i,j,k} = \begin{cases} 1-(2k+j), & \text{if } i = 2,4 \text{ and } 1 \le 2k+j \le d, \\ 0, & \text{otherwise.} \end{cases}$$

(3) *If $\eta_3 < 0$, then for any $r, \epsilon_1, \epsilon_2 > 0$, there exist constants $b_{(4)}^{i,j,k}$ such that*

$$|c_{i,d-(2k+j)}(a)| \le b_{(4)}^{i,j,k} a_1^{1+\lambda_1} a_2^{1-l_i\lambda_2} \left(\frac{a_1}{a_2}\right)^{\beta^{(2)}_{i,j,k}(\epsilon_1)} \times \exp\left\{((-1)^{i+1}|\eta_0| + \epsilon_2)\frac{a_1}{a_2} - l_i\eta_3 a_2^2\right\}, \quad \text{for } r \ge a_1 > 0,\ a_2 > 0,$$

where we set for each $1 \le i \le 4$ and $0 \le k \le d$ and any fixed $\epsilon > 0$

$$\beta^{(2)}_{i,j,k}(\epsilon_1) = \begin{cases} 1-(2k+j)-\epsilon_1, & \text{if } i = 2,4 \text{ and } 1 \le 2k+j \le d, \\ 1-2\lambda_2-(2k+j)-\epsilon_1, & \text{if } i = 1,3 \text{ and } -2\lambda_2 \le 2k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1. Firstly F_i is defined as a linear combination of the series solutoin. Then from Theorem1.2 we obtain the folloing result.

Corollary 1.1. *If $i = 1,3$, for any $r, \epsilon_1, \epsilon_2 > 0$, there exist constants b_i such that*

$$|c_{i,k}(a)| \le b_i a_1^{1+\lambda_1} a_2^{1-\lambda_2} \exp\left\{(|\eta_0| + \epsilon_1)\frac{a_1}{a_2} + (2|\eta_3| + \eta_3 + \epsilon_2)a_2^2\right\} \quad \text{for } r \ge a_1 > 0, a_2 > 0.$$

2. Poincaré series

We assume $\eta_3 < 0$.

2.1. The Convergence of the Poincaré series. We denote by α_1, α_2 the functions

$$\alpha_i(g) = a_i, \quad (i = 1, 2),$$
$$\text{for } g = n \cdot diag(a_1, a_2, a_1^{-1}, a_2^{-1}) \cdot k, \quad n \in N, k \in K, a_1, a_2 > 0$$

and denote by Γ the group $Sp(2; \mathbb{Z})$.

Then we know the following result.

Lemma 2.1 ([?] **B.Diehl**). *If* $\Re(s_2) > 2$ *and* $\Re(s_1) > \Re(s_2) + 2$, *then the sum.*

$$\sum_{N \cap \Gamma \backslash \Gamma} \alpha_1(\gamma g)^{s_1} \alpha_2(\gamma g)^{s_2}$$

is absolutely convergent.

We set $\eta_i' = \dfrac{\eta_i}{2\pi}$ $i = 0, 3$. Then we define the following functions.

Definition 2.1. For $s_1, s_2 \in \mathbb{C}$, we define the function $f_i^{s_1, s_2}$ $(1 \leq i \leq 4)$ by

$$f_i^{s_1,s_2}(g) = \exp\left\{-\left(s_1|\eta_0|\frac{a_1}{a_2} + s_2|\eta_3|a_2^2\right)\right\} F_i(g), \quad \text{for} g \in G$$

For $\eta_0', \eta_3' \in \mathbb{Z}$, we define the Poincar'e series $P_{s_1,s_2}(g)$ by

$$P_{s_1,s_2}(g) = \sum_{P \cap \Gamma \backslash \Gamma} f_1^{s_1,s_2}(\gamma g)$$

Then we obtain the following result from Lemma2.1.

Theorem 2.1. *For* $\Lambda_2 < -1, \Lambda_1 + \Lambda_2 > 1$ *and* $\Re(s_i) > 1$ $(i = 1, 2)$, *the Poincar'e series* $P_{s_1,s_2}(g)$ *is absolutely convergent.*

2.2. The Fourier coefficients. We investigate the Fourier coefficients of $P_{s_1,s_2}(g)$ with respect to N. Here we consider only in the case of unitary character of N.

Let W be the Wyel group of G,

$$W = \{w_i \mid 0 \leq i \leq 7\},$$

where we put

$$w_1 = \left(\begin{array}{cc|cc} & & 1 & \\ & 1 & & \\ \hline -1 & & & \\ & & & 1 \end{array}\right), w_2 = \left(\begin{array}{cc|cc} 1 & & & \\ & & & 1 \\ \hline & & 1 & \\ & -1 & & \end{array}\right), w_3 = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right),$$

$w_4 = w_1 w_3, w_5 = w_2 w_3, w_6 = w_1 w_2 w_3, w_7 = w_1 w_2.$

We set $M = Z_K(\mathfrak{a})$, $\Gamma(w) = \Gamma \cap MNAwN$ for $w \in W$. Let for $t_0, t_3 \in \mathbb{R}$,

$$\eta_{t_0,t_3} \; : \; n \in N \to \exp\{2\pi\sqrt{-1}(t_0 n_0 + t_3 n_3)\} \in \mathbb{C}^\times$$

be a unitary character of N. For $(g, \gamma) \in G \times \Gamma$ and $l_0, l_3 \in \mathbb{Z}$, we set

$$\begin{aligned}
\Phi^\gamma(g; l_0, l_3) &= \int_{N \cap P^\gamma \cap \Gamma \backslash N} F_{s_1,s_2}(\gamma n g)\eta_{l_0,l_3}(n^{-1})dn, \\
\Phi_w(g; l_0, l_3) &= \sum_{(P\cap\Gamma)\backslash\Gamma(w)/(N\cap\Gamma)} \Phi^\gamma(g; l_0, l_3), \\
b_{l_0,l_3}(g) &= \int_{N\cap\Gamma\backslash N} P_{s_1,s_2}(ng)\eta_{l_0,l_3}(n^{-1})dn.
\end{aligned}$$

where we put $P^\gamma = \gamma^{-1}P\gamma$.

Then we obtain the result.

Theorem 2.2. *We assume that* $\eta_3 < 0$ *and* $\eta_i' = \dfrac{\eta_i}{2\pi} \in \mathbb{Z}$ $(i = 0, 3)$. *Then we obtain the following results.*

(1) *For* $l_0, l_3 \in \mathbb{Z}$, *we have*

$$b_{l_0,l_3}(g) = \sum_{0\le i\le 7} \Phi_{w_i}(g; l_0, l_3).$$

(2) *If* $i = 2, 3, 4, 5$, *we have*

$$\Phi_{w_i}(g; l_0, l_3) = 0 \qquad \text{for } g \in G, l_0, l_3 \in \mathbb{Z}.$$

If $i = 0$, *we have*

$$\Phi_{w_0}(g; l_0, l_3) = \begin{cases} 0, & \text{if } (l_0, l_3) \neq (\eta_0', \eta_3'), \\ f_1^{s_1,s_2}(g), & \text{if } (l_0, l_3) = (\eta_0', \eta_3'). \end{cases}$$

If $i = 1$, *there exist constants* $C_1(l_0, a(\gamma))$ *such that*

$$\Phi_{w_1}(g; l_0, l_3) = \begin{cases} 0, & \text{if } \eta_3'\beta^2 - l_3 \in \mathbb{Z}\backslash\{0\}, \\ C_1(l_0, a(\gamma)) \exp(\eta_0' n_0 + \eta_3' n_3 + \eta_3'\beta^2 u_3 + l_0 u_0) f_2^{s_1,s_2}(g; l_0, \eta_3'\beta(\gamma)^2) & \\ \times \begin{cases} 1, & \text{if } \eta_3'\beta^2 - l_3 = 0, \\ \dfrac{\exp(\eta_3'\beta^2 - l_3) - 1}{\eta_3'\beta^2 - m_3}, & \text{if } \eta_3'\beta^2 - l_3 \in \mathbb{Q}\backslash\mathbb{Z}. \end{cases} & \end{cases}$$

If $i=6$, *there exist constants* $C_2(l_3, a(\gamma))$ *such that*

$$\Phi_{w_6}(g;l_0,l_3)=\begin{cases} 0, & \text{if } \eta_0'\frac{\alpha}{\beta}-l_0\in\mathbb{Z}\backslash\{0\}, \\ C_2(l_3,a(\gamma))\exp\left(\eta_0'n_0+\eta_3'n_3+\eta_0'\frac{\alpha}{\beta}u_0+l_3u_3\right)f_3^{s_1,s_2}\left(g;\eta_0'\frac{\alpha}{\beta},l_3\right) & \\ \quad\times\begin{cases} 1, & \text{if } \eta_0'\frac{\alpha}{\beta}-l_0=0, \\ \dfrac{\exp\left(\eta_0'\frac{\alpha}{\beta}-l_0\right)-1}{\eta_0'\frac{\alpha}{\beta}-l_0}, & \text{if } \eta_0'\frac{\alpha}{\beta}-l_0\in\mathbb{Q}\backslash\mathbb{Z}. \end{cases} \end{cases}$$

If $i=7$, *there exist constants* $C_3(l_0,l_3,a(\gamma))$ *such that*

$$\Phi_{w_7}(g;l_0,l_3)=C_3(l_0,l_3,a(\gamma))\exp(\eta_0'n_0+\eta_3'n_3+l_0u_0+l_3u_3)f_4^{s_1,s_2}(g;l_0,l_3).$$

Where we know $\gamma\in\Gamma$ *has a unique decomposition* $\gamma=mnau$ $(m\in M, a\in A, n,u\in N)$ *and put*

$$a(\gamma)=a=diag(\alpha,\beta,\alpha^{-1},\beta^{-1}),\quad u=\left(\begin{array}{cc|cc} 1 & u_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -u_0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & u_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right),$$

$$n=\left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & n_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right).$$

We denote by $f_i^{s_1,s_2}(\cdot;t_0,t_3)$ *the function* $f_i^{s_1,s_2}(\cdot)$ *for the unitary character* η_{t_0,t_3} $(t_0,t_3\in\mathbb{Q})$ *of* N.

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