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Fourier expansion of holomorphic Siegel modular forms of genus 3 along the minimal parabolic subgroup

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1. Introduction.

We are constructing a certain type of Fourier expansion of holomorphic Siegel modular forms of genus 3, different from the two expansions already known, i.e. classical Fourier expansion and Fourier Jacobi expansion. More precisely, our expansion is along the minimal parabolic subgroup of a symplectic group, while the other two are along the Siegel parabolic subgroup or Jacobi parabolic subgroup. We already obtained the Fourier expansion for the case of genus 2, which is the master thesis of the author (cf. [N]). In these days, we have constructed the expansion for the case of genus 3. From this work, we hope to obtain some hints to get the expansion for the case of arbitrary genus. In the case of genus 2, we got some relations among our Fourier expansion and the other two ones in terms of their Fourier coefficient and obtained certain informations on the other two expansions. For the case of genus 3, we are also going to do the same work after the construction of the Fourier expansion. And we expect that such work will give a new result on the two known expansions for the case of genus 3, as the work for the case of genus 2 did. We think that our work as above is meaningful since holomorphic Siegel modular forms of higher genus are not studied so much, even of genus 3.

In the construction of our Fourier expansion, it is crucial to compute the following two associated to some irreducible unitary representations of the maximal unipotent subgroup N:
1) generalized Whittaker function for holomorphic discrete series,
2) theta series on N constructed from the Hermite function.
The function as in 1) is defined to be the image of an embedding of a holomorphic discrete series into the space of the representation induced from the irreducible unitary representation of N. By computing it, we see what kind of a function occurs in our Fourier expansion. The theta series mentioned as above plays a primary role to obtain the realization of the Whittaker functions in the Fourier expansion. If these two are computed, we get our Fourier expansion.

The first object is computed by solving the differential equations arising from the "Cauchy Riemann condition". The second object is computed by calculating the Hermite differential equations rewritten by the coordinate of N and the differential equations coming from the actions of the infinitesimal character of the irreducible unitary representaton.

2. Notations for Lie groups and Lie algebras.

Let $G = Sp(3;\mathbb{R})$ be the real symplectic group of degree 3, given by

$$\{g \in SL_6(\mathbb{R}) \mid {}^t g\, J g = J\},$$

where $J = \begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix} \in M_6(\mathbb{R})$ and K a maximal compact subgroup of G, which is isomorphic to $U(3)$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of G and K respectively. The Cartan involution θ (i.e. $\theta(X) = -\,{}^t X$) induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here $\mathfrak{p}$ is the eigenspace of $\mathfrak{g}$ with the eigenvalue -1 and $\mathfrak{k}$ coincides with that with the eigenvalue 1.

Here we introduce the two root system of $\mathfrak{g}$, i.e. the restricted root system and the root system with respect to a compact Cartan subalgebra. For the former one, we give a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, specified by

$$\left\{ \begin{pmatrix} A & \\ & -A \end{pmatrix} \;\middle|\; A = \mathrm{diag}(t_1, t_2, t_3),\ t_i \in \mathbb{R} \right\}.$$

Let E_{ij} denote the ij-th matrix unit with $1 \le i, j \le 6$ and $\{e_i\}_{1 \le i \le 3}$ the standard basis of 3-dimensional Euclidean space, and let $A_i = E_{ii} - E_{i+3,i+3}$ with $1 \le i \le 3$. The set $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm e_i \pm e_j,\ \pm 2e_k \mid 1 \le i < j \le 3,\ 1 \le k \le 3\}$ gives the restricted root system. Let E_α denote the root vector corresponding to a root α. The root vecters for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ are as follows:

$$\begin{aligned} E_{e_i+e_j} &= E_{i,j+3} + E_{j,i+3},\ E_{-e_i-e_j} = E_{i+3,j} + E_{j+3,i}, \\ E_{2e_i} &= E_{i,i+3},\ E_{-2e_i} = E_{i+3,i}, \\ E_{e_i-e_j} &= E_{ij} - E_{j+3,i+3},\ E_{-e_i+e_j} = E_{ji} - E_{i+3,j+3}. \end{aligned}$$

Here the notation $E_{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3}$ means that $[A_i, E_{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3}] = \alpha_i E_{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3}$ with $1 \le i \le 3$. Using these vectors, we have a following root space decomposition of $(\mathfrak{g}, \mathfrak{a})$:

$$\begin{aligned} \mathfrak{g} = \mathfrak{g}_0 &\oplus \{\bigoplus_{1 \le i < j \le 3} (\mathbb{R}E_{e_i+e_j} + \mathbb{R}E_{-e_i-e_j})\} \oplus \{\bigoplus_{1 \le i \le 3} (\mathbb{R}E_{2e_i} + \mathbb{R}E_{-2e_i})\} \\ &\oplus \{\bigoplus_{1 \le i < j \le 3} (\mathbb{R}E_{e_i-e_j} + \mathbb{R}E_{-e_i+e_j})\}, \end{aligned}$$

where $\mathfrak{g}_0$ denotes the space of vectors with their eigenvalues 0.

The set $\Delta(\mathfrak{g}, \mathfrak{a})^+ = \{e_i \pm e_j,\ 2e_k \mid 1 \le i < j \le 3,\ 1 \le k \le 3\}$ forms a set of positive roots of $\Delta(\mathfrak{g}, \mathfrak{a})$. Then we have a Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})^+} \mathbb{R}E_\alpha$.

Next, we consider the root system of the other type and set $\mathfrak{h} = \bigoplus_{1 \le i \le 3} \mathbb{R}(E_{i,i+3} - E_{i+3,i})$, which is the Lie algebra of a compact Cartan subgroup. We think of the root decomposition of $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes \mathbb{C}$. The set $\{T_i = E_{i,i+3} - E_{i+3,i}\}_{1 \le i \le 3}$ forms a basis of $\mathfrak{h}_\mathbb{C}$. Let $F_\alpha \in \mathfrak{g}_\mathbb{C}$ be the root vector corresponding to a root α. Then the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is of the same type as the restricted root system and the root vectors are given as follows:

$$\begin{aligned}
F_{e_i+e_j} &= E_{ij} + E_{ji} - E_{i+3,j+3} - E_{j+3,i+3} + \sqrt{-1}(E_{i,j+3} + E_{j,i+3} + E_{i+3,j} + E_{i,j+3}),\\
F_{2e_k} &= E_{kk} - E_{k+3,k+3} + \sqrt{-1}(E_{k,k+3} + E_{k+3,k}),\\
F_{e_i-e_j} &= E_{ij} - E_{ji} + E_{i+3,j+3} - E_{j+3,i+3} - \sqrt{-1}(E_{i+3,j} + E_{j+3,i} - E_{i,j+3} - E_{j,i+3}),\\
F_{-e_i-e_j} &= E_{ij} + E_{ji} - E_{i+3,j+3} - E_{j+3,i+3} - \sqrt{-1}(E_{i,j+3} + E_{j,i+3} + E_{i+3,j} + E_{i,j+3}),\\
F_{-2e_k} &= E_{kk} - E_{k+3,k+3} - \sqrt{-1}(E_{k,k+3} + E_{k+3,k}),\\
F_{-e_i+e_j} &= E_{ij} - E_{ji} + E_{i+3,j+3} - E_{j+3,i+3} + \sqrt{-1}(E_{i+3,j} + E_{j+3,i} - E_{i,j+3} - E_{j,i+3}),
\end{aligned}$$

where $1 \le i < j \le 3$ and $1 \le k \le 3$. Here the notation $F_{\beta_1 e_1+\beta_2 e_2+\beta_3 e_3}$ means that $[T_i, F_{\beta_1 e_1+\beta_2 e_2+\beta_3 e_3}] = \beta_i F_{\beta_1 e_1+\beta_2 e_2+\beta_3 e_3}$ with $1 \le i \le 3$. The set $\Delta^+ = \{e_i \pm e_j,\ 2e_k \mid 1 \le i < j \le 3,\ 1 \le k \le 3\}$ give the standard positive root system and $\Delta_n^+ = \{e_i + e_j,\ 2e_k \mid 1 \le i < j \le 3,\ 1 \le k \le 3\}$ the set of non-compact positive roots. Put

$$\mathfrak{p}^+ = \bigoplus_{\alpha\in\Delta_n^+} \mathbb{R}F_\alpha, \quad \mathfrak{p}^- = \bigoplus_{\alpha\in\Delta_n^+} \mathbb{R}F_{-\alpha}.$$

Then, in $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}\otimes\mathbb{C}$, these two subspaces gives the holomorphic part and the anti-holomorphic part of it and we have a decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Next, we give Iwasawa decompositions of the generator of $\mathfrak{p}^-$. For that purpose, we introduce an element $X_{ij} \in \mathfrak{k}_{\mathbb{C}}$ $(1 \le i < j \le 3)$, specified by

$$-E_{ij} + E_{ji} - E_{i+3,j+3} + E_{j+3,i+3} + \sqrt{-1}(E_{i,j+3} + E_{j,i+3} - E_{i+3,j} - E_{j+3,i}).$$

Then the decompositions are as follows:

$$F_{-e_i-e_j} = X_{ij} + 2E_{e_i-e_j} - 2\sqrt{-1}E_{e_i+e_j}, \quad F_{-2e_k} = \sqrt{-1}T_i + A_i - 2\sqrt{-1}E_{2e_i}.$$

3. Representation of the maximal compact subgroup K

The maximal compact subgroup K is isomorphic to the unitary group $U(3)$ of degree 3, so the complexifications of K and $\mathfrak{k}$ are isomorphic to $GL(3;\mathbb{C})$ and $\mathfrak{gl}(3;\mathbb{C})$ respectively. In terms of highest weight theory, the equivalent classes of irreducible finite dimensional representations of $GL(3;\mathbb{C})$ can be parametrized by the set of the dominant weights, which is given by

$$D(3) = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \lambda_1 \ge \lambda_2 \ge \lambda_3\}.$$

We denote by τ_λ the irreducible finite dimensional representation of $GL(3;\mathbb{C})$ with highest weight $\lambda \in D(3)$

Here, for the irreducible representation $(\tau_\lambda, V_\lambda)$ of $GL(3;\mathbb{C})$, we explicitly give the infinitesimal actions of generators of $\mathfrak{gl}(3;\mathbb{C})$ by τ_λ. For that purpose, we introduce the notion of Gel'fand Tsetlin scheme. The following argument is given in [V-K], §§18.11.

It can be shown that there is a basis of V_λ parametrized by the following diagrams:

$$Q = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \lambda_{12} & & \lambda_{22} & \\ & & \lambda_{11} & & \end{pmatrix},$$

where $(\lambda_{12}, \lambda_{22}, \lambda_{11}) \in \mathbb{Z}^{\oplus 3}$ is such that $\lambda_1 \geq \lambda_{12} \geq \lambda_2 \geq \lambda_{22} \geq \lambda_3 \in \mathbb{Z}^{\oplus 2}$ and $\lambda_{12} \geq \lambda_{11} \geq \lambda_{22}$. We call these diagrams the *Gel'fand Tsetlin schemes* and the basis $\{v_Q\}$ parametrized by the diagrams $\{Q\}$ the *Gel'fand Tsetlin basis.* Using this basis, we give the explicit formulas of infinitesimal action of $\mathfrak{gl}(3,\mathbb{C})$ by the differential $d\tau_\lambda$ of τ_λ. The Lie algebra is generated by the ij-th matrix units E_{ij} with $1 \leq i,j \leq 3$. First we write the formulas for $E_{j,j+1}$ and E_{jj}:

$$d\tau_\lambda(E_{j,j+1})v_Q = \sum_{i=1}^{j} a_{ij}(Q) v_{Q^+_{(ij)}}, \quad d\tau_\lambda(E_{jj})v_Q = (\sum_{i=1}^{j} \lambda_{ij} - \sum_{i=1}^{j-1} \lambda_{i,j-1}) v_Q,$$

where $a_{ij}(Q) = \sqrt{\left| \frac{\prod_{k=1}^{j+1}(\lambda_{k,j+1}-\lambda_{ij}-k+i)\prod_{k=1}^{j-1}(\lambda_{k,j-1}-\lambda_{ij}-k+i-1)}{\prod_{k=1,\neq i}^{j}(\lambda_{kj}-\lambda_{ij}-k+i)\prod_{k=1,\neq i}^{j}(\lambda_{kj}-\lambda_{ij}-k+i-1)} \right|}$, and $Q^+_{(ij)}$ is the diagram with $\lambda_{ij} \to \lambda_{ij}+1$ and $\lambda_{kl} \to \lambda_{kl}$ for $(k,l) \neq (i,j)$. In the subsequent argument, we need the formulas only for E_{ij} with $1 \leq i \leq j \leq 3$. Furthermore, note that the general E_{ij} with $i \leq j$ can be expressed by the bracket product of $E_{j,j+1}$'s. In fact, $E_{13} = [E_{12}, E_{23}]$. Here we give the infinitesimal actions of X_{ij} and T_i (for notations see the previous section), which are members of generators of $\mathfrak{k}_\mathbb{C}$. Via the map $\mathfrak{k}_\mathbb{C} \ni \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in \mathfrak{gl}(3,\mathbb{C})$, X_{ij} and T_i are mapped to $-2E_{ij}$ and $\sqrt{-1}E_{ii}$ respectively. Under these preparation, the explicit formulas are as follows:

$$\begin{aligned}
d\tau_\lambda(X_{12})v_Q &= -2a_{11}(Q)v_{Q^+_{(11)}}, \\
d\tau_\lambda(X_{13})v_Q &= -2(a_{12}(Q)a_{11}(Q^+_{(12)}) - a_{11}(Q)a_{12}(Q^+_{(11)}))v_{Q^+_{(11,12)}} \\
&\quad - 2(a_{22}(Q)a_{11}(Q^+_{(22)}) - a_{11}(Q)a_{22}(Q^+_{(11)})v_{Q^+_{(11,22)}}, \\
d\tau_\lambda(X_{23})v_Q &= -2a_{12}(Q)v_{Q^+_{(12)}} - 2a_{22}(Q)v_{Q^+_{(22)}}, \\
d\tau_\lambda(T_1)v_Q &= \sqrt{-1}\lambda_{11}v_Q, \\
d\tau_\lambda(T_2)v_Q &= \sqrt{-1}(\lambda_{12}+\lambda_{22}-\lambda_{11})v_Q, \\
d\tau_\lambda(T_3)v_Q &= \sqrt{-1}(\lambda_1+\lambda_2+\lambda_3-\lambda_{12}-\lambda_{22})v_Q,
\end{aligned}$$

where $Q^+_{(ij;kl)}$ means that $\lambda_{ij} \mapsto \lambda_{ij}+1$, $\lambda_{kl} \mapsto \lambda_{kl}+1$ and the other components remain the same.

4. Holomorphic discrete series of $Sp(3;\mathbb{R})$.

We introduce the notion of holomorphic discrete series representation of $Sp(3;\mathbb{R})$. For that purpose, we use the Harish-Chandra's characterization of discrete series representation of semi-simple Lie group (cf. [Kn], Chap.IX, §7, Chap.XII, §5). First, consider the totality of continuous characters on the compact Cartan subgroup $T = \exp(\mathfrak{h})$. Their derivations are parametrized by $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3}$. The set $\mathbb{Z}^{\oplus 3}$ gives a weight lattice in $\mathrm{Hom}(\mathfrak{h}, \mathbb{C})$. Note that $\{e_1 - e_2,\ e_1 - e_3,\ e_2 - e_3\}$ gives the set of compact positive roots, and let ρ and ρ_c denote halves the sums of positive roots and compact positive roots respectively. Taking into account that $\Lambda + \rho$ is analytically integral for each $\Lambda \in \mathbb{Z}^{\oplus 3}$ and due to the Harish-Chandra's theory on discrete series, we see that the holomorphic discrete series representations of $Sp(3;\mathbb{R})$ can be parametrized by

$$\begin{aligned}\Xi &= \{\Lambda \in \mathbb{Z}^{\oplus 3} |\ \text{strictly dominant with respect to } \Delta^+\} \\ &= \{\Lambda \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > 0,\ \Lambda_2 > 0,\ \Lambda_3 > 0,\ \Lambda_1 > \Lambda_2 > \Lambda_3\}.\end{aligned}$$

Such Λ's are called the Harish-Chandra parameters for the holomorphic discrete series. We denote by π_Λ the holomorphic discrete series with the parameter Λ. The highest weight of the minimal K-type of π_Λ is given by the special weight $\lambda = \Lambda + \rho - 2\rho_c$, which we call the Blattner parameter. More precisely, $\lambda = (\Lambda_1 + 1, \Lambda_2 + 2, \Lambda_3 + 3)$ if $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$. On the other hand, we will also treat the contragredient π_Λ^* of π_Λ. Its Harish-Chandra parameter (resp. Blattner parameter) is given by $(-\Lambda_3, -\Lambda_2, -\Lambda_1)$ (resp. $(-\Lambda_3 - 3, -\Lambda_2 - 2, -\Lambda_1 - 1)$). It is obtained by the actions of the two elements of the Weyl group, specified by the permutation $1 \mapsto 3$ and the change of signs of Λ_1, Λ_2 and Λ_3.

5. Representation of the maximal unipotent subgroup

Let $N = \exp(\mathfrak{n})$, which is the standard maximal unipotent subgroup of G. In this section, we construct the irreducible unitary representations of N, using the Kirillov theory on the unitary representations of nilpotent Lie group (cf. [C-G], Chap.2). First we give some preparations. Every element $x \in N$ can be written as

$$\begin{aligned} x &= (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x'_{12}, x'_{13}, x'_{23}) \\ &= \left(\begin{array}{ccc|ccc} 1 & & & x_1 & x_{12} & x_{13} \\ & 1 & & x_{12} & x_2 & x_{23} \\ & & 1 & x_{13} & x_{23} & x_3 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \left(\begin{array}{ccc|ccc} 1 & x'_{12} & x'_{13} & & & \\ & 1 & x'_{23} & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & -x'_{12} & 1 & \\ & & & x'_{12}x'_{23} - x'_{13} & -x'_{23} & 1 \end{array}\right), \end{aligned}$$

where $x_{ij}, x'_{ij}, x_k \in \mathbb{R}$ for $1 \le k \le 3$ and $1 \le i, j \le 3$. Let $\mathfrak{n}^*$ be the dual space of $\mathfrak{n}$ and $\{l_k, l_{ij}, l'_{ij}\}$ with $1 \le k \le 3$ and $1 \le i, j \le 3$ the dual basis of $\mathfrak{n}^*$, where l_k, l_{ij} and l'_{ij} are dual to $E_{2e_k}, E_{e_i+e_j}$ and $E_{e_i-e_j}$ respectively. We write every linear form l as $l = \sum_{1\le i<j\le 3}(\xi_{ij} l_{ij} + \xi'_{ij} l'_{ij}) + \sum_{1\le k\le 3} \xi_k l_k$ with $\xi_{ij},\ \xi'_{ij},\ \xi_k \in \mathbb{R}$

We denote by Ad^* the coadjoint actions of N on $\mathfrak{n}^*$. As one of the main statement of the Kirillov theory, we have

Proposition 5.1. (1) *Any* $\eta \in \hat{N}$ *is of the form:*

$$\eta_l = \mathrm{L}^2\text{-}\mathrm{Ind}_{M_l}^N \chi_l$$

with some $l \in \mathfrak{n}^*$, *where* $M_l = \exp(\mathfrak{M}_l)$ *with* $\mathfrak{M}_l$ *a polarization subalgebra for* l, *and* χ_l *is the character on* M_l *defined by*

$$\chi_l(m) = \exp(2\pi\sqrt{-1}l(\log(m))) \quad m \in M_l.$$

(2) *Two representations* η_l *and* $\eta_{l'}$ *are equivalent if and only if* $l' = \mathrm{Ad}^*(n) \cdot l$ *with some* $n \in N$. *In other word, we have a bijection:*

$$\hat{N} \simeq \mathfrak{n}^*/\mathrm{Ad}^*(N).$$

Here, we introduce an $\mathrm{Ad}^*(N)$-stable filtration of $\mathfrak{n}^*$. Since $\mathfrak{n}$ is a 5-step nilpotent Lie algebra, it has the following descending central series, which is a $\mathrm{Ad}(N)$-stable filtration of $\mathfrak{n}$:

$$\mathfrak{n} \supset \mathfrak{n}^{(1)} = [\mathfrak{n}, \mathfrak{n}] \supset \mathfrak{n}^{(2)} = [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \supset \mathfrak{n}^{(3)} \supset^{(4)} \supset \{0\}.$$

Take the annihilators of each component in the dual space $\mathfrak{n}^*$, then we have a $\mathrm{Ad}^*(N)$-stable filtration of $\mathfrak{n}^*$ as follows:

$$\mathfrak{n}^* = \{0\}^\perp \supset \{\mathfrak{n}^{(4)}\}^\perp \supset \{\mathfrak{n}^{(3)}\} \supset \{\mathfrak{n}^{(2)}\}^\perp \supset \{\mathfrak{n}^{(1)}\}^\perp \supset \{0\}.$$

Taking this into account, we can divide the choices of representatives of $\mathfrak{n}^*/\mathrm{Ad}^*(N)$ into the following 5 ways:(i) $l \in \{\mathfrak{n}^{(1)}\}^\perp$ (ii) $l \in \{\mathfrak{n}^{(2)}\}^\perp \backslash \{\mathfrak{n}^{(1)}\}^\perp$ (iii) $l \in \{\mathfrak{n}^{(3)}\}^\perp \backslash \mathfrak{n}^{(2)}\}^\perp$ (iv) $l \in \{\mathfrak{n}^{(4)}\}^\perp \backslash \{\mathfrak{n}^{(3)}\}^\perp$ (v) $l \in \mathfrak{n}^* \backslash \{\mathfrak{n}^{(4)}\}^\perp$.

6. Generalized Whittaker function for holomorphic discrete series.

6.1. Definition. In this section, we recall the definition of generalized Whittaker functions for holomorphic discrete series, calculate the differential equations which they satisfy and give the explicit formulas of them. First, we recall the definition. For that purpose, we introduce the following two spaces associated to fixed $(\tau, V_\tau) \in \hat{K}$ and $(\eta, H_\eta) \in \hat{N}$.

$$\begin{aligned} C_\eta^\infty(N\backslash G) :=& \{f : \text{smooth } H_\eta^\infty\text{-valued function on } G \\ & \mid f(ng) = \eta(n)f(g) \quad (n, g) \in N \times G\}, \\ C_{\eta,\tau}^\infty(N\backslash G/K) :=& \{F : \text{smooth } H_\eta^\infty \otimes V_\tau\text{-valued function on } G \\ & \mid F(ngk) = \eta(n) \otimes \tau^{-1}(k)F(g) \quad (n, g, k) \in N \times G \times K\}, \end{aligned}$$

where H_η^∞ denotes the space of C^∞-vectors in H_η.

Definition 6.1. *Let* π_Λ *be the holomorphic discrete series with Harish-Chandra parameter* Λ. *Consider the space* $\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G))$ *and the restriction map of it to the minimal* K*-type* τ_λ *of* π_Λ:

$$res_{\tau_\lambda} : \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G)) \ni F \mapsto F \cdot \iota \in \mathrm{Hom}_K(\tau_\lambda, C_\eta^\infty(N\backslash G)),$$

where ι denotes the inclusion of τ_λ into π_Λ. A generalized Whittaker function with K-type τ_λ for π_Λ is defined to be an element of images by res_{τ_λ}.

Note that there is a canonical identification:

$$\mathrm{Hom}_K(\tau_\lambda, C_\eta^\infty(N\backslash G)) \simeq C^\infty_{\eta,\tau_\lambda^*}(N\backslash G/K),$$

where τ_λ^* denotes the contragredient of τ_λ. Furthermore, from the Iwasawa decomposition of G, one obtains a bijection of the former space with $C^\infty(A; V_\lambda \otimes H_\eta^\infty)$ (the space of smooth $V_\lambda \otimes H_\eta^\infty$-valued functions). Then the space of generalized Whittaker functions for π_Λ is under the bijection with

$$\{F \in C^\infty_{\eta,\tau_\lambda^*}(N\backslash G/K) \mid dR_X \cdot F = 0 \quad \forall X \in \mathfrak{p}^+\},$$

where dR denotes the differential of the right translation R (cf. [Y], Proposition 10.1). The condition characterizing this space is called the *Cauchy Riemann condition.*

6.2. Differential equations and explicit formulas of the Whittaker functions. Let π_Λ be the holomorphic discrete series on $Sp(3;\mathbb{R})$ with Harish-Chandra parameter $\Lambda = (\lambda_1 - 1,\ \lambda_2 - 2,\ \lambda_3 - 3)$, where $(\lambda_1, \lambda_2, \lambda_3)$ is the Blattner parameter for π_Λ. And let $W(a) = \sum w_Q(a) \cdot v_Q$ be the restriction of a generalized Whittaker function for π_Λ to the radial part A, where $\{v_Q\}$ denotes the Gel'fand Tsetlin basis for $(\tau_\lambda^*, V_\lambda^*)$. Note that the highest weight of τ_λ^* is $(-\lambda_3, -\lambda_2, -\lambda_1)$. From the Cauchy Riemann condition, we see that $W(a)$ is characterized by the following 6 differential equations:

(i) $dR_{F_{-e_1-e_2}} W(a) = 0 \Leftrightarrow$
$$a_{11}(Q^-_{(11)}) w_{Q^-_{(11)}}(a) + \varphi^-_{12} w_Q(a) = 0,$$
(ii) $dR_{F_{-e_1-e_3}} W(a) = 0 \Leftrightarrow$
$$(a_{12}(Q^-_{(12,11)}) a_{11}(Q^-_{(11)}) - a_{11}(Q^-_{(12,11)}) a_{12}(Q^-_{(12)})) w_{Q^-_{(12,11)}}(a)$$
$$+(a_{22}(Q^-_{(22,11)}) a_{11}(Q^-_{(11)}) - a_{11}(Q^-_{(22,11)}) a_{22}(Q^-_{(22)})) w_{Q^-_{(22,11)}}(a) + \varphi^-_{13} w_Q(a) = 0,$$
(iii) $dR_{F_{-e_2-e_3}} W(a) = 0 \Leftrightarrow$
$$a_{12}(Q^-_{(12)}) w_{Q^-_{(12)}}(a) + a_{22}(Q^-_{(22)}) w_{Q^-_{(22)}}(a) + \varphi^-_{23} w_Q(a) = 0,$$
(iv) $dR_{F_{-2e_1}} W(a) = 0 \Leftrightarrow$
$$\mathcal{L}_1^- w_Q(a) + \lambda_{11} w_Q(a) = 0,$$
(v) $dR_{F_{-2e_2}} W(a) = 0 \Leftrightarrow$
$$\mathcal{L}_2^- w_Q(a) + (\lambda_{12} + \lambda_{22} - \lambda_{11}) w_Q(a) = 0,$$
(vi) $dR_{F_{-2e_3}} W(a) = 0 \Leftrightarrow$
$$\mathcal{L}_3^- w_Q(a) - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_{12} + \lambda_{22}) w_Q(a) = 0,$$

where

$$\partial_i := a_i \frac{\partial}{\partial a_i} \ (1 \le i \le 3),$$
$$\mathcal{L}_i^- := \partial_i - 2\sqrt{-1} a_i^2 d\eta(E_i) \ (1 \le i \le 3),$$
$$\varphi_{ij}^- := a_i a_j^{-1} d\eta(E'_{ij}) - \sqrt{-1} a_i a_j d\eta(E_{ij}) \ (1 \le i < j \le 3).$$

By solving these differential equations, we obtain

Theorem 6.2. *(I) For every $\eta \in \hat{N}$,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G)) \le 1.$$

In particular, the equality holds if and only if $\eta \in \hat{N}$ is one of the following four:

(1) *η corresponding to $l \in \{\mathfrak{N}^{(1)}\}^\perp$ such that $\xi'_{12} = \xi'_{23} = 0$, which is a unitary character.*

(2) *η corresponding to $l \in \{\mathfrak{N}^{(2)}\}^\perp \backslash \{\mathfrak{N}^{(1)}\}^\perp$ such that $\xi_2 > 0$, $\xi_{13} = \xi'_{12} = \xi'_{13} = 0$, which has $\underline{l = \xi_2 l_2 + \xi_3 l_3 \text{ with } \xi_2 > 0}$, as a representative of its coadjoint orbit. This representation has $L^2(\mathbb{R})$ as a model.*

(3) *η corresponding to $l \in \mathfrak{N}^* \backslash \{\mathfrak{N}^{(4)}\}^\perp$ such that $\xi_1 > 0$ and $\xi_2 = \xi_{12} = \xi_{23} = \xi'_{23} = 0$, which has $\underline{l = \xi_1 l_1 + \xi_3 l_3 \text{ with } \xi_1 > 0}$, as a representative of its coadjoint orbit. This representation has $L^2(\mathbb{R}^2)$ as a model.*

(4) *η corresponding to $l \in \mathfrak{N}^* \backslash \{\mathfrak{N}^{(4)}\}^\perp$ such that $\xi_1 > 0$, $\begin{vmatrix} 2\xi_1 & \xi_{12} \\ \xi_{12} & 2\xi_2 \end{vmatrix} > 0$ and $(\xi_2, \xi_{12}, \xi_{23}) \neq (0,0,0)$, which has $\underline{l = \xi_1 l_1 + \xi_2 l_2 + \xi_3 l_3 \text{ with } \xi_1 > 0 \text{ and } \xi_2 > 0}$, as a representative of its coadjoint orbit. This representation has $L^2(\mathbb{R}^3)$ as a model.*

Furthermore, we set

$$\mathcal{A}_\eta(N\backslash G) := \{f \in C_\eta^\infty(N\backslash G) \mid f|_A \text{ is of moderate growth}\}.$$

Then, for any $\eta \in \hat{N}$ as above,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, \mathcal{A}_\eta(N\backslash G)) = 1 \Leftrightarrow \xi_3 \ge 0.$$

Here we remark that the coefficients ξ_1, ξ_2 and ξ_3 in the representatives of the orbits may be different from the original ξ_1, ξ_2, and ξ_3.
(II) We add the condition $\xi_3 \ge 0$ to the above four cases. Then, for these cases, the explicit formulas of Whittaker functions are given as follows:
(i) When η is as in (1),

$$w_Q(\xi_3; a) = \begin{cases} C a_1^{\lambda_3} a_2^{\lambda_2} a_3^{\lambda_1} e^{-2\pi \xi_3 a_3^2}, & Q = \begin{pmatrix} -\lambda_3 - \lambda_2 - \lambda_1 \\ -\lambda_3 - \lambda_2 \\ -\lambda_2 \end{pmatrix}, \\ 0, & Q \text{:otherwise.} \end{cases}$$

(ii) When η is as in (2),

$$w_Q(\xi_2,\xi_3;a,t) = \begin{cases} Ca_1(Q)a_1^{\lambda_3}a_2^{\lambda_1-l}a_3^{\lambda_2+l}t^l \\ \times\exp\{-2\pi(a_2^2\xi_2+a_3^2\xi_3+a_3^2\xi_2t^2)\}, & Q=\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 & -\lambda_1+l \\ & & -\lambda_3 \end{pmatrix}, \\ 0, & Q\text{:otherwise}, \end{cases}$$

where t denotes the coordinate of $\mathbb{R}$.
(iii) When η as in (3),

$$w_Q(\xi_1,\xi_3;a,s,u) = \begin{cases} Ca_2(Q)a_1^{\lambda_1-l-m}a_2^{\lambda_3+m}a_3^{\lambda_2+l}s^mu^l\times \\ \exp\{-2\pi(a_1^2\xi_1+a_3^2\xi_3+a_2^2\xi_1s^2+a_3^2\xi_1u^2)\}, & Q=\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 & -\lambda_1+l \\ & & -\lambda_1+l+m \end{pmatrix}, \\ 0, & Q\text{:otherwise}, \end{cases}$$

where (s,u) denotes the coordinate of $\mathbb{R}^2$.
(iv) When η is as in (4),

$$w_Q(\xi_1,\xi_2,\xi_3;a,s,t,u) = \begin{cases} Ca_2(Q)a_1^{\lambda_1-l-m}a_2^{\lambda_3+m}a_3^{\lambda_2+l}s^mu^l\times \\ \exp\{-2\pi(a_1^2\xi_1+a_2^2(\xi_1s^2+\xi_2) \\ +a_3^2(\xi_1u^2+\xi_2t^2+\xi_3)\}, & Q=\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 & -\lambda_1+l \\ & & -\lambda_1+l+m \end{pmatrix}, \\ 0, & Q\text{:otherwise}, \end{cases}$$

where (s,t,u) denotes the coordinate of $\mathbb{R}^3$.
Here C denotes an arbitrary constant and, in (ii) (iii) and (iv),

$$a_1(Q)=(-1)^l\sqrt{\frac{\prod_{0<i<l}(\lambda_1-\lambda_2-i+1)}{l!}},$$

$$a_2(Q)=(-1)^m\sqrt{\frac{(\lambda_1-\lambda_3)!\prod_{0<i<l}(\lambda_1-\lambda_2-i+1)(\lambda_1-\lambda_3-i-m+1)}{l!m!(\lambda_1-\lambda_3-m)!\prod_{0\le i\le l}(\lambda_1-\lambda_3-i+1)}}.$$

7. Formulation of the Fourier expansion

In this section, we introduce two notations: $\Gamma = Sp(3;\mathbb{Z})$ and $N_{\mathbb{Z}} = N\cap\Gamma$. We first recall the definition of holomorphic Siegel modular form on G.

Definition 7.1. *Let π_Λ, τ_λ and τ_λ^* as in the argument before. A C^∞-function $f : G \to V_\lambda^*$ is called a holomorphic Siegel modular form of weight τ_λ with respect to Γ if it satisfies the following conditions:*
(i) For $\gamma \in \Gamma$ and $k \in K$,

$$f(\gamma g k) = \tau_\lambda^*(k)^{-1} f(g) \quad (g \in G).$$

(ii) Let V_f be the subspace of $C^\infty(\Gamma\backslash G)$ generated by the right translation of the coefficient $c_{f,v}(g) = \langle f(g), v\rangle$, where $v \in V_\lambda$ and $\langle\cdot,\cdot\rangle : V_\lambda^* \times V_\lambda \to \mathbb{C}$ is the canonical pairing. Then it is isomorphic to π_Λ as $(\mathfrak{g}_\mathbb{C}, K)$-module, and each $c_{f,v}$ satisfies the Cauchy Riemann condition:

$$dR_X \cdot c_{f,v} = 0 \quad \forall X \in \mathfrak{p}^-.$$

For a fixed $g \in G$, $f(xg)$ $(x \in N)$ belongs to $L^2(N_\mathbb{Z}\backslash N) \underset{\mathbb{C}}{\otimes} V_\lambda^*$. Since $N_\mathbb{Z}\backslash N$ is compact, from Gel'fand Graev Piatetski-Shapiro's Theorem, we have

$$L^2(N_\mathbb{Z}\backslash N) = \bigoplus_{\eta\in\hat{N}} m(\eta)\cdot H_\eta \simeq \bigoplus_{\eta\in\hat{N}} \mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N)) \underset{\mathbb{C}}{\otimes} H_\eta,$$

where $m(\eta) = \dim_\mathbb{C} \mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N)) < \infty$. Let $\{\Phi_M^\eta\}_{1\le M\le m(\eta)}$ denote a basis of $\mathrm{Hom}_N(\eta, L^2(N_\mathbb{Z}\backslash N))$. Then the Fourier expansion of $f(xg)$ along the minimal parabolic subgroup is given as follows:

$$f(xg) = \sum_{\{Q\}}\sum_{\eta}\sum_{M=1}^{m(\eta)} (\Phi_M^\eta \otimes W_f^{(\eta,Q)}(g))(x) \otimes v_Q,$$

where $\{Q\}$ denotes the set of Gel'fand Tsetlin schemes for τ_λ^*, $\{v_Q\}$ the Gelfand Tsetlin basis for V_λ^*, and $W_f^{(\eta,Q)}(g) \in H_\eta^\infty$ for $g \in G$. Set $W_f^\eta(g) := \sum_{\{Q\}} W_f^{(\eta,Q)}(g)\cdot v_Q$. Then we observe that

$$W_f^\eta \in C^\infty_{\eta,\tau_\lambda^*}(N\backslash G/K)$$

and that this satisfies the Cauchy Riemann condition since f does. Hence we see that W_f^η is a generalized Whittaker function with K-type τ_λ for π_Λ, which is given at §6.

Consider the η-component of the decomposition as above. Let $\{h_i\}_{i\in I}$ be a complete orthogonal basis of H_η and $W_f^{(\eta,Q)}(g) = \sum_{i\in I} c_i^{\eta,Q}(g) h_i$ the expansion of $W_f^{(\eta,Q)}$ by this basis. Then the η-component of the Fourier expansion is

$$\sum_{\{Q\}}\{\sum_{i\in I} c_i^{\eta,Q}(g)\cdot \Phi_M^\eta(h_i)(x)\}\cdot v_Q.$$

The remaining work for the construction of our Fourier expansion is to compute $c_i^{\eta,Q}$ and $\Phi_M^\eta(h_i)$ as above. The coefficient $c_i^{\eta,Q}(g)$ can be obtained by computing $\langle W_f^{(\eta,Q)}(g), h_i\rangle$ with $\langle *,*\rangle$ denoting the scalar product on H_η. Our H_η is isomorphic to $\mathbb{C}$ or $L^2(\mathbb{R}^n)$ with $n = 1, 2$ or 3. For η as in (2) (3) and (4) of Theorem 6.2, we take the totality of Hermite functions as the above $\{h_i\}_{i\in I}$ and the Hermite inner product as the scalar product on H_η. The explicit formula of $c_i^{\eta,Q}$ will be given in

Theorem 9.1 (see also Remark 9.2). In the next section, we determine a basis of $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ by giving the functions $\Phi^{\eta}_M(h_i)$ explicitly.

8. Generalized theta series.

Let $h_i(t) = e^{\frac{t^2}{2}}\frac{d^i}{dt^i}e^{-t^2}$ ($i \in \mathbb{N}$) be the i-th Hermite function. The space $L^2(\mathbb{R}^n)$ has $\{h_{i_1}(t_1)\cdots h_{i_n}(t_n)\}_{i\geq 0,\ldots,i_n\geq 0}$ as a complete orthogonal basis for it. We may consider the case $n = 1, 2, 3$ now. Let $\eta \in \hat{N}$ be one of the four representations as in (1),(2),(3) and (4) of Theorem 6.2 (I). We find a basis $\{\Phi^{(\eta)}_M\}_{1\leq M\leq m(\eta)}$ of $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ for them. It is settled by determining the images of Hermite functions (resp. $1 \in \mathbb{C}$) by such intertwining operators for the case (2) (3) and (4) of Theorem 6.2 (I) (resp. the case (1)). They are characterized by the differential equations mentioned in the introduction, except for the case (1). As to the case (1), the image of $1 \in \mathbb{C}$ is characterized by the differential equations arising from the infinitesimal actions of the generator of $\mathfrak{n}$. Here we explain these differential equations in detail. Let $\mathfrak{u}(\mathfrak{n})$ be the universal enveloping algebra of $\mathfrak{n}$. We define

$$\mathfrak{z}u(\eta) := \{X \in \mathfrak{u}(\mathfrak{n}) \mid d\eta(X) = \text{constant multiple}\},$$

which are given as follows:

(2)′ If $\eta \in \hat{N}$ is as in (2) of the theorem,

$$\mathfrak{z}u(\eta) \simeq \mathbb{C}\left[E_{2e_1}, E_{2e_2}, E_{e_1+e_2}, E_{e_1+e_3}, E_{e_1-e_2}, E_{e_1-e_3}, \begin{vmatrix} 2E_{2e_2} & E_{e_2+e_3} \\ E_{e_2+e_3} & 2E_{2e_2} \end{vmatrix}\right].$$

(3)′ If η is as in (3) of the theorem,

$$\mathfrak{z}u(\eta) \simeq \mathbb{C}\left[E_{2e_1}, \begin{vmatrix} 2E_{2e_1} & E_{e_1+e_3} \\ E_{e_1+e_2} & E_{e_2+e_3} \end{vmatrix}, \begin{vmatrix} 2E_{2e_1} & E_{e_1+e_2} \\ E_{e_1-e_3} & E_{e_2-e_3} \end{vmatrix}, \begin{vmatrix} 2E_{2e_1} & E_{e_1+e_3} \\ E_{e_1+e_3} & 2E_{2e_1} \end{vmatrix}\right].$$

(4)′ If η is as in (4) of the theorem,

$$\mathfrak{z}u(\eta) \simeq \mathbb{C}\left[E_{2e_1}, \begin{vmatrix} 2E_{2e_1} & E_{e_1+e_2} \\ E_{e_1+e_2} & 2E_{2e_1} \end{vmatrix}, \begin{vmatrix} 2E_{2e_1} & E_{e_1+e_2} & E_{e_1+e_3} \\ E_{e_1+e_2} & 2E_{2e_2} & E_{e_2+e_3} \\ E_{e_1+e_3} & E_{e_2+e_3} & 2E_{2e_3} \end{vmatrix}\right].$$

These are obtained by calculating algebraic relations of $d\eta(X)$ for generators X of $\mathfrak{n}$. In particular, $\mathfrak{z}u(\eta)$ as in (4)' is isomorphic to the center of $\mathfrak{u}(\mathfrak{n})$.

For η as in (2)' (3)' and (4)', the differential equations characterizing $\Phi(h)$ with the Hermite function h and $\Phi \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ are given as

(A) Hermite differential equation rewritten by the coordinate of N via the Φ,

(B) differential equations arising from the infinitesimal actions of $\mathfrak{z}u(\eta)$.

In order to give their explicit formulas, we introduce the following notations:

$$X'_{12} := x'_{12} + \frac{1}{4\pi\sqrt{-1}\xi_1}\frac{\partial}{\partial x_{12}}, \quad X'_{13} := x'_{13} + \frac{1}{4\pi\sqrt{-1}\xi_1}x'_{23}\frac{\partial}{\partial x_{12}} + \frac{1}{4\pi\sqrt{-1}\xi_1}\frac{\partial}{\partial x_{13}},$$

$$X'_{23} := \frac{1}{16\pi^2\xi_1\xi_2}\left(x'_{23}\frac{\partial^2}{\partial x_{12}^2} + \frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x_{13}}\right) + \frac{1}{2\pi\sqrt{-1}\xi_2}x'_{13}\frac{\partial}{\partial x_2} + \frac{1}{4\pi\sqrt{-1}\xi_2}\frac{\partial}{\partial x_{23}}.$$

Under the preparations as above, the differetail equations for the four cases can be expressed as follows:
I) When $\eta \in \hat{N}$ is as in (1) of Theorem 6.2 (I), $H_\eta \simeq \mathbb{C}$. We can choose $\{1\}$ as a basis of H_η. For $\Phi \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$, $\Phi(1)$ is characterized by the follwoing differential equation:

$$(1)\ d\eta(E_{2e_3})\Phi(1) = 2\pi\sqrt{-1}\xi_3\Phi(1), \quad (2)\ d\eta(X)\Phi(1) = 0 \quad \text{for } X(\neq E_{2e_3}) \in \mathfrak{n}.$$

II) When $\eta \in \hat{N}$ is as in the above $(2)'$, $H_\eta \simeq L^2(\mathbb{R})$, whose basis can be chosen as the totality of Hermite functions $\{h_i(t)\}_{i\geq 0}$. For $\Phi \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$, $\Phi(h_i(t))$ is characterized by the following equations:

$$(3)\ dr_N(X)\Phi(h_i(t)) = 0 \quad \text{for} \quad X = E_{2e_1}, E_{e_1+e_2}, E_{e_1+e_3}, E_{e_1-e_2}, \text{ or }, E_{e_1-e_3},$$
$$(4)\ dr_N(E_{2e_2})\Phi(h_i(t)) = 2\pi\sqrt{-1}\xi_2\Phi(h_i(t)),$$
$$(5)\ dr_N(4E_{2e_2}E_{2e_3} - E^2_{e_1+e_2})\Phi(h_i(t)) = -16\pi^2\xi_2\xi_3\Phi(h_i(t)),$$
$$(6)\ (\frac{\partial^2}{\partial {x'_{23}}^2} + \frac{1}{4\pi\sqrt{-1}\xi_2}\frac{\partial}{\partial x_{23}})^2)\Phi(h_0(t)) = -\Phi(h_0(t)),$$
$$(7)\ \Phi(h_i(t)) = \{\frac{\partial}{\partial x'_{23}} - (x'_{23} + \frac{1}{4\pi\sqrt{-1}\xi_2}\frac{\partial}{\partial x_{23}})\}^i\Phi(h_0(t)).$$

III) When $\eta \in \hat{N}$ is as in the above $(3)'$, $H_\eta \simeq L^2(\mathbb{R}^2)$, for which we can take the totality of Hermite functions $\{h_{i_1}(s)h_{i_2}(u)\}_{i_1\geq 0, i_2\geq 0}$. For $\Phi \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$, $\Phi(h_{i_1}(s)h_{i_2}(u))$ is characterized by

$$(8)\ dr_N(E_{2e_1})\Phi(h_{i_1}(s)h_{i_2}(u)) = 2\pi\sqrt{-1}\xi_1\Phi(h_{i_1}(s)h_{i_2}(u)),$$
$$(9)\ dr_N(2E_{2e_1}E_{e_2+e_3} - E_{e_1+e_3}E_{e_1+e_2})\Phi(h_{i_1}(s)h_{i_2}(u)) = 0,$$
$$(10)\ dr_N(2E_{2e_1}E_{e_2-e_3} - E_{e_1+e_2}E_{e_1-e_3})\Phi(h_{i_1}(s)h_{i_2}(u)) = 0,$$
$$(11)\ dr_N(4E_{2e_1}E_{2e_3} - E^2_{e_1+e_3})\Phi(h_{i_1}(s)h_{i_2}(u)) = -16\pi^2\xi_1\xi_3\Phi(h_{i_1}(s)h_{i_2}(u)),$$
$$(12)\ \{(\frac{\partial^2}{\partial {x'_{12}}^2} - {X'_{12}}^2) + (\frac{\partial^2}{\partial {x'_{13}}^2} - {X'_{13}}^2)\}\Phi(h_0(s)h_0(u)) = -2\Phi(h_0(s)h_0(u)),$$
$$(13)\ \Phi(h_{i_1}(s)h_{i_2}(u)) = (\frac{\partial}{\partial x'_{12}} - X'_{12})^{i_1}(\frac{\partial}{\partial x'_{13}} - X'_{13})^{i_2}\Phi(h_0(s)h_0(u)).$$

IV) When $\eta \in \hat{N}$ is as in the above $(4)'$, $H_\eta \simeq L^2(\mathbb{R}^3)$, for which we can choose the totality of Hermite functions $\{h_{i_1}(s)h_{i_2}(u)h_{i_3}(t)\}_{i_1\geq 0, i_2\geq 0, i_3\geq 0}$.
For $\Phi \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$, $\Phi(h_{i_1}(s)h_{i_2}(u)h_{i_3}(t))$ is characterized by

$$(14)\ dr_N(E_{2e_1})\Phi(h_{i_1}(s)h_{i_2}(t)h_{i_3}(u)) = 2\pi\sqrt{-1}\xi_1\Phi(h_{i_1}(s)h_{i_2}(t)h_{i_3}(u)),$$
$$(15)\ dr_N(4E_{2e_1}E_{2e_2} - E^2_{e_1+e_2})\Phi(h_{i_1}(s)h_{i_2}(t)h_{i_3}(u))$$
$$= -16\pi^2\xi_1\xi_2\Phi(h_{i_1}(s)h_{i_2}(t)h_{i_3}(u)),$$

(16) $dr_N\left(\begin{vmatrix} 2E_{2e_1} & E_{e_1+e_2} & E_{e_1+e_3} \\ E_{e_1+e_2} & 2E_{2e_2} & E_{e_2+e_3} \\ E_{e_1+e_3} & E_{e_2+e_3} & 2E_{2e_3}\end{vmatrix}\right)\Phi(h_{i_1}(s)h_{i_2}(u)h_{i_3}(t)) =$

$-32\pi^3\xi_1\xi_2\xi_3\Phi(h_{i_1}(s)h_{i_2}(u)h_{i_3}(t))$,

(17) $\{(\frac{\partial^2}{\partial {x'_{12}}^2} - {X'_{12}}^2) + (\frac{\partial^2}{\partial {x'_{13}}^2} - {X'_{13}}^2) + ((\frac{\partial}{\partial x'_{23}} - \frac{1}{4\pi\sqrt{-1}}\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x'_{13}})^2 - {X'_{23}}^2)\}$

$\times\, \Phi(h_0(s)h_0(u)h_0(t)) = -3\Phi(h_0(s)h_0(u)h_0(t))$,

(18) $\Phi(h_{i_1}(s)h_{i_2}(u)h_{i_3}(t)) = (\frac{\partial}{\partial x'_{12}} - X'_{12})^{i_1}\times$

$(\frac{\partial}{\partial x'_{13}} - X'_{13})^{i_2}((\frac{\partial}{\partial x'_{23}} - \frac{1}{4\pi\sqrt{-1}\xi_1}\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x'_{13}}) - X'_{23})^{i_3}\Phi(h_0(s)h_0(u)h_0(t))$.

By $N_{\mathbb{Z}}$-invariance and the above equations, we get

Proposition 8.1. (1) *When $\eta \in \hat{N}$ is as in* (1) *of Theorem* 6.2 *(I),*

$$\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N)) = \mathbb{C}\cdot\Phi_0,$$

where $\Phi_0 : \mathbb{C} \to \mathbb{C}\exp 2\pi\sqrt{-1}\xi_3 x_3$.
(2) *When $\eta \in \hat{N}$ is as in* (2) *of the theorem, we introduce a set*

$$\mathfrak{M}(\xi_2,\xi_3) = \{M \in \mathbb{Z} \mid \frac{M^2}{4\xi_1} + \xi_2 \in \mathbb{Z}\}/\sim,$$

where $M \sim M' \leftrightarrow M \equiv M' \mod 2\xi_1$. For a $M \in \mathfrak{M}(\xi_2,\xi_3)$, we define $\Phi^\eta_M \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ by

$$\Phi^\eta_M(h_i(t))(x) = \phi^i_{\xi_2,\xi_3}(M;x) = \sum_{m\in\mathbb{Z}} h_i(x'_{23} + \frac{2\xi_2 m + M}{2\xi_2})$$
$$\times \exp 2\pi\sqrt{-1}(\xi_2 x_2 + \frac{(2\xi_2 m + M)^2 + 4\xi_2\xi_3}{4\xi_2}x_3 + (2\xi_2 m + M)x_{23}).$$

The set $\{\Phi^\eta_M\}_{M\in\mathfrak{M}(\xi_2,\xi_3)}$ gives a basis of $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$.
(3) *When $\eta \in \hat{N}$ is as in* (3) *of the theorem, we introduce a set*

$$\mathfrak{M}'(\xi_1,\xi_3) = \{M = (M_{12}, M_{13}) \in \mathbb{Z}^2 \mid \frac{M_{12}^2}{4\xi_1} \in \mathbb{Z},\ \frac{M_{13}^2}{4\xi_1} + \xi_3 \in \mathbb{Z},\ \frac{M_{12}M_{13}}{2\xi_1} \in \mathbb{Z}\}/\sim,$$

where $M \sim M' \leftrightarrow M'_{12} = 2\xi_1 n'_{12} + M_{12}$, $M'_{13} = 2\xi_1 n'_{13} + M_{12}n'_{23} + M_{13}$ with some $(n'_{12}, n'_{13}, n'_{23}) \in \mathbb{Z}^3$. For a $M = (M_{12}, M_{13}) \in \mathfrak{M}'(\xi_1,\xi_3)$, we define $\Phi^\eta_M \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ by

$$\Phi^\eta_M(h_{i_1}(s)h_{i_2}(u))(x)$$
$$= \phi^{i_1,i_2}_{\xi_1,\xi_3}(M;x) = \sum_{(m_{12},m_{13},m_{23})\in\mathbb{Z}^3} h_{i_1}(x'_{12} + \frac{m'_{12}}{2\xi_1})h_{i_2}(x'_{13} + \frac{m'_{12}}{2\xi_1}x'_{23} + \frac{m'_{13}}{2\xi_1})\times$$

$$\exp 2\pi\sqrt{-1}(\xi_1 x_1 + \frac{{m'_{12}}^2}{4\xi_1}x_2 + (\frac{{m'_{13}}^2}{4\xi_1} + \xi_3)x_3$$
$$+ m'_{12}x_{12} + m'_{13}x_{13} + \frac{m'_{12}m'_{13}}{2\xi_1}x_{23}).$$

The set $\{\Phi^\eta_M\}_{M\in\mathfrak{M}'(\xi_1,\xi_3)}$ *gives a basis of* $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$.
(4) *When* $\eta \in \hat{N}$ *is as in* (4) *of the theorem, we introduce a set*

$$\mathfrak{M}(\xi_1,\xi_2,\xi_3) =$$
$$\{M = (M_{12}, M_{13}, M_{23}) \in \mathbb{Z}^3 \mid \frac{M_{12}^2}{4\xi_1} + \xi_2, \frac{M_{13}^2}{\xi_1} + \frac{2\xi_1 M_{23} - M_{12}M_{13})^2}{16\xi_1^2\xi_2} + \xi_3 \in \mathbb{Z}\}/\sim,$$

where
$M \sim M' \leftrightarrow M'_{12} = 2\xi_1 n'_{12} + M_{12},\ M'_{13} = 2\xi_1 n'_{13} + M_{12}n'_{23} + M_{13},\ M'_{23} = 2\xi_1 n'_{12}n'_{13} + 2(\frac{M_{12}^2}{4\xi_1} + \xi_2)n'_{23} + M_{12}(n'_{13} + n'_{12}n'_{23}) + M_{13}n'_{12} + M_{23}$ *with some* $(n'_{12}, n'_{13}, n'_{23}) \in \mathbb{Z}^3$.
For a $M = (M_{12}, M_{13}, M_{23}) \in \mathfrak{M}(\xi_1,\xi_2,\xi_3)$, *we define* $\Phi^\eta_M \in \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$ *by*

$$\Phi^\eta_M(h_{i_1}(s)h_{i_2}(t)h_{i_3}(u))(x) =$$
$$\phi^{i_1,i_2,i_3}_{\xi_1,\xi_2,\xi_3}(M;x) = \sum_{(m_{12},m_{13},m_{23})\in\mathbb{Z}^3} h_{i_1}(x'_{12} + \frac{m'_{12}}{2\xi_1})h_{i_2}(x'_{13} + \frac{m'_{12}}{2\xi_1}x'_{23} + \frac{m'_{13}}{2\xi_1}) \times$$
$$h_{i_3}(x'_{23} + \frac{2\xi_1 m'_{23} - m'_{12}m'_{13}}{4\xi_1\xi_2}) \exp 2\pi\sqrt{-1}(\xi_1 x_1 + (\frac{{m'_{12}}^2}{4\xi_1} + \xi_2)x_2$$
$$+ (\frac{(2\xi_1 m'_{23} - m'_{12}m'_{13})^2 + 4\xi_1\xi_2 {m'_{13}}^2}{16\xi_1^2\xi_2} + \xi_3)x_3 + m'_{12}x_{12} + m'_{13}x_{13} + m'_{23}x_{23}).$$

where, $m'_{12} = 2\xi_1 m_{12} + M_{12},\ m'_{13} = 2\xi_1 m_{13} + M_{12}m_{23} + M_{13}$,
$m'_{23} = 2\xi_1 m_{12}m_{13} + (\frac{M_{12}^2}{2\xi_1} + 2\xi_2)m_{23} + M_{12}(m_{13} + m_{12}m_{23}) + M_{13}m_{12} + M_{23}$. *The set* $\{\Phi^\eta_M\}_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)}$ *gives a basis of* $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$.

Remark 8.2. From direct computation, we see that the equivalent relations on $\mathfrak{M}(\xi_2,\xi_3)$, $\mathfrak{M}'(\xi_1,\xi_3)$ and $\mathfrak{M}(\xi_1,\xi_2,\xi_3)$ are well-defined and that these sets are finite.

9. **Main result.**

Summarizing the previous argument, we obtain our Fourier expansion using the theta series computed at the previous section.

Theorem 9.1. *The Fourier expansion of a holomorphic Siegel modular form* f *of weight* τ_λ *on* G *is as follows:*

$$f(na) = \sum_{\xi_3\in\mathbb{N}\cup 0} C^f_{\xi_3} a_1^{\lambda_3} a_2^{\lambda_2} a_3^{\lambda_1} \exp 2\pi\sqrt{-1}(\xi_3(x_3 + \sqrt{-1}a_3^2)) \cdot v_{Q_H}$$
$$+ \sum_{Q\in\Lambda_1}(\sum_{\substack{\xi_2\in\mathbb{N},\ \xi_3\geq 0 \\ \mathfrak{M}(\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_2,\xi_3)} C^{f,M}_{\xi_2,\xi_3} a_1(Q) a_1^{\lambda_3} a_2^{\lambda_1 - l} a_3^{\lambda_2 + l} e^{-2\pi(a_2^2\xi_2 + a_3^2\xi_3)}$$

$$\times \sum_{i\geq 0} \alpha_i(l; \frac{1}{2} - 2\pi\xi_2 a_2^2)\phi^i_{\xi_2,\xi_3}(M;n))v_Q$$

$$+ \sum_{Q\in\Lambda_2} (\sum_{\substack{\xi_1\in\mathrm{N},\xi_3\geq 0 \\ \mathfrak{M}'(\xi_1,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}'(\xi_1,\xi_3)} C^{f,M}_{\xi_1,\xi_3} a_2(Q) a_1^{\lambda_1-l-m} a_2^{\lambda_3+m} a_3^{\lambda_2+l} e^{-2\pi(a_1^2\xi_1+a_3^2\xi_3)}$$

$$\times \sum_{i_1\geq 0,\ i_2\geq 0} \alpha_{i_1}(m; \frac{1}{2} - 2\pi a_2^2\xi_1)\alpha_{i_2}(l; \frac{1}{2} - 2\pi a_3^2\xi_2)\phi^{i_1,i_2}_{\xi_1,\xi_3}(M;n))v_Q$$

$$+ \sum_{Q\in\Lambda_2} (\sum_{\substack{\xi_1\in\mathrm{N},\ \xi_2>0,\ \xi_3\geq 0 \\ \mathfrak{M}(\xi_1,\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)} C^{f,M}_{\xi_1,\xi_2,\xi_3} a_2(Q) a_1^{\lambda_1-l-m} a_2^{\lambda_3+m} a_3^{\lambda_2+l}$$

$$\times e^{-2\pi(a_1^2\xi_1+a_2^2\xi_2+a_3^2\xi_3)} \sum_{i_1\geq 0,\ i_2\geq 0,\ i_3\geq 0} \alpha_{i_1}(m; \frac{1}{2} - 2\pi a_2^2\xi_1)\alpha_{i_2}(l; \frac{1}{2} - 2\pi a_3^2\xi_1)$$

$$\times \alpha_{i_3}(0; \frac{1}{2} - 2\pi a_3^2\xi_2)\phi^{i_1,i_2,i_3}_{\xi_1,\xi_2,\xi_3}(M;n))v_Q$$

Notations for this:
(1) $C^f_{\xi_3}, C^{f,M}_{\xi_2,\xi_3}, C^{f,M}_{\xi_1,\xi_3}$ *and* $C^{f,M}_{\xi_1,\xi_2,\xi_3}$ *are Fourier coefficients,*
(2) $\alpha_i(k;\rho) = \{(-1)^{k+i+[\frac{i}{2}]} + (-1)^{[\frac{i}{2}]}\}2^{2[\frac{i}{2}]+\delta-1}\left(\frac{\delta-k}{2}\right)_{[\frac{i}{2}]}\Gamma\left(\frac{k+1-\delta}{2}\right) \times$
${}_2F_1\left(\frac{k+1}{2}, \frac{k+2}{2}, \frac{k+1-\delta}{2} - [\frac{i}{2}]; \rho\right)$, *where* $\delta = 0$ *or* 1 *when* i *is even or odd respectively.*
(3) $Q_H = \begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 \quad -\lambda_2 & \\ & -\lambda_3 & \end{pmatrix}$,
(4) $Q \in \Lambda_1$ *means that* Q *run through Gel'fand Tsetlin schemes of the form*
$\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 \quad -\lambda_1+l & \\ & -\lambda_3 & \end{pmatrix}$ *with* $0 \leq l \leq \lambda_1 - \lambda_2$,
(5) $Q \in \Lambda_2$ *means that* Q *run through Gel'fand Tsetlin schemes of the form*
$\begin{pmatrix} -\lambda_3 & -\lambda_2 & -\lambda_1 \\ & -\lambda_3 \quad -\lambda_1+l & \\ & -\lambda_1+l+m & \end{pmatrix}$ *with* $0 \leq l \leq \lambda_1 - \lambda_2$ *and* $0 \leq m \leq \lambda_1 - \lambda_3 - l$.

Remark 9.2. The coefficient $c_i^{\eta,Q}$ mentioned in §6 is explicitly given as those of $\phi^{i_1}_{\xi_2,\xi_3}$, $\phi^{i_1,i_2}_{\xi_1,\xi_3}$ and $\phi^{i_1,i_2,i_3}_{\xi_1,\xi_2,\xi_3}$ when η is not a character.

Remark 9.3. Seemingly, this expansion may be strange since there are Gel'fand Tsetlin schemes Q such that the coefficients of v_Q is zero. But the reason is that we evaluate f at $na \in NA \subset G$. By a certain $k \in K$, all coefficients of $f(nak)$ are non-zero.

Essentially, this series is obtained by expanding the Whittaker functions with respect to the Hermite functions. Hence, giving a certian change of the summation to the expansion, we obtain another expansion in terms of generalized Whittaker functions:

Theorem 9.4.

$$f(na) = \sum_{\xi_3 \in \mathbb{N}\cup 0} C^f_{\xi_3} w_Q(\xi_3, a) \exp 2\pi\sqrt{-1}(\xi_3 x_3) \cdot v_{Q_H}$$

$$+ \sum_{Q\in\Lambda_1} \Big(\sum_{\substack{\xi_2\in\mathbb{N},\ \xi_3\geq 0\\ \mathfrak{M}(\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_2,\xi_3)} C^{f,M}_{\xi_2,\xi_3} \sum_m w_Q(\xi_2,\xi_3; a, x'_{23} + \frac{2\xi_2 m + M}{2\xi_1})$$

$$\times \exp 2\pi\sqrt{-1}(\xi_2 x_2 + (\frac{(2\xi_1 m + M)^2}{4\xi_2} + \xi_3)x_3 + (2\xi_2 m + M)x_{23}))v_Q$$

$$+ \sum_{Q\in\Lambda_2} \Big(\sum_{\substack{\xi_1\in\mathbb{N},\xi_3\geq 0\\ \mathfrak{M}'(\xi_1,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}'(\xi_1,\xi_3)} C^{f,M}_{\xi_1,\xi_3} \sum_{(m_{12},m_{13})\in\mathbb{Z}^2} w_Q(\xi_1,\xi_3; a, x'_{12} + \frac{m'_{12}}{2\xi_1},$$

$$x'_{13} + \frac{m'_{12}}{2\xi_1}x'_{23} + \frac{m'_{13}}{2\xi_1}) \exp 2\pi\sqrt{-1}(\xi_1 x_1 + \frac{{m'_{12}}^2}{4\xi_1}x_2 + (\frac{{m'_{13}}^2}{4\xi_1} + \xi_3)x_3$$

$$+ m'_{12}x_{12} + m'_{13}x_{13} + \frac{m'_{12}m'_{13}}{4\xi_1}x_{23})v_Q$$

$$+ \sum_{Q\in\Lambda_2} \Big(\sum_{\substack{\xi_1\in\mathbb{N},\ \xi_2>0,\ \xi_3\geq 0\\ \mathfrak{M}(\xi_1,\xi_2,\xi_3)\neq\emptyset}} \sum_{M\in\mathfrak{M}(\xi_1,\xi_2,\xi_3)} C^{f,M}_{\xi_1,\xi_2,\xi_3} w_Q(\xi_1,\xi_2,\xi_3; a, x'_{12} + \frac{m'_{12}}{2\xi_1}, x'_{13}$$

$$+ \frac{m'_{12}}{2\xi_1}x'_{23} + \frac{m'_{13}}{2\xi_1}, x'_{23} + \frac{2\xi_1 m'_{23} - m'_{12}m'_{13}}{4\xi_1\xi_2}) \exp 2\pi\sqrt{-1}(\xi_1 x_1 + (\frac{{m'_{12}}^2}{4\xi_1} + \xi_2)x_2$$

$$+ (\frac{(2\xi_1 m'_{23} - m'_{12}m'_{13})^2 + 4\xi_1\xi_2{m'_{13}}^2}{16\xi_1^2\xi_2} + \xi_3)x_3 + m'_{12}x_{12} + m'_{13}x_{13} + m'_{23}x_{23}))v_Q.$$

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