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Matrix coefficients of the large discrete series representations of $Sp(2;R)$ as hypergeometric series of two variables II

by

織田 孝幸 (東京大学・数理科学) =Takayuki ODA (Univ. of Tokyo)

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Introduction

これは1994年度の集会の、数理解析研究所講究録909:

「$Sp(2;\mathbf{R})$ 上と $SU(2,2)$ 上の保型形式」、95年5月

に書いた同じ表題の記事の継続・完結編である (90ページより101ページ)。

前回のノーとでは、極小 K-type の行列要素のうち、端のウエイトに対応する成分の A 動径成分しか明示公式を得られなかった。今回、全体の可積分系のもつ単純な構造に気が付き、全ての成分について、簡単なオイラー型の積分表示が得られた。

This is a continuation of a former artilcle by the same author with the same tilte in the RIMS Kokyuroku

"Automorphic Forms on $Sp(2;\mathbf{R})$ and $SU(2,2)$" , May, 1995

Also it solves the main problem, not solved completed in the former note.

In a previous paper, we had a partial result on an explicit formula for the A-radial part of the matrix coefficients with minimal K-type of a large discrete series representation of $Sp(2;\mathbf{R})$.

Our former result was not satisfactory, because we had the result only for the extremal components of the minimal K-type. At that time I do not yet understand the structure of the holonomic system (i.e. a system of difference-differential equations) for the matrix coefficients of a large discrete series, perplexed by the apparently combinatorial complexity of the equations.

Now we can see that that system has rather simple structure, discussed in Section 4 of this note. This enable us to have a complete set of rather simple integral (and power sereis) expression of the A-radial parts of the matrix coefficients. They are written as a kind of Eulerian integral, and a special case of Appell's classical F_2-type hyeprgeometric function of two variables, though the origin of the system is different from the classical one (the classical origin is thrown up to infinity by birationa; transformation).

Because of the laziness of the author and also to lessen the burden to him, general setting explained in the former article and other papers are not refrained. The symbols for the Lie group and Lie algbra of $Sp(2;\mathbf{R})$ should be the same as Oda's former paper in Tôhoku Journal, 1994, and also they are compatible with that of Iida, Publ. of RIMS, Kyoto University, 1996.

In a paper in preparation, Section 1 should contains such generalities on the structure of Lie grous and Lie algbras, and Harish-Chandra's parametrization etc. Section 2 should explain the gradient operators and Schmid operators etc. These are the same as other papers, mutatis mutandis.

So we shall start from §3 !

Before starting that, I would like to comment that a simliar result is obtained for the large discrete series of $SU(2,2)$. There appear almost the same functions discussed here, and the proof is almost parallel. So together with the classical result by Hua on the Bergmann kernel of the holomorphic discrete series, we have some explicit formulae for the A-radial parts of the minimal K-type matrix coefficients of all the discrete series of $SU(2,2)$, not only those of $Sp(2;\mathbf{R})$.

We do not attach a complete reference. Related papers up to the time of the former article may be found in its reference, or in the reference of Iida's paper. Some of development on the spherical functions on $Sp(2;\mathbf{R})$ or on $SU(2,2)$ is reviewed in a joint paper with Hayata and Koseki, refered below. The paper of Takayama below, is not logically related to this paper itself. But it has some important connection internally, and strongly related to (near) future work(s).

Added references

Hayata, T., Koseki,H. and Oda, T.: 'Matrix coefficients of the principal P_J-series and the middle discrete series of $SU(2,2)$. *submitted for publication at the summer of 1998*

Iida, M.: 'Spherical functions of the principal series representations as hypergeometric functions of C_2-type', *Publ. RIMS, Kyoto University*, **32** (1996), 689-727

Takayama, N.: 'Prolongation of singularities of solutions of the Euler-Darboux equation and a global structure of the space of holonomic solutions II', *Funkcial. Ekvac.*, **36**, (1993), 187-234

3 The explicit formula of Schmid operator

これは、前回の数理解析研究所講究録の記事の第3節：
"§明示的な方程式系 "
の書換である。内容は整理して基本的にはより少なくなっている。

3.1 Presentation of the holonomic system

We prepare some macro symbols to denote our difference-differential equations. With the help of these symbols, some symmetries of the system become apparent.

Here are some symbols of "hyperbolic triangular functions". The functions are in the variables $(a_1, a_2) \in A$, i.e. on the split Catan subgroup A. The are really hyperbolic triangular functions in the coordinates of the Lie algebra of A.

Notation For $(a_1, a_2) \in A$,

$$sh(a_i) = (a_i - a_i^{-1})/2, \quad ch(a_i) = (a_i + a_i^{-1})/2 \qquad (i = 1, 2)$$

$$th(a_i) = sh(a_i)/ch(a_i), \quad ct(a_i) = ch(a_i)/sh(a_i)$$

$$D(a_1, a_2) = sh^2(a_1) - sh^2(a_2) = ch^2(a_1) - ch^2(a_2).$$

注意. 最後に定義した D は前回のものの 1/4 である。従って以下の式も度々 4 の因子が、たいていの場合なくなる等変更点がある。。

Now we introduce some symbols concerning the indices $I = (i, j) = (i_1, i_2)$ which belong to the product set $\{0, d\}^2$, $\{0, d\}$ being the set of numbers $\{0, 1, \cdots, d-1, d\}$.

Notation For I, we set

$$s(I) = \frac{1}{2}(d-i-j), \quad c_1(I) = \frac{1}{2}(i-j-L), \quad c_2(I) = \frac{1}{2}(j-i-L), w(I) = |s(I)|.$$

Here we recall that we set $L = l_1 + l_2$. The last number $w(I)$ is called *the weight* of the index I. Moreover for $p \in \{1, 2\}$. we set

$$\partial_p^{\pm}(I) := \partial_p \mp s(I)ct(a_p) - c_p(I)th(a_p).$$

The functions $e_{pq}(a)$ $(p, q \in \{1, 2\})$ are given by

$$e_{pq}(a) = (-)^{p-1} sh(a_p)ch(a_q)/D(a_1, a_2).$$

Proposition Here are the differential-difference equations given by Schmid operators:
(a): *chirality equations*

$$(1.+): \quad \{\partial_1^+(I(0,2)) + (d-j-1)e_{11}(a)\}c_{I(0,2)} + (d-i)e_{12}(a)c_{I(1,1)}$$
$$+(j+1)e_{22}c_I(a) + ie_{21}c_{I(-1,1)} = 0.$$
$$(1.-): \quad \{\partial_1^-(I)) + (i+1)e_{11}(a)\}c_I + je_{12}(a)c_{I(1,-1)}$$
$$+(d-j)e_{21}c_{I(1,1)}(a) + (d-i-1)e_{22}c_{I(2,0)} = 0.$$
$$(2.+): \quad \{\partial_2^+(I(2,0)) + (d-i-1)e_{22}(a)\}c_{I(2,0)} + (d-j)e_{21}(a)c_{I(1,1)}$$
$$+je_{12}(a)c_{I(1,-1)} + (i+1)e_{11}(a)c_I = 0.$$
$$(2.-): \quad \{\partial_2^-(I)) + (j+1)e_{22}(a)\}c_I + ie_{21}(a)c_{I(-1,1)}$$
$$+(d-i)e_{12}c_{I(1,1)}(a) + (d-j-1)e_{11}c_{I(0,2)} = 0.$$

(b) *adjoint equations*

$$(3r): \quad \{\partial_2^- + 2c_2(I)th(a_2) + 2(i+1)e_{22}(a)\}c_I + 2je_{21}c_{I(1,-1)}$$
$$+\{\partial_1^+(I(2,0))+2c_1(I(2,0))th(a_1)+2(d-i-1)e_{11}(a)\}c_{I(2,0)}+2(d-j)e_{12}c_{I(1,1)} = 0.$$
$$(3l): \quad \{\partial_1^- + 2c_1(I)th(a_1) + 2(j+1)e_{11}(a)\}c_I + 2ie_{12}c_{I(-1,1)}$$
$$+\{\partial_2^+(I(0,2))+2c_2(I(0,2))th(a_2)+2(d-j-1)e_{22}(a)\}c_{I(0,2)}+2(d-i)e_{21}c_{I(1,1)} = 0.$$

3.2 Symmetry with respect to the indices I

Definition We define an involutive automorphism on the set $\{1,d\}^2$ of indices by

$$I = (i,j) \to I' = (d-j, d-i).$$

Obviously we have

$$s(I') = -s(I), c_k(I') = c_k(I) quad (k = 1,2).$$

Hence

$$\partial_k^{\pm}(I') = \partial_k^{\mp}(I) \quad (k = 1,2).$$

Also for $\varepsilon_1, \varepsilon_2 \in \{0, \pm 1, \pm 2\}$,

$$\{I(\varepsilon_1, \varepsilon_2)\}' = I'(-\varepsilon_2, -\varepsilon_1).$$

Apply the involution "$'$" to the equations of our holonomic system, say, to (1.$\pm$). Then it is transformed to similar equations (1.$\mp$). Therefore we have the following

Observation If we replace the functions $\{c_I\}$ by another system $\{\tilde{c}_I\}$ defined

$$\tilde{c}_I = c_{I'} = c_{d-j,d-i} \quad \text{for each } I = (i,j).$$

Then they satisfy the same holonomic system as for $\{c_I\}$.

3.3 The order of zeros of c_I at the origin

Lemma Let $\{c_I\}$ be a system of solutions of our holonomic system in (3.1), such that all the c_I are regular at the origin $(a_1, a_1) = (1, 1)$. Then each c_I is divisible by $\{sh(a_1)sh(a_2)\}^{w(I)}$ in the ring of germs of analytic functions at the origin. Here $w(I) = |s(I)|$ is the weight of I.

4 Reduction of the holonomic system

4.1 Change of functions and change of variables

Up to the previous section, we had an explicit system of differential equations in terms of the coefficients $\{c_I\}$. In order to make this holonomic system simpler to handle, we introduce new functions $\{h_I\}$, which differ from $\{c_I\}$ by simple multiplicators. Also we change the varibles a_i by $-sh^2(a_i)$.

Definition (multiplicator) We set

$$\mu_I^{\pm}(a) = \{sh(a_1)sh(a_2)\}^{\pm s(I)} \prod_{i=1}^{2} ch(a_i)^{c_i(I)}$$

$$= \{sh(a_1)sh(a_2)\}^{\pm\frac{1}{2}(d-i-j)}\{ch(a_1)ch(a_2)\}^{-\frac{1}{2}L}(ch(a_1)/ch(a_2))^{\frac{1}{2}(i-j)}.$$

Definition (new functions) We set

$$c_I(a) = \mu_I^+(a)h_I^+(a) = \mu_I^-(a)h_I^-(a).$$

We consider mainly only h_I^+ in the later sections for the reason of symmetry between the systems $\{h_i^{\pm}\}$. Therefore we drop the superscript "+" in the symbol from now on.

Remark. (Observation on symmetry) For $I' = (d-j, d-i)$,

$$\mu_{I'}^-(a) = \mu_I^+(a).$$

Definition (change of variables) We set

$$x_i = -sh^2(a_i) \qquad (i = 1, 2).$$

Note that

$$ch^2(a_i) = 1 - x_i = -(x_i - 1), \text{ and } \frac{\partial}{\partial x_i} = -\frac{1}{sh(a_i^2)}\partial_i$$

$$= -\frac{1}{2sh(a_i)ch(a_i)}\partial_i \qquad (i = 1, 2).$$

Now we rewrite the equations for $\{c_I\}$ in terms of new functions $\{h_I\}$ and new variables x_i.

Remark (Symmetry) By the involution $I \to I' = (d-j, d-i)$, the system of equations for $\{h_I\}$ $(s(I) \geq 0)$ can be regarded as a system of equations for $\{h_I^-\}$ $(s(I') \leq 0)$.

4.2 The holonomic system for $\{h_I\}$

Proposition

(a) < *chirality equations* >

$$(1+): \quad \{\frac{\partial}{\partial x_1} + \frac{d-j-1}{2(x_1-x_2)}\}h_{I(0,2)} + \frac{d-i}{2(x_1-x_2)}h_{I(1,1)}$$
$$+\frac{(j+1)x_2}{2(x_1-x_2)}h_I + \frac{ix_2}{2(x_1-x_2)}h_{I(-1,1)} = 0.$$
$$(1-): \quad \{x_1\frac{\partial}{\partial x_1} + s(I) + \frac{(i+1)x_1}{2(x_1-x_2)}\}h_I + \frac{jx_1}{2(x_1-x_2)}h_{I(1,-1)}$$
$$+\frac{d-j}{2(x_1-x_2)}h_{I(1,1)} + \frac{d-i-1}{2(x_1-x_2)}h_{I(2,0)} = 0.$$
$$(2+): \quad \{\frac{\partial}{\partial x_2} - \frac{d-i-1}{2(x_1-x_2)}\}h_{I(2,0)} - \frac{d-j}{2(x_1-x_2)}h_{I(1,1)}$$
$$-\frac{jx_1}{2(x_1-x_2)}h_{I(1,-1)} - \frac{(i+1)x_1}{2(x_1-x_2)}h_I = 0$$
$$(2-): \quad \{x_2\frac{\partial}{\partial x_2} + s(I) - \frac{(j+1)x_2}{2(x_1-x_2)}\}h_I - \frac{ix_2}{2(x_1-x_2)}h_{I(-1,1)}$$
$$-\frac{d-i}{2(x_1-x_2)}h_{I(1,1)} - \frac{d-j-1}{2(x_1-x_2)}h_{I(0,2)} = 0.$$

(b) < *adjoint equations* >

$$(3r): (x_2-1)\{x_2\frac{\partial}{\partial x_2} + s(I) + c_2(I)\frac{x_2}{x_2-1} - \frac{(i+1)x_2}{x_1-x_2}\}h_I$$
$$-j\frac{(x_1-1)x_2}{x_1-x_2}h_{I(1,-1)} - (d-j)\frac{x_2-1}{x_1-x_2}h_{I(1,1)}$$
$$-(x_1-1)\{\frac{\partial}{\partial x_1} + \frac{c_1(I(2,0))}{x_1-1} + \frac{d-i-1}{x_1-x_2}\}h_{I(2,0)} = 0.$$
$$(3l): (x_1-1)\{x_1\frac{\partial}{\partial x_1} + s(I) + c_1(I)\frac{x_1}{x_1-1} + \frac{(j+1)x_1}{x_1-x_2}\}h_I$$
$$+i\frac{x_1(x_2-1)}{x_1-x_2}h_{I(-1,1)} + (d-i)\frac{x_1-1}{x_1-x_2}h_{I(1,1)}$$
$$-(x_2-1)\{\frac{\partial}{\partial x_2} + \frac{c_2(I(0,2))}{x_2-1} - \frac{d-j-1}{x_1-x_2}\}h_{I(0,2)} = 0.$$

Proof) It is immediately derived from the system for $\{c_I\}$. We should note that identity

$$\mu_I^+(a)^{-1}\cdot\partial_i\{\mu_I^+(a)f(a)\} = \{\partial_i + s(I)ct(a_i) + c_i(I)th(a_i)\}f.$$

The other tasks are careful computation. (q.e.d)

4.3 Inductive equations

We derive from the basic system of the previous section for $\{h_I\}$, some simpler equations in this subsection. We call them inductive equations because they will be used to find solutions for general I from the peripheral entries like $h_{0,0}$, $h_{0,1}$.

Lemma We have

$$(i):\quad \frac{\partial}{\partial x_1}h_{I(0,2)}+\frac{\partial}{\partial x_2}h_{I(1,1)}-\frac{i}{2}h_{I(-1,1)}-\frac{j+1}{2}h_I=0$$
$$(i\in\{0,d-1\},j\in\{-1,d-2\}),$$
$$(ii):\quad \frac{\partial}{\partial x_1}h_{I(1,1)}+\frac{\partial}{\partial x_2}h_{I(2,0)}-\frac{i+1}{2}h_I-\frac{j}{2}h_{I(-1,1)}=0$$
$$(i\in\{-1,d-2\},j\in\{0,d-1\}).$$

Proof) Shift the index I in the formula (2.+) by $I\to I(-1,1)$, and substract the obtained fromula from (1.+). The other (ii) is just the index shift $I\to I(1,-1)$ in (i). It is written explicitly for the reference. (q.e.d)

Proposition (indutive equations of type I) We have

$$(i):\quad \frac{\partial}{\partial x_1}h_{I(0,2)}=-(x_2\frac{\partial}{\partial x_2}+s(I))h_I\qquad (j\in\{0,d-2\})$$

$$(ii):\quad \frac{\partial}{\partial x_2}h_{I(2,0)}=-(x_1\frac{\partial}{\partial x_1}+s(I))h_I\qquad (i\in\{0,d-2\})$$

$$(iii):\quad \frac{\partial}{\partial x_1}h_{I(1,1)}-\frac{j}{2}h_{I(1,-1)}-(x_1\frac{\partial}{\partial x_1}+s(I)+\frac{i+1}{2})h_I=0$$
$$(i\in\{0,d-2\},j\in\{0,d-1\})$$
$$(iv):\quad \frac{\partial}{\partial x_2}h_{I(1,1)}-\frac{i}{2}h_{I(-1,1)}-(x_2\frac{\partial}{\partial x_2}+s(I)+\frac{j+1}{2})h_I=0$$
$$(i\in\{0,d-1\},j\in\{0,d-2\})$$

Proof) Add (1.+) and (2.−) to get (i); add (1.−) and (2.+) to get (ii). To show (iii) (resp. (iv)), substract (ii) (resp. (i)) of the present Proposition from (ii) (resp. (i)) of the previous Lemma. (q.e.d)

Reamrk Note that up to this point, we use only the chirality equations.

We have more complicated inductive equations derived from the right and left adjoint equations.

Proposition (inductive equations of type II)

$$(v): \quad (x_2-1)\{\frac{\partial}{\partial x_2}+\frac{d-j-1+c_2(I(0,2))}{x_2-1}\}h_{I(0,2)}+ih_{I(-1,1)}$$

$$-(x_1-1)\{x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+3s(I)+j+1+\frac{c_1(I)x_1}{x_1-1}\}h_I=0.$$

$$(vi): \quad (x_1-1)\{\frac{\partial}{\partial x_1}+\frac{d-i-1+c_1(I(0,2))}{x_1-1}\}h_{I(2,0)}+jh_{I(1,-1)}$$

$$-(x_2-1)\{2x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+3s(I)+i+1+\frac{c_2(I)x_2}{x_2-1}\}h_I=0.$$

Proof) Substract (3l) (resp. (3r)) from (2.$-$) $\times$ $2(x_1-1)$ (resp. (1.$-$) $\times$ $2(x_2-1)$) to get (v) (resp. (iv)). (q.e.d)

4.4 Linear relations for $\{h_I\}$

By eleminating the derivative terms in the equations by addition and substraction among them, we have various linear relations of h_I's.

Lemma (6-term relations) We have

$$(i): \quad s(I)x_2h_I+s(I)x_1h_{I(1,-1)}+\frac{j-1}{2}x_1x_2h_{I(0,-2)}+\frac{i}{2}x_1x_2h_{I(-1,-1)}$$

$$-\frac{d-j}{2}h_{I(1,1)}-\frac{d-i-1}{2}h_{I(2,0)}=0 \quad (i\in\{0,d-1\}, j\in\{1,d\})$$

Proof) Apply the shift $I\to I(0,-2)$ to (1.+). Then we have

$$\{\frac{\partial}{\partial x_1}+\frac{d+1-j}{2(x_1-x_2)}\}h_I+\frac{d-i}{2(x_1-x_2)}h_{I(1,-1)}+\frac{(j-1)x_2}{2(x_1-x_2)}h_{I(0,2)}$$

$$+\frac{ix_2}{2(x_1-x_2)}h_{I(-1,-1)}=0 \quad (j\in\{1,d\}).$$

Multiply x_1 to this, and substract (1.$-$). Note that $d-i-j=2s(I)$. Then we have our Lemma. (q.e.d)

Remark By eliminating ∂_2 in place of ∂_1 of the proof of above lemma, we have a similar 6-terms linear relation. It is identical with the above lemma up to the shift. For the convenience of reference, we write it explicitly here:

$$(ii): \quad s(I)x_2h_{I(-1,1)}+s(I)x_1h_I+\frac{j}{2}x_1x_2h_{I(-1,-1)}+\frac{i-1}{2}x_1x_2h_{I(-2,0)}$$

$$-\frac{d-j-1}{2}h_{I(0,2)}-\frac{d-i}{2}h_{I(1,1)}=0 \quad (i\in\{1,d\}, j\in\{0,d-1\})$$

Lemma (9-terms linear relations)

$$
\begin{aligned}
(i):\quad & x_1x_2\{2(i-1)(x_2-1)h_{I(-2,0)}-(j-1)(x_1-1)h_{I(0,-2)} \\
& +\{(2j-i)(x_1-1)+(i-j-L)(x_1-x_2)\}h_{I(-1,-1)}\} \\
& +2s(I)(x_1+x_2)(x_2-1)h_{I(-1,1)}+2s(I)(x_1-1)x_2h_I-s(I)(x_1-1)x_2h_{I(1,-1)} \\
& +(d-i-1)(x_1-1)h_{I(2,0)}-2(d-j-1)(x_2-1)h_{I(0,2)} \\
& -\{(d-2i+j)(x_1-1)+(i-j-L)(x_1-x_2)\}h_{I(1,1)}=0. \quad (i\in\{1,d-1\}, j\in\{1,d-1\})
\end{aligned}
$$

$$
\begin{aligned}
(ii):\quad & x_1x_2\{2(j-1)(x_1-1)h_{I(0,-2)}-(i-1)(x_2-1)h_{I(-2,0)} \\
& +\{(j-2i)(x_2-1)+(j-i-L)(x_1-x_2)\}h_{I(-1,-1)}\} \\
& +2s(I)x_1(x_2-1)h_{I(-1,1)}-2s(I)x_1(x_2-1)h_I-s(I)(x_1-1)(x_1+x_2)h_{I(1,-1)} \\
& +2(d-i-1)(x_1-1)h_{I(2,0)}-(d-j-1)(x_2-1)h_{I(0,2)} \\
& +\{(d-2j+i)(x_2-1)-(j-i-L)(x_1-x_2)\}h_{I(1,1)}=0. \quad (i,j\in\{1,d-1\})
\end{aligned}
$$

Proof)

Theorem (5-terms relation) For $i,j\in\{1,d-1\}$ we have

$$
\begin{aligned}
& (d-i-j)\{2h_I+(x_1-1)h_{I(1,-1)}+(x_2-1)h_{I(-1,1)}\} \\
& +(i+j-L)x_1x_2h_{I(-1,-1)}+(i+j+L-2d)h_{I(1,1)}=0.
\end{aligned}
$$

Remark We may rewrite the above relation as

$$
\begin{aligned}
& s(I)\{2h_I(x_1-1)h_{I(1,-1)}+(x_2-1)h_{I(-1,1)}\} \\
& +\{(d-L)/2-s(I)\}x_1x_2h_{I(-1,-1)}-\{(d-L)/2+s(I)\}h_{I(1,1)}=0.
\end{aligned}
$$

Proof) Compute first a combination of 6-terms relations:

$$2(x_1-1)\times(6-i)4(x_2-1)\times(6-ii),$$

which equals to

少し長いので省略

**

After that, add the 9-terms relation (i) of the previous lemma. (q.e.d)

Corollary (initial values) For $i,j\in\{1,d-1\}$, we have

$$s(I)\{2h_I(0)-h_{I(1,-1)}(0)-h_{I(-1,1)}(0)\}=\{s(I)+\frac{d-L}{2}\}h_{I(1,1)}(0).$$

4.5 The initial values $h_I(0,0)$

We determine the values of $h_I(x_1, x_2)$ at the origin $(0,0)$. We start with the case of diagonal elements.

Lemma If $s(I) = 0$ $(i.e. j = d - i)$, then

$$h_I(0,0) = h_{i,d-i}(0) = c_0(-1)^{c_1(L)}\binom{d}{i}^{-1} = c_0(-1)^{c_1(I)}\frac{i!j!}{d!},$$

where c_0 is a constant independent of i.

Proof The normalization condition for the matrix coefficient $\Phi(a_1, a_2)$ should be

$$\Phi(1,1) = 1_{d+1} \quad (\text{ the unit matrix of size } d\ ),$$

if Φ is written in term of some basis in the representation space of the minimal K-type and its dual basis. In our formulation, we use the standard basis in both sides of K-types. In of the relation between the standard basis and the dual basis for them, this condition is equivalent to

$$c_{i,d-i} = c_0(-1)\binom{d}{i}^{-1}, \text{ if } i + j = d,$$

$$c_{i,j} = 0, \text{ if } i + j \neq d.$$

Note finally that

$$h_{i,d-i}(0) = c_{i,d-i}(0) \text{ if } s(I) = 0.$$

Remark The constant c_0 depends on the choice of the identitification of the standard basis and dual standard basis. The is canonical way to specify it completely. Even if one fix the length of the standard basis and the dual standard basis, $c = 0$ still has ambiguity up to a complex number of modulus 1.

Next we consider the relation between $h_I(0)$ with the same $s(I)$.

Lemma

$$(d-j)h_{i,j+1}(0) + (d-i)h_{i+1,j}(0) = 0.$$

Proof Input $x_1 = 0, x_2 = 0$ in the 6-terms linear relation (Lemma (***)). Then

$$(d-j)h_{i+1,j+1}(0) + (d-i-1)h_{i+2,j}(0) = 0.$$

Replace i by $i - 1$ to get our lemma. (q.e.d)

Now we determine the value $h_I(0)$ for general I if $s(I) \geq 0$. By the Corollary of Theorem $(* * *)$ (5-terms relation), we have

$$s(I)\{2h_I(0) - h_{I(-1,1)}(0) - h_{I(1,-1)}(0)\} = (\frac{1}{2}(d-L) + s(I))h_{I(1,1)}(0).$$

On ther other hand, the previous lemma shows that

$$2h_I(0) - h_{I(-1,1)} - h_{I(1,-1)}$$

$$= \frac{1}{(d-i)(d-j)}\{2(d-i)(d-j)+(d-i)(d-i+1)+(d-j+1)(d-j)\}h_I(0)$$

$$= \frac{(2d+1-i-j)(2d-i-j)}{(d-i)(d-j)}h_I(0).$$

Hence for $I=(i,j)$,

$$h_I(0) = \frac{(s(I)+\frac{1}{2}(d-L))(d-i)(d-j)}{s(I)(2d+1-i-j)(2d-i-j)}h_{I(1,1)}(0).$$

Thus we have the following

Proposition If $s(I) \geq 0$,

$$\frac{1}{d+1}h_I(0) = c_0(0)(-1)^{(i-j+d)/2}\binom{\frac{1}{2}(d-L)+s(I)}{s(I)}\frac{(d-i)!(d-j)!}{(2d+1-i-j)!}.$$

Proof When $s(I)=0$, the right hand side $RHS(I)$, which is a function in I, equals to

$$c_0(-1)^i\frac{(d-i)!(d-j)!}{(d+1)!} = \frac{1}{d+1}h_{i,d-i}(0).$$

So this case is settled. For the case $s(I)>0$, consider the ratio

$$RHS(I(-1,-1))/RHS(I) = \frac{s(I)+\frac{1}{2}(d-L)+1}{s(I)+1}\frac{(d+1-i)(d+1-j)}{(2d+3-i-j)(2d+2-i-j)}$$

$$= h_{I(-1,-1)}(0)/h_I(0),$$

as shown in the Collorary (***). Hence our Proposition is shown by induction with respect to $s(I)$. (q.e.d)

5 Modified system of F_2 for extremal entries

We want to show that the extremal entries h_I with $s(I) = \pm[\frac{d}{2}]$ are solution of certain holonomic system of rank 4 with singularities along the divisor $x_1x_2(x_1-1)(x_2-1)(x_1-x_2) = 0$ and at inifinity, which is called the *modified system of (Appell's)* F_2 in [Takayama II, §2, pp. ***), that consists of an Euler-Darboux equation and Poisson equation. These last two equations are deduced from a part of the inductive equations in Section (*.*).

We treat the Euler-Darboux equation first.

5.1 Euler-Darboux equation for extremal entries

If the index I attains the psossible highest weight, i.e. if $w(I) = |s(I)| = [\frac{d}{2}]$,

$$I = (0,0) \text{ or } I = (d,d) \quad \text{if } d \text{ is even, and}$$

$$I = (0,1), = (1,0), = (d, d-1), \text{ or } = (d-1,d) \quad \text{if } d \text{ is odd.}$$

By symmetry, it suffices to consider only positive $s(I)$.

Proposition (Euler-Darboux equations)

(a) If d is even, both $h_{0,0}$ and $h_{1,1}$ satisfy the eqaution:

$$\{\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d+1}{2}\frac{1}{x_1 - x_2}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\}h_{i,i} = 0 \qquad (i = 0,1).$$

(b) If d is odd, $h_{0,1}$ and $h_{1,0}$ satisfy the equations:

$$\{\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d+2}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_1} + \frac{d}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_2})\}h_{0,1} = 0$$

and

$$\{\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_1} + \frac{d+2}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_2})\}h_{1,0} = 0,$$

respectivly.

Proof If d is even, we have

$$\frac{\partial}{\partial x_i}h_{1,1} = (x_i\frac{\partial}{\partial x_i} + \frac{d+1}{2})h_{0,0} \quad (i = 1,2),$$

by setting $I = (0,0)$ in the formula (iii) among the inductive equations of type I (Porposition (*.*)).

Personal Memo to Recall:

$$\frac{\partial}{\partial x_1}h_{I(1,1)} - \frac{j}{2}h_{I(1,-1)} - (x_1\frac{\partial}{\partial x_1} + s(I) + \frac{i+1}{2})h_I = 0.$$

For $h_{1,1}$, we have

$$\frac{\partial^2}{\partial x_1 \partial x_2}h_{1,1} = (x_1\frac{\partial}{\partial x_1}+\frac{d+1}{2})\frac{\partial}{\partial x_2}h_{0,0} = x_2^{-1}(x_1\frac{\partial}{\partial x_1}+\frac{d+1}{2})(\frac{\partial}{\partial x_2}h_{1,1}-\frac{d+1}{2}h_{0,0}).$$

Multiply x_2 to the above formula to have

$$x_2\frac{\partial^2}{\partial x_1 \partial x_2}h_{1,1} = (x_1\frac{\partial}{\partial x_1} + \frac{d+1}{2})\frac{\partial}{\partial x_2}h_{1,1} - \frac{d+1}{2}\frac{\partial}{\partial x_1}h_{1,1}$$

$$= x_1\frac{\partial}{\partial x_1 \partial x_2}h_{1,1} - \frac{d+1}{2}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})h_{1,1},$$

which is the equation in question.

For $h_{0,0}$ we have

$$\frac{\partial^2}{\partial x_1 \partial x_2}h_{1,1} = \frac{\partial}{\partial x_1}(x_2\frac{\partial}{\partial x_2} + \frac{d+1}{2})h_{0,0} = (x_2\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2}\frac{\partial}{\partial x_1})h_{0,0}.$$

Change the role of the variables x_1, x_2 to have a symmetric formula, and substract it from the original one. Then we have asymmetric equality:

$$\{(x_1\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2}\frac{\partial}{\partial x_2}) - (x_2\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2}\frac{\partial}{\partial x_1})\}h_{0,0} = 0,$$

which is the desired equation.

Now we settle the case d odd. Set $I = (0,-1)$ in (1.+) and $I = (-1,0)$ in (2.+). Then

$$(\frac{\partial}{\partial x_1} + \frac{d}{2}\frac{1}{x_1 - x_2})h_{0,1} + \frac{d}{2}\frac{1}{x_1 - x_2}h_{1,0} = 0$$

and

$$(\frac{\partial}{\partial x_2} - \frac{d}{2}\frac{1}{x_1 - x_2})h_{1,0} - \frac{d}{2}\frac{1}{x_1 - x_2}h_{0,1} = 0.$$

Eliminate $h_{1,0}$ in the second formula by using the first one. Then we have the equation for $h_{0,1}$. Eliminate $h_{0,1}$ in the first formula by the second formula. The we have the other equation for $h_{1,0}$. (q.e.d)

5.2 Poisson equations for the peripheral and extremal entries

We deduce the other partial differential equations for the extremal h_I. We start with an equation valid for the peripheral entries, i.e. for h_I with $i = 0$ (or $j = 0$ by symmetry).

Lemma If $I = (0, j)$ i.e. $i = 0$, we have an equation

$$(\#)_j : \{\sum_{k=1}^{2} x_i(x_i - 1)\frac{\partial^2}{\partial x_i^2} + 2(x_1 - 1)x_2\frac{\partial^2}{\partial x_1 \partial x_2}\}h_I$$

$$\sum_{k=1}^{2}\{(\frac{d-L}{2}+d-j+3)x_i-(s(I)+1)\}\frac{\partial}{\partial x_i}h_I$$

$$-(d+1)\frac{\partial}{\partial x_1}h_I+(\frac{d-j}{2}+1)(\frac{d-L}{2}+\frac{d-j}{2}+1)h_I=0.$$

Proof Set $i=0$ in the formula (v) of the indutive equation (Proposition). Then we have

$$(x_2-1)(\frac{\partial}{\partial x_2}+\frac{d-j-1+c_2(I(0,2))}{x_2-1})h_{I(0,2)}$$

$$-(x_1-1)(x_1\frac{\partial}{\partial x_1}2x_2\frac{\partial}{\partial x_2}+3s(I)+j+1+\frac{c_1(I)x_1}{x_1-x_2})h_I=0.$$

Apply the operator $\frac{\partial}{\partial x_1}$ to the above formula. Then, since the operator on the function $h_{I(0,2)}$ depends only on the variable x_2, it commutes with $\frac{\partial}{\partial x_1}$. Recalling the inductive equation

$$\frac{\partial}{\partial x_1}h_{I(0,2)}=-(x_2\frac{\partial}{\partial x_2}+s(I))h_I,$$

we have an equation:

$$\frac{\partial}{\partial x_1}\{(x_1-1)(x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+3s(I)+j+1+\frac{c_1(I)x_1}{x_1-1})h_I\}$$

$$+\{(x_2-1)\frac{\partial}{\partial x_2}+d-j-1+c_2(I(0,2))\}(x_2\frac{\partial}{\partial x_2}+s(I))h_I=0.$$

Note here

$$3s(I)+j+1+c_1(I)=2s(I)+j+1+\{s(I)+c_1(I)\}$$

$$=2s(I)+\frac{d-L}{2}+1=\frac{d-L}{2}+d-j+1.$$

Then the last equation is found to be the desired equation by direct computation. (q.e.d)

The following Poisson equations are obtained from the above lemma.

Theorem

(a) If d is even,

$$\{\sum_{k=1}^{2}x_k(x_k-1)\frac{\partial^2}{\partial x_i^2}+(d+1)\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1)\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_2}$$

$$\sum_{k=1}^{2}\{(\frac{d-L}{2}+2)x_k-(\frac{d}{2}+1)\}\frac{\partial}{\partial x_k}+(\frac{d}{2}+1)(\frac{d-L}{2}+\frac{d}{2}+1)\}h_{0,0}=0.$$

(b) If d is odd,

$$\{\sum_{k=1}^{2}x_k(x_k-1)\frac{\partial^2}{\partial x_i^2}+(d+2)\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-d\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_2}$$

$$\sum_{k=1}^{2}\{(\frac{d-L}{2}+2k-2)x_k-(\frac{d+2k-3}{2})\}\frac{\partial}{\partial x_k}+(\frac{d-1}{2}+1)(\frac{d-L}{2}+\frac{d-1}{2}+1)\}h_{0,1}=0.$$

and

$$\{\sum_{k=1}^{2}x_k(x_k-1)\frac{\partial^2}{\partial x_i^2}+d\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+2)\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_2}$$

$$\sum_{k=1}^{2}\{(\frac{d-L}{2}+4-2k)x_k-(\frac{d+3-2k}{2})\}\frac{\partial}{\partial x_k}+(\frac{d-1}{2}+1)(\frac{d-L}{2}+\frac{d-1}{2}+1)\}h_{1,0}=0.$$

Proof When d is even (resp. odd), set $j=0$ (resp. $j=1$) in the formula $(\#)_j$ of the above lemma, and applying Euler-Darboux equation, replace the term $2(x_1-1)x_2\frac{\partial^2}{\partial x_1\partial x_2}$ by

$$(d+1+j)\frac{(x_1-1)x_2}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1-j)\frac{(x_1-1)x_2}{x_1-x_2}\frac{\partial}{\partial x_1}$$

$$=(d+1+j)\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1-j)\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_1}$$

$$-(d+1+j)\frac{x_1-1}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1-j)\frac{x_2}{x_1-x_2}\frac{\partial}{\partial x_2}$$

with $j=0$ (resp. $j=1$). (q.e.d)

前回はこの節の方程式と、この”extremal entries”の解の級数表示・積分表示が可能であることを注意して終わった。今回のノートの、”inductive equations”の部分が全く分からなかった。気が付けばコロンブスの卵である。

6 Solutions $\{h_I\}$

6.1 The solutions for the modified system of F_2

We recall some basic facts on the regular solutions of the modified system of F_2 (cf.[Iida,§**]).

Definition We define an operator Q_{B_1,B_2}, called *Euler-Darboux operator*, by

$$Q_{B_1,B_2} = \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{B_2}{x_1 - x_2}\frac{\partial}{\partial x_1} + \frac{B_1}{x_1 - x_2}\frac{\partial}{\partial x_2}.$$

Here B_1, B_2 are constants.

Lemma If a function $h(x_1, x_2)$, analytic around the origin, satisfies an equation

$$Q_{B_1,B_2}h = 0 \qquad (B_1 > 0, B_2 > 0),$$

it has a power series expansion of the form:

$$h(x_1, x_2) = \sum_{m_1,m_2\geq} \frac{(B_1)_{m_1}(B_2)_{m_2}\xi(m_1 + m_2)}{m_1!m_2!} x_1^{m_1} x_2^{m_2}.$$

with some series $\{\xi(k)\}_{k\in\mathrm{N}}$. Here we set

$$(B)_k = \frac{\Gamma(B + k)}{\Gamma(B)}.$$

Moreover, let $F_h(z)$ be a function in one variable z, regular around zero, defined by the restriction h to the diagonal:

$$h(z, z) = \frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1 + B_2)} F_h(z) = \sum_{k\geq 0} \frac{(B_1 + B_2)_k}{k!} \xi(k) z^k.$$

Then the Eulerian integral formula for the beta function implies an integral

$$h(x_1, x_2) = \int_0^1 F_h(tx_1 + (1 - t)x_2) t^{B_1 - 1}(1 - t)^{B_2 - 1} dt$$

if the integral of the right hand side converges.

The modified F_2 system consists of the above Q_{B_1,B_2} and a Poisson operator $P = P_{A,B_1,B_1,C;\lambda}$. The role of the Poisson operator is the following "intertwining property" for the series of the above type.

Lemma Assume that h satisfies the condition in the previous lemma :$Q_{B_1,B_2}h = 0$, and let $F_h(z)$ be the associated power series of one variable defined there. Let

$$P = \sum_{i=1}^{i} x_i(x_i - 1)\frac{\partial^2}{\partial x_i^2} + \{(A + B_1 - B_2 + 1)x_1 + B_2 - C + 2B_2\frac{x_1(x_1 - 1)}{x_1 - x_2}\}\frac{\partial}{\partial x_1}$$

$$+\{(A-B_1+B_2+1)x_2+B_1-C+2B_1\frac{x_2(x_2-1)}{x_1-x_2}\}\frac{\partial}{\partial x_2}-\lambda.$$

Then $h_1 = Ph$ also satisfies $Q_{B_1,B_2}h_1 = 0$ and for the operator

$$L = z(z-1)\frac{d^2}{dz^2} - \{C-(A+B_1+B_2+1)z\}\frac{d}{dz} - \lambda,$$

we have an interesting intertwining property:

$$F_{Ph}(z) = L \cdot F_h(z).$$

Reamrk This is an analogy of Lemma(8.6) of [Iida, §8, p.***]. He formulated this for the integral expression of h. But this expression requires extra-work to verify its convergence in the definite interval $[0,1]$, strictly speaking. So we prefer to formulate in terms of (formal) power sereis.

Proof Since $Q_{B_1,B_2}h = 0$, we have an equality

$$Ph = \{P + 2(x_1-1)x_2Q_{B_1,B_2} - (x_1-x_2)Q_{B_1,B_2}\}h.$$

Set

$$P_0h = \sum_{i=1}^{2}\sum_{j=1}^{2} x_ix_j\frac{\partial^2}{\partial x_i\partial x_j} + (A+B_1+B_2+1)\sum_{i=1}^{2} x_i\frac{\partial}{\partial x_i}\lambda,$$

and

$$P_1 = -\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2} - (x_1+x_2)\frac{\partial^2}{\partial x_1\partial x_2} - C(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2})$$
$$= -(\sum_{i=1}^{2} x_i\frac{\partial}{\partial x_i} + C)\cdot(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}).$$

Then by direct computation, we have an equality between operators:

$$P + 2(x_1-1)x_2Q_{B_1,B_2} - (x_1-x_2)Q_{B_1,B_2} = P_0 + P_1.$$

The power sereis P_0h is given by

$$\sum \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}\{(m_1+m_2)^2+(A+B_1+B_2)(m_1+m_2)-\lambda\}\xi(m_1+m_2)x_1^{m_1}x_2^{m_2}$$

and P_1h by

$$-\sum \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}(m_1+m_2+B_1+B_2)(m_1+m_2+C)\xi(m_1+m_2+1)x_1^{m_1}x_2^{m_2}.$$

Hence $h_1 = Ph$ is equal to

$$\sum \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}\eta(m_1+m_2)x_1^{m_1}x_2^{m_2}$$

with

$\eta(k) = \{k^2+(A+B_1+B_2)k-\lambda\}\xi(k)-(k+B_1+B_2)(k+C)\xi(k+1)$ for each natural number k.

Now it is obvious that $Q_{B_1,B_2}h_1 = 0$ from the last power series expression of h_1, and for F_{h_1} we have

$$\frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1+B_2)}F_{h_1}(z) = \sum_{k\geq 0}\frac{(B_1+B_2)_k}{k!}\eta(k)z^k$$

by definition. In turn it is equal to the sum of

$$\sum\frac{B_1+B_2)_k}{k!}\{k^2+(A+B_1+B_2)k-\lambda\}\xi(k)z^k = \{(z\frac{d}{dz}+(A+B_1+B_2)(z\frac{d}{dz})-\lambda\}\{\frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1+B_2)}F_h(z)\}$$

and

$$-\sum_{k\geq 0}\frac{(B_1+B_2)_k}{k!}(k+B_1+B_2)(k+C)\xi(k+1)z^k$$

$$= -\sum_{k=0}^{\infty}\frac{(B_1+B2)_{k+1}}{(k+1)!}(k+1+C-1)\xi(k+1)\frac{d}{dz}(z^{k+1})$$

$$= -\frac{d}{dz}\{\sum_{n=0}^{\infty}(n+C-1)\xi(n)z^n\}$$

$$= -\frac{d}{dz}(z\frac{d}{dz}+C-1)\frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1+B_2)}F_h(z).$$

Therefore, cancelling the same factor $\Gamma(B_1)\Gamma(B_2)/\Gamma(B_1+B_2)$, we have

$$F_{h_1}(z) = \{(z\frac{d}{dz}^2 + (A+B_1+B_2)(z\frac{d}{dz} - \lambda - \frac{d}{dz}(z\frac{d}{dz}+C-1)\}F_h(z),$$

as desired. (q.e.d)

6.2 The solutions for the extremal entries

Now we can desribe the solutions for extremal entries $H_{0,0}$ (when d is even) and $h_{0,1}, h_{1,0}$ (when d is odd). We saw in the previous section (§§(*.*)) that

$$Q_{(d+1)/2,(d+1)/2}h_{0,0} = 0 \text{ and } Q_{(d+1)/2,(d+1)/2}h_{1,1} = 0 \text{ for even } d,$$

and

$$Q_{d/2,(d+2)/2}h_{0,1} = 0 \text{ and } Q_{(d+2)/2,d/2}h_{1,0} = 0 \text{ for odd } d,$$

respectively. Thus we have the following

Lemma

(i) If d is even, $h_{0,0}$ and $h_{1,1}$ are constant mutiple of formal power series $h^P_{0,0}$ and $h^P_{1,1}$ of the form:

$$h^P_{i,i} = \sum_{m_1 \geq 0, m_2 \geq 0} \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1! m_2!} \xi_{i,i}(m_1 + m_2) x_1^{m_1} x_2^{m_2} \quad (i = 0, 1)$$

with $B_1 = B_2 = \frac{d+1}{2}$, for some series $\{\xi_{i,i}(k) \mid k \in \mathrm{N}\}$ $(i = 0, 1)$.

(ii) If d is odd, $h_{0,1}$ and $h_{1,0}$ have power series expressions:

$$h^P_{i,1-i} = \sum_{m_1, m_2 \geq 0} \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1! m_2!} \xi_{i,1-i}(m_1 + m_2) x_1^{m_1} x_2^{m_2} \quad (i = 0, 1)$$

with $B_1 = \frac{d}{2}+i, B_2 = \frac{d}{2}+1-i$, for some series $\{\xi_{i,1-i}(k) \mid k \in \mathrm{N}\}$ $(i = 0, 1)$.

Since $h_{0,0}$ satsfies the Poisson equation

$$Ph_{0,0} = 0$$

with parameters

$$A = \frac{d-L}{2}+1,\ B_1 = B_2 = (d+1)/2,\ C = \frac{3}{2}d+2,\ \lambda = -(\frac{d}{2}+1)(\frac{d-L}{2}+\frac{d}{2}+1),$$

the associated function $F_{h_{0,0}}(z)$ should be a constant multiple of the hypergeometric series

$${}_2F_1(\frac{d}{2}, \frac{d-L}{2} + \frac{d}{2} + 1; \frac{3}{2}d + 2; z).$$

6.3 The power series solutions for general h_I

We determine the (unique) power series expansion of the solutions of the holonomic system for $\{h_I | s(I) \geq 0\}$, which is regular at the origin.

Theorem The holonomic system for $\{h_I | d \equiv i + j (\text{mod } 2)\}$ given by Proposition (*.*.*) has a unique system of solutions regular at the origin $(x_1, x_2) = (0, 0)$, up to constant multiple. Moreover these power series expansions are given as follows:
(i) If d is even, and $i + j \leq d$, for $I = (i, j)$

$$h_I(x_1, x_2) = c_0'(-1)^{(i-j+d)/2} \sum_{m_1, m_2 \geq 0} (m_1! m_2!)^{-1} (\frac{d+1}{2})_{m_1 - [j/2]} (\frac{d+1}{2})_{m_2 - [i/2]}$$

$$\times \prod_{\ell=1}^{[i/2]} (m_1 + s(I) + \ell) \prod_{\ell=1}^{[j/2]} (m_2 + s(I) + \ell) \xi_{00}(m_1 + m_2 - (i+j)/2) x_1^{m_1} x_2^{m_2}.$$

Here

$$\xi_{00}(k) = \frac{(\frac{d+2}{2})_k (\frac{d+2}{2} + \frac{d-L}{2})_k}{(d + \frac{3}{2})_k (d+1)_k} \quad \text{for } k \in \{-d/2, \infty\}.$$

(ii) If d is odd and $s(I) \geq 0$,

$$h_I(x_1, x_2) = c_1'(-1)^{(i-j+d)/2} \sum_{m_1, m_2 \geq 0} (m_1! m_2!)^{-1} (\frac{d}{2})_{m_1 + 1 - [(j+1)/2]} (\frac{d}{2})_{m_2 + 1 - [(i+1)/2]} \cdot$$

$$\times \prod_{\ell=1}^{[i/2]} (m_1 + s(I) + \ell) \prod_{\ell=1}^{[j/2]} (m_2 + s(I) + \ell) \xi_{01}(m_1 + m_2 - (i+j-1)/2) x_1^{m_1} x_2^{m_2}.$$

Here

$$\xi_{01}(k) = \frac{(\frac{d+1}{2})_k (\frac{d+1}{2} + \frac{d-L}{2})_k}{(d + \frac{1}{2})_k (d+1)_k} \quad \text{for } k \in \{-d/2, \infty\}.$$

Finally, c_0', c_1' are constants independent of i, j.

Proof We have to prepare two lemmas to show our Theorem. Let $\{h_I^P\}$ denotes the system of the power series defined by the right hand sides of our Theorem. Then, the first one claims that the initial values $h_I(0)$ coincide with the constants terms $h_I^P(0)$ of the power series solutions h_I^P. The second one claims that the system $\{h_I^P\}$ is a special solution of our holonomic system (Proposition).

These are the substantial works in the proof. We omit exact formulation of these lemmata in this review.

6.4 Integral expression of solutions

In this subsection, we give an integral expression of Euler type for the system of solutions for $\{h_I\}$. It is deduced from the power series expression given in the previous subsection. The first step in this procedure is to represent h_I's by simpler power series.

Lemma Assume that $s(I) \geq 0$. Let c_0' (resp. c_1') be the constant defined in the previous Theorem for even (resp. odd) d. We set for $I = (i_1, i_2)$

$$H_I(x_1, x_2) = \sum_{m_1, m_2 \geq 0} \frac{1}{m_1! m_2!} \times$$

$$\times \begin{cases} (\frac{d+1}{2})_{m_1 - [i_2/2]} (\frac{d+1}{2})_{m_2 - [i_1/2]} \xi_{00}(m_1 + m_2 - \frac{1}{2}(i+j)) x_1^{m_1} x_2^{m_2} (\text{ if } d \text{ is even}) \\ (\frac{d}{2})_{m_1+1-[(i_2+1)/2]} (\frac{d}{2})_{m_2+1-[(i_1+1)/2]} \xi_{01}(m_1 + m_2 - \frac{1}{2}(i+j-1)) x_1^{m_1} x_2^{m_2} (\text{ if } d \text{ is odd }). \end{cases}$$

Then

$$h_I = c_*' \times (x_1 x_2)^{-s(I)} (\frac{\partial}{\partial x_1})^{[i_1/2]} (\frac{\partial}{\partial x_2})^{[i_2/2]} \{(x_1 x_2)^{s(I)} x_1^{[i_1/2]} x_2^{[i_2/2]} H_I\}.$$

Proof It suffices to check the corresponding statement termwise:

$$(x_1 x_2)^{-s(I)} (\frac{\partial}{\partial x_1})^{[i_1/2]} (\frac{\partial}{\partial x_2})^{[i_2/2]} \{(x_1 x_2)^{s(I)} x_1^{[i_1/2]} x_2^{[i_2/2]} x_1^{m_1} x_2^{m_2}\}$$

$$= \prod_{\ell=1}^{[i_1/2]} (m_1 + s(I) + \ell) \prod_{\ell=1}^{[i_2/2]} (m_2 + s(I) + \ell) x_1^{m_1} x_2^{m_2}$$

(q.e.d)

The next step is to find an integral expression for H_I. Note that now it satifies Euler-Darboux equations with

$$B_1 = (d+1)/2 - [i_2/2], B_2 = (d+1)/2 - [i_1/2] \text{ if } d \text{ is even.}$$

$$B_1 = (d+2)/2 + [-i_2/2], B_2 = (d+2)/2 + [-i_1/2] \text{ if } d \text{ is odd.}$$

Theorem We have the follwing integral expression for $\{H_I\}$:
(a) The case of even d: (i) If both i_1 and i_2 are even,

$$H_I(x_1, x_2) = H_I(0) \frac{\Gamma(d+1-\frac{i_1+i_2}{2})}{\Gamma(\frac{d+1-i_1}{2}) \Gamma(\frac{d+1-i_2}{2})} \times$$

$$\times \int_0^1 {}_2F_1(s(I)+1, \frac{1}{2}(d-L)+s(I)+1; \frac{d+3}{2}+s(I); tx_1+(1-t)x_2) t^{\frac{d+1-i_2}{2}-1} (1-t)^{\frac{d+1-i_1}{2}-1} dt.$$

Here the constant $H_I(0)$is given by

$$H_I(0) \frac{\Gamma(d+1-\frac{i_1+i_2}{2})}{\Gamma(\frac{d+1-i_1}{2}) \Gamma(\frac{d+1-i_2}{2})} = c_0'(-1)^{(i-j+d)/2} \frac{\Gamma(d+1-\frac{i_1+i_2}{2})}{\Gamma(\frac{d+1}{2}) \Gamma(\frac{d+1}{2})} \xi_{00}(-\frac{i_1+i_2}{2})$$

$$= c_0'(-1)^{(i-j+d)/2}\frac{\Gamma(d+1)}{\Gamma(\frac{d+1}{2})^2}\frac{(\frac{d+3}{2}+s(I))_{(i_1+i_2)/2}}{(s(I)+1)_{(i_1+i_2)/2}(\frac{1}{2}(d-L)+s(I)+1)_{(i_1+i_2)/2}}$$

(ii) If both i_1 and i_2 are odd,

$$H_I(x_1,x_2) = H_I(0)\frac{\Gamma(d+1-\frac{i_1+i_2}{2}+1)}{\Gamma(\frac{d+2-i_1}{2})\Gamma(\frac{d+2-i_2}{2})G(0)} \times$$

$$\times \int_0^1 G(tx_1+(1-t)x_2)t^{\frac{d+2-i_2}{2}-1}(1-t)^{\frac{d+2-i_1}{2}-1}dt.$$

where

$$G(z) = (z\frac{d}{dz}+d+1-\frac{i_1+i_2}{2})_2F_1(s(I)+1,\frac{1}{2}(d-L)+s(I)+1;\frac{d+3}{2}+s(I);z).$$

Moreover the constant $H_I(0)$ is given by

$$H_I(0)\frac{\Gamma(d+2-\frac{i_1+i_2}{2})}{\Gamma(\frac{d+2-i_1}{2})\Gamma(\frac{d+2-i_2}{2})G(0)} = c_0'(-1)^{(i-j+d)/2}\frac{\Gamma(d+1-\frac{i_1+i_2}{2})}{\Gamma(\frac{d+1}{2})\Gamma(\frac{d+1}{2})}\xi_{00}(-\frac{i_1+i_2}{2})$$

$$= c_0'(-1)^{(i-j+d)/2}\frac{\Gamma(d+1)}{\Gamma(\frac{d+1}{2})^2}\frac{(\frac{d+3}{2}+s(I))_{(i_1+i_2)/2}}{(s(I)+1)_{(i_1+i_2)/2}(\frac{1}{2}(d-L)+s(I)+1)_{(i_1+i_2)/2}}$$

(b) The case of odd d:

$$H_I(x_1,x_2) = H_I(0)\frac{\Gamma(d+1-\frac{i_1+i_2}{2}+\frac{1}{2})}{\Gamma(\frac{d}{2}-[\frac{i_1}{2}])\Gamma(\frac{d}{2}-[\frac{i_2}{2}])} \times$$

$$\times \int_0^1 {}_2F_1(s(I)+1,\frac{1}{2}(d-L)+s(I)+1;\frac{d+2}{2}+s(I);tx_1+(1-t)x_2)$$

$$t^{\frac{d+1-i_2}{2}-1}(1-t)^{\frac{d+2-i_1}{2}-1}dt. \quad (\text{ if } i_1 \text{ even, } i_2 \text{ odd})$$

$$t^{\frac{d+2-i_2}{2}-1}(1-t)^{\frac{d+1-i_1}{2}-1}dt. \quad (\text{ if } i_1 \text{ odd, } i_2 \text{ even})$$

The constant $H_I(0)$ is given by

$$H_I(0)\frac{\Gamma(d+\frac{3}{2}-\frac{i_1+i_2}{2})}{\Gamma(\frac{d}{2}-[\frac{i_1}{2}])\Gamma(\frac{d}{2}-[\frac{i_2}{2}])} = c_1'(-1)^{(i-j+d)/2}\frac{\Gamma(d+\frac{3}{2}-\frac{i_1+i_2}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2})}\xi_{01}(-\frac{i_1+i_2}{2}+\frac{1}{2})$$

END