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# Partition properties of subsets of $\mathcal{P}_\kappa\lambda$

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Abstract. Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. The following partition property is shown to be consistent relative to supercompactness: For any $f : \bigcup_{n<\omega}[X]^n_\subset \to \gamma$ with $X \subset \mathcal{P}_\kappa\lambda$ unbounded and $1 < \gamma < \kappa$ there is an unbounded $Y \subset X$ with $|f``[Y]^n_\subset| = 1$ for any $n < \omega$.

Let $\kappa$ be a regular cardinal $> \omega$, $\lambda$ a cardinal $\geq \kappa$ and $F$ a filter on $\mathcal{P}_\kappa\lambda$. Partition properties of the form $\mathcal{P}_\kappa\lambda \to (F^+)^2_2$ (see below for the definition) were introduced by Jech [Je] and successfully used to characterize supercompactness: Menas [Me] proved $\mathcal{P}_\kappa\lambda \to (\mathcal{C}^+_{\kappa\lambda})^2_2$ for $\kappa$ $2^{\lambda^{<\kappa}}$-supercompact via a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ with the partition property. As noted by Kamo [Kam], Menas' argumnt can be modified to derive $\mathcal{P}_\kappa\lambda \to (\mathcal{C}^+_{\kappa\lambda})^2_2$ directly from $\lambda$-supercompactness of $\kappa$. For the converse direction Di Prisco-Zwicker [DZ] and others refined the global result of Magidor [Mag]: $\lambda$-supercompactness of $\kappa$ follows from $\mathcal{P}_\kappa 2^{\lambda^{<\kappa}} \to (\mathcal{C}^+_{\kappa 2^{\lambda^{<\kappa}}})^2_2$.

Johnson [Jo] studied properties of the form $X \to (F^+)^2_2$ for $X \in F^+$, which means that for any $f : [X]^2_\subset \to 2$ there is $Y \in F^+$ with $Y \subset X$ and $|f``[Y]^2_\subset| = 1$. In this note we are concerned with the case where $F$ is canonically defined, in particular $\mathcal{F}_{\kappa\lambda}$, the minimal fine filter on $\mathcal{P}_\kappa\lambda$.

We generally follow the terminology of Kanamori [Kan] with the following exception: For a cardinal $\mu \geq \omega$ we set $[X]^\mu = \{x \subset X : |x| = \mu\}$, $[X]^{<\mu} = \{x \subset X :$

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$|x| < \mu\}$ and $\lim A = \{\alpha < \mu : \sup(A \cap \alpha) = \alpha > 0\}$ for $A \subset \mu$. By $F^+ \to (F^+)^2_2$ we mean $X \to (F^+)^2_2$ for any $X \in F^+$. We understand $\bigcup a \subsetneq \bigcap b$ when the union $a \cup b$ of $a \in [\mathcal{P}_\kappa\lambda]^m_\subset$ and $b \in [\mathcal{P}_\kappa\lambda]^n_\subset$ with $m, n < \omega$ is formed.

Abe [A1] proved $\mathcal{F}^+_{\kappa\lambda} \not\to (\mathcal{F}^+_{\kappa\lambda})^2_2$ under $\lambda^{<\kappa} = 2^\lambda$. Matet [Mat] used Laver's idea (see [JS]) to get the same conclusion from an opposite assumption:

**Proposition 1.** *Assume $\lambda^\kappa = \lambda$. Then $\mathcal{F}^+_{\kappa\lambda} \not\to (\mathcal{F}^+_{\kappa\lambda})^2_2$.*

*Proof.* First set $\mathcal{P}_\kappa\lambda = \{x_\xi : \xi < \lambda\}$ and $[\mathcal{P}_\kappa\lambda]^\kappa = \{Y_\alpha : \alpha < \lambda\}$. By induction on $\xi < \lambda$ construct $x_\xi \subset z_\xi \in \mathcal{P}_\kappa\lambda$ mutually distinct and $y^{\alpha i}_\xi \in Y_\alpha$ with $y^{\alpha i}_\xi \subsetneq z_\xi$ mutually distinct for $\alpha \in z_\xi$ and $i < 2$ as follows: At stage $\xi < \lambda$ by induction on $n < \omega$ take $z_{\xi n} \in \mathcal{P}_\kappa\lambda$ and $y^{\alpha i}_\xi \in Y_\alpha$ for $\alpha \in z_{\xi n}$ and $i < 2$ so that $x_\xi \subset z_{\xi 0} \not\subset \bigcup_{\zeta<\xi} z_\zeta$ and $z_{\xi n} \cup \bigcup\{y^{\alpha i}_\xi : \alpha \in z_{\xi n} \wedge i < 2\} \subsetneq z_{\xi n+1}$. Finally set $z_\xi = \bigcup_{n<\omega} z_{\xi n}$. We claim that $f$ defined by $f(\{y^{\alpha i}_\xi, z_\xi\}) = i$ witnesses $\{z_\xi : \xi < \lambda\} \not\to (\mathcal{F}^+_{\kappa\lambda})^2_2$.

Fix an unbounded set $X \subset \{z_\xi : \xi < \lambda\}$. We show $f``[X]^2_\subset = 2$. Take $\alpha < \lambda$ with $Y_\alpha \in [X]^\kappa$, and $\xi < \lambda$ with $\alpha \in z_\xi \in X$. Then $f(\{y^{\alpha i}_\xi, z_\xi\}) = i$ for $i < 2$ by definition, as desired. $\square$

The above proof yields in fact for any $\gamma < \kappa$ $f : [X]^2_\subset \to \gamma$ with $X \in \mathcal{F}^+_{\kappa\lambda}$ and $f``[Y]^2_\subset = \gamma$ for any $Y \in \mathcal{F}^+_{\kappa\lambda}$ with $Y \subset X$.

It is natural to ask, as did Abe [A1], if $\mathcal{F}^+_{\kappa\lambda} \not\to (\mathcal{F}^+_{\kappa\lambda})^2_2$ holds in general. His answer [A2] to the analogous question would make it more interesting: $\mathcal{C}^+_{\kappa\lambda} \not\to (\mathcal{C}^+_{\kappa\lambda})^2_2$. Appealing more directly to Magidor's idea [Mag], we give a canonical witness to Abe's observation:

**Proposition 2.** *Let $\mu < \kappa$ be regular. Then $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}\,(x \cap \kappa) = \mu\} \not\to (\mathcal{C}^+_{\kappa\lambda})^2_2$.*

*Proof.* Set $S = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}\,(x \cap \kappa) = \mu\}$ and for $x \in S$ fix an unbounded set $c_x \subset$

$x \cap \kappa$ of order type $\mu$. For $\{x, y\} \in [S]^2_{\subset}$ let $f(\{x, y\})$ be 0 when $\min(c_x \Delta c_y) \in c_x$, and 1 otherwise. Fix a stationary set $T \subset S$. We show $f``[T]^2_{\subset} = 2$.

First we have $\gamma < \kappa$ such that for any $w \in \mathcal{P}_\kappa\lambda$ there are $w \subset x, y \in T$ with $\gamma \in c_x - c_y$: Suppose to the contrary that we have $g : \kappa \to \mathcal{P}_\kappa\lambda$ such that for any $\gamma < \kappa$ and $g(\gamma) \subset x, y \in T$, $\gamma \in c_x$ iff $\gamma \in c_y$. Take $x, y \in C(g) \cap T$ with $x \cap \kappa < y \cap \kappa$ by the stationarity of $\{z \cap \kappa : z \in C(g) \cap T\}$. Then $c_x = c_y \cap x \cap \kappa$ has order type $\mu$, contradicting the choice of $c_y$.

Now let $\gamma < \kappa$ be the minimal as above. Then for $\alpha < \gamma$ we have $w_\alpha \in \mathcal{P}_\kappa\lambda$ such that for any $w_\alpha \subset x, y \in T$, $\alpha \in c_x$ iff $\alpha \in c_y$. Set $w = \bigcup_{\alpha<\gamma} w_\alpha \in \mathcal{P}_\kappa\lambda$. Take $w \subset x \subset y \subset z$ from $T$ with $\gamma \in c_x \cap c_z - c_y$. Then $\min(c_x \Delta c_y) = \min(c_y \Delta c_z) = \gamma$ by $w_\alpha \subset x \subset y \subset z$ for any $\alpha < \gamma$, and hence $f(\{x, y\}) = 0$ and $f(\{y, z\}) = 1$ by defintion, as desired. □

The rest of the paper is devoted to a negative answer to Abe's question in the strong sense. We refer to Baumgartner's expository paper [B] for the rudiments of iterated forcings. We are indebted for the definition of the poset $Q_f$ below to Galvin (see [JS]), who proved under $\mathrm{MA}_{\omega_1}$ that for any $f : [X]^m_{\subset} \to n$ with $X \subset [\omega_1]^{<\omega}$ unbounded and $1 < m, n < \omega$ there is an unbounded $Y \subset X$ with $|f``[Y]^m_{\subset}| = 1$.

Assume for the moment that $\kappa$ is a compact cardinal and $\lambda \leq 2^\kappa$. Fix a coloring $f : \bigcup_{n<\omega} [S]^n_{\subset} \to \gamma$ with $S \subset \mathcal{P}_\kappa\lambda$ unbounded and $1 < \gamma < \kappa$. We define a poset $Q_f$ and establish its basic properties.

Fix a fine ultrafilter $U$ on $S$ and define inductively a $\kappa$-complete ultrafilter $U_n$ on $[S]^n_{\subset}$ by $U_0 = \{\{\emptyset\}\}$ and $U_{n+1} = \{X : \{x : \{a : \{x\} \cup a \in X\} \in U_n\} \in U\}$. For $n < \omega$ let $\beta_n$ be the unique $\beta < \gamma$ with $\{a \in [S]^n_{\subset} : f(a) = \beta\} \in U_n$. Let

$Q_f = \{p \in [S]^{<\kappa} : \forall m, n < \omega \forall a \in [p]^m_{\subset}(\{b \in [S]^n_{\subset} : f(a \cup b) = \beta_{m+n}\} \in U_n)\}$ and $q \leq p$ iff $q \supset p$ and $y \not\subset x$ for any $x \in p$ and $y \in q - p$.

First for a generic filter $G \subset Q_f$, $\bigcup G$ is unbounded in $\mathcal{P}_\kappa\lambda$ by the density of $\{q \in Q_f : \exists y \in q(x \subset y)\}$ for any $x \in \mathcal{P}_\kappa\lambda$, and homogeneous for $f$: $f``[\bigcup G]^n_{\subset} = \{\beta_n\}$ for any $n < \omega$.

Next $Q_f$ is $\kappa$-centered closed (hence in particular $\kappa$-directed closed): A centered subset $D$ of $Q_f$ of size $< \kappa$ has a lower bound $\bigcup D$.

Finally we show that $Q_f$ is $\kappa$-linked. Fix an injection $\pi : \mathcal{P}_\kappa\lambda \to {}^\kappa 2$. For $A \subset {}^\alpha 2$ with $\alpha < \kappa$ set $Q_{f,A} = \{p \in Q_f : \{\pi(x)|\alpha : x \in p\} = A \wedge \langle \pi(z)|\alpha : z \in \bigcup_{x \in p} \mathcal{P}x\rangle$ is injective$\}$. Then $Q_f = \bigcup\{Q_{f,A} : \exists \alpha < \kappa(A \subset {}^\alpha 2)\}$ by inaccessibility of $\kappa$. To see linkedness of $Q_{f,A}$, fix $p, q \in Q_{f,A}$. Then $x \not\subset y$ for any $x \in p - q$ and $y \in q$: Otherwise we would have $x = z$ for some $x \in p - q$, $y \in q$ with $x \subset y$ and $z \in q$ with $\pi(x)|\alpha = \pi(z)|\alpha$. Similarly $y \not\subset x$ for any $x \in p$ and $y \in q - p$. Thus $p \cup q \leq p, q$, as desired.

A minor modification of the original proof [B] for $\kappa = \omega_1$ yields the following

**Lemma.** *Assume $2^{<\kappa} = \kappa$. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \beta\rangle$ be a $< \kappa$-support iteration such that $\Vdash_\alpha$ "$\dot{Q}_\alpha$ is $\kappa$-linked and $\kappa$-centered closed" for any $\alpha < \beta$. Then $P_\beta$ satisfies $\kappa^+$-c.c.*

*Proof.* Fix $X \in [P_\beta]^{\kappa^+}$. For $\alpha < \beta$ let $\Vdash_\alpha$ "$\dot{Q}_\alpha = \bigcup_{\gamma<\kappa} \dot{Q}_{\alpha\gamma}$ with $\dot{Q}_{\alpha\gamma}$ linked for any $\gamma < \kappa$." For $p \in X$ by induction on $\xi < \kappa$ take $p_\xi \leq p$, $\alpha^p_\xi \in \mathrm{supp}(p_\xi)$ and $\gamma^p_\xi < \kappa$ so that $p_\zeta \leq p_\xi$ and $p_{\xi+1}|\alpha^p_\xi \Vdash_{\alpha^p_\xi}$ "$p_\xi(\alpha^p_\xi) \in \dot{Q}_{\alpha^p_\xi \gamma^p_\xi}$" for any $\xi < \zeta < \kappa$, and $\{\xi < \kappa : \alpha^p_\xi = \alpha\}$ is unbounded for any $\alpha \in \bigcup_{\xi<\kappa} \mathrm{supp}(p_\xi)$. Take $Y \in [X]^{\kappa^+}$ and $\delta < \kappa$ so that $\delta \in \Delta_{\xi<\kappa} \bigcap\{\lim\{\xi < \kappa : \alpha^p_\xi = \alpha\} : \alpha \in \mathrm{supp}(p_\xi)\}$ for any

$p \in Y$. Next take $Z \in [Y]^{\kappa^+}$ so that $\{\{\alpha^p_\xi : \xi < \delta\} : p \in Z\}$ forms a $\Delta$-system with root $d \in [\beta]^{<\kappa}$. Finally take $W \in [Z]^{\kappa^+}$ and $H \in [\delta \times d \times \kappa]^{<\kappa}$ so that $H = \{(\xi, \alpha^p_\xi, \gamma^p_\xi) : \xi < \delta \wedge \alpha^p_\xi \in d\}$ for any $p \in W$.

To see that $W$ is linked, fix $p, q \in W$. Inductively we construct a lower bound $r$ of $\{p_\xi, q_\xi : \xi < \delta\} \subset P_\beta$ with support $s = \bigcup_{\xi<\delta} \mathrm{supp}(p_\xi) \cup \bigcup_{\xi<\delta} \mathrm{supp}(q_\xi)$. At stage $\alpha \in s$ it suffices to show $r|\alpha \Vdash_\alpha$ "$\{p_\xi(\alpha), q_\xi(\alpha) : \xi < \delta\} \subset \dot{Q}_\alpha$ is centered."

When $\alpha \in d = \bigcup_{\xi<\delta} \mathrm{supp}(p_\xi) \cap \bigcup_{\xi<\delta} \mathrm{supp}(q_\xi)$, for unboundedly many $\xi < \delta$ $\alpha = \alpha^p_\xi$ by the choice of $\delta$, and hence $r|\alpha \le p_{\xi+1}|\alpha, q_{\xi+1}|\alpha$ forces $p_\xi(\alpha), q_\xi(\alpha) \in \dot{Q}_{\alpha\gamma}$, where $(\xi, \alpha, \gamma) \in H$, as desired. Otherwise the claim follows, since $r|\alpha \Vdash_\alpha$ "$\{p_\xi(\alpha), q_\xi(\alpha) : \xi < \delta\} \subset \dot{Q}_\alpha$ is descending." □

We are now ready to prove our main result. By $F^+ \to (F^+)^{<\omega}_\gamma$ we mean that for any $f : \bigcup_{n<\omega}[X]^n_\subset \to \gamma$ with $X \in F^+$ there is $Y \in F^+$ with $Y \subset X$ and $|f``[Y]^n_\subset| = 1$ for any $n < \omega$. Note that $\mathcal{F}^+_{\kappa\kappa} \to (\mathcal{F}^+_{\kappa\kappa})^{<\omega}_\gamma$ iff $\kappa$ is Ramsey for any $1 < \gamma < \kappa$.

**Theorem.** *Let $\kappa$ be a supercompact cardinal and $\lambda$ a cardinal $> \kappa$. Then there is a $\kappa^+$-c.c. poset forcing supercompactness of $\kappa$ and $\mathcal{F}^+_{\kappa\lambda} \to (\mathcal{F}^+_{\kappa\lambda})^{<\omega}_\gamma$ for any $1 < \gamma < \kappa$.*

*Proof.* First we force with the Laver poset [L] for $\kappa$ and then add $\lambda$ Cohen subsets of $\kappa$ to ensure supercompactness of $\kappa$ and $\lambda \le 2^\kappa$ in the further extensions. Next we perform a $< \kappa$-support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < 2^{\lambda^{<\kappa}}\rangle$ with $\Vdash_\alpha$ "$\dot{Q}_\alpha = Q_{\dot{f}}$" for some canonical $P_\alpha$-name $\dot{f}$ for a coloring. The standard inductive argument, together with $\kappa$-closure and $\kappa^+$-c.c. of $P_\alpha$ shows that $P_\alpha$ is of size $\le 2^{\lambda^{<\kappa}}$, and so is the set of canonical $P_\alpha$-names for colorings for any $\alpha < 2^{\lambda^{<\kappa}}$, whose union can be identified with that of canonical $P_{2^{\lambda^{<\kappa}}}$-names for colorings. Thus the iteration can be arranged so that a homogeneous set for a coloring in the final model by $P_{2^{\lambda^{<\kappa}}}$ appears in an intermediate model, which remains unbounded by absoluteness of $\mathcal{P}_\kappa\lambda$, as desired. □

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