数理解析研究所講究録
1098 巻 1999 年 1-13

On integrability test for ultradiscrete equations

Daisuke Takahashi†and Kenji Kajiwara‡

†Department of Mathematical Sciences, Waseda University, Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, JAPAN
daisuke@mse.waseda.ac.jp
‡Department of Electrical Engineering, Doshisha University, Kyotanabe, Kyoto 610-0321, JAPAN
kaji@elrond.doshisha.ac.jp

Abstract. We consider an integrability test for ultradiscrete equations based on the singularity confinement analysis for discrete equations. We show how singularity pattern of the test is transformed into that of ultradiscrete equation. The ultradiscrete solution pattern can be interpreted as a perturbed solution. We can also check an integrability of a given equation by a perturbation growth of a solution, namely Lyapunov exponent. Therefore, singularity confinement test and Lyapunov exponent are related each other in ultradiscrete equations and we propose an integrability test from this point of view.

1. Introduction

Integrability is an important concept in nonlinear equations. If a given equation turns out to be integrable, we can get many exact structures from the system, for example, conserved quantities, symmetries, exact solutions, and so on. Therefore, it has been an important problem to test integrability of equations. For differential equations, it is well known that the Painlevé test is powerful to detect integrability. [1] However, for difference equations, the Painlevé test cannot be applied due to discreteness of independent variables, and the singularity confinement (SC) test is proposed instead. [2]

Let us consider the following multiplicative type of difference equation

$$x_{n+1}x_n^{\sigma}x_{n-1}=\alpha\lambda^n x_n+1. \tag{1}$$

This equation for $\sigma=0,1,2$ with $\lambda=1$ belongs to so-called Quispel-Roberts-Thompson (QRT) system [3, 4] which is a large family of integrable second order ordinary difference equations. The term "integrability" is somewhat more delicate in discrete system than continuous one. For QRT system, it is integrable in a sense that it has a quartic conserved quantity and thus the general solution is expressed by elliptic functions. For generic λ, equation (1) with $\sigma = 0,\, 1,\, 2$ are known as discrete analogues to Painlevé I equation, which are considered to be integrable in a sense that it passes the SC test. [5] It is considered to be non-integrable for other σ.

The SC test is applied to equation (1) as follows. Let us assume $\sigma = 2$ and $\lambda = 1$ to make discussions easier. If initial x's(assume x_0 and x_1) are $x_0 = f$ (non-zero finite) and $x_1 = -\frac{1}{\alpha}$ respectively, then we have $x_2 = 0$, and x_3 becomes singular. In order to see the behaviour of this singularity, we introduce a small parameter δ (~ 0) and put $x_0 = f$ and $x_1 = -\frac{1}{\alpha} + \delta$. Then, successive iteration and asymptotic evaluation in δ gives a singularity pattern as follows:

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
f	$-\frac{1}{\alpha}+\delta$	$\frac{\alpha^3}{f}\delta$	$-\frac{f^2}{\alpha^5}\delta^{-2}$	$-\frac{\alpha^3}{f}\delta$	$-\frac{1}{\alpha}-\delta$	f	$-\frac{\alpha+\alpha^2 f}{f^2}$

(2)

The above pattern shows;(a) A singularity due to x_1 occurs at $n = 3$, (b) The singularity is confined, that is, does not spread on the whole lattice, (c) Information on x_0 pass through the singularity to x_n at $n \geq 4$. Following to SC test, both locality of singularity and preservation of information on initial data strongly indicate an integrability of the equation.

There is another singularity pattern using initial data $x_0 = f$ and $x_1 = \delta$:

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
f	δ	$\frac{1}{f}\delta^{-2}$	$\alpha\, f\, \delta$	$\frac{1}{\alpha^2 f}$	$\alpha^2(1+\alpha\, f)\delta^{-1}$	$\frac{\alpha\, f}{1+\alpha\, f}\delta$	$\frac{1+\alpha\, f}{\alpha^4 f^2}\delta^{-1}$	$\alpha^4 f$	$\frac{1+\alpha^5 f}{\alpha^4(1+\alpha\, f)}\delta$

(3)

The above SC pattern shows; (a) x_{8n} = finite and $x_{8n+1} = O(\delta)$ for any n, (b) Information of x_0 pass through periodic singularities to x_n at large n. In this pattern, singularity spreads on the whole lattice but singular points are confined by finite values which have information on initial x_0. Although such periodic singularity pattern has not been discussed anywhere to the authors' knowledge, we may consider that this pattern also suggests integrability.

Next let us consider an ultradiscrete analogue to equation (1). [6, 7, 8, 9, 5, 10, 11, 12, 13, 14] Applying the following transformation,

$$x_n = e^{X_n/\varepsilon}, \qquad \alpha = e^{A/\varepsilon}, \qquad \lambda = e^{L/\varepsilon}, \tag{4}$$

we obtain

$$X_{n+1} + \sigma\, X_n + X_{n-1} = \varepsilon \log(1 + e^{(X_n + n\, L + A)/\varepsilon}). \tag{5}$$

If we take a limit $\varepsilon \to +0$, we get

$$X_{n+1} + \sigma\, X_n + X_{n-1} = \max(0,\, X_n + n\, L + A), \tag{6}$$

using a formula

$$\lim_{\varepsilon \to +0} \varepsilon\, \log(e^{a/\varepsilon} + e^{b/\varepsilon} + \cdots) = \max(a,\, b,\, \cdots). \tag{7}$$

If initial data X_0, X_1 and parameters L and A of (6) are all integer, X_n for any n is always integer. In this sense, the dependent variable of equation (6) is discretized through a limit (7). The above discretization process on dependent variable is called 'ultradiscretization' and equation (6) is an ultradiscrete analogue to equation. (1) [9] We obtain a cellular automaton from an ultradiscrete equation, if it is possible to restrict values of dependent variable in a range of integers. Thus, it is expected that ultradiscrete equations connect the continuous and the discrete worlds, that is, differential or difference equations and cellular automata.

A large difference between ultradiscrete equations and discrete ones is an existence of singularities. Since ultradiscrete equations are piecewise linear, singularities in normal sense do not exist. Therefore, we cannot use SC test as it is.

In this letter, we demonstrate our trial to detect integrability of ultradiscrete equations based on the SC test for discrete equations. In the next section, we show how SC test is transformed in equation (6). Then, we discuss differences between ultradiscrete analogous to integrable difference equations and those to nonintegrable ones. In section 3, we show that SC test is related to Lyapunov exponent in ultradiscrete equations. Lyapunov exponent is a growth rate of perturbation of solution, by which we can test integrability of a given equation. [15] In the last section, we give concluding remarks.

2. Ultradiscrete analogue to singularity pattern

In this section, we discuss how the singularity pattern of the difference equation (1) is transformed to that of ultradiscrete analogue (6). First we note that the case of $\sigma = 0, 1, 2$ in autonomous case ($L = 0$) of equation (6) should be considered to be integrable, since;(a) Each of them has a conserved quantity (b) Each of them admits a general solution which is expressed by ultradiscrete analogue of elliptic functions. [10] Therefore, we can examine whether a transformed singularity pattern is useful to test an integrability of ultradiscrete equation or not by comparing the pattern for $\sigma = 0, 1, 2$ with that for other cases.

When we ultradiscretize equation (1) to obtain equation (6), we use transformation of variables (4). In this transformation, if L and A are not 0, this means that we introduce singularities into parameters α and λ as well as x in equation (1). In order to discuss effects of singular parameters separately, first we consider a case of finite α and λ, that is, $A = L = 0$. Moreover, let us assume $\sigma = 2$ case,

$$X_{n+1} + 2\,X_n + X_{n-1} = \max(0,\, X_n), \tag{8}$$

as a concrete example. In the transformation (4), x (also α and λ) must be always positive. Therefore, we cannot obtain an ultradiscrete pattern corresponding to pattern (2) because x_n necessarily becomes negative at certain n's.

On the other hand, if f and δ are positive, x_n in the singularity pattern (3) is always positive and can be ultradiscretized. In the limiting process, relation between those values and ultradiscretization parameter ε is important. Let us assume that f is finite independent of ε and $\delta = e^{-K/\varepsilon}$ where K is positive and finite. Then, we get the following pattern on X_n from pattern (3) through (4).

X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
0	$-K$	$2K$	$-K$	0	K	$-K$	K	0	$-K$

(9)

This pattern is strictly periodic with period 8. Note that this pattern can also be obtained from equation (8) with initial data $X_0 = 0$ and $X_1 = -K$ directly. We can easily see that positive, zero, negative X_n correspond to infinite, finite, infinitesimal x_n respectively. Periodic confinement of singularities of x_n is transformed into separation of positive values by 0's in X_n. However, only K corresponding to δ survives and information on f is lost in the pattern.

It is important in the SC test whether the information of initial data survives or not. Therefore, next we take $f = e^{\rho/\varepsilon}$ and $\delta = e^{-K/\varepsilon}$ where ρ is finite and $|\rho| \ll K$. The initial data become $X_0 = \rho$ and $X_1 = -K$, and the following pattern is obtained from pattern (3),

X_0	X_1	X_2	X_3	X_4	X_5
ρ	$-K$	$2K-\rho$	$-K+\rho$	$-\rho$	$K+\max(0,\rho)$

X_6	X_7	X_8	X_9
$-K+\rho$ $-\max(0,\rho)$	$K-2\rho$ $+\max(0,\rho)$	ρ	$-K$

(10)

This pattern is also periodic with period 8. The information of initial data, namely ρ, clearly survives for large n. It is easy to see that we can obtain this pattern directly by successive iteration of equation (8). Therefore, we can conclude that the equation and its singular solution pattern are consistently transformed from difference equation to ultradiscrete one at least in this example. Similar results are obtained for the other integrable cases ($\sigma = 0$ and 1).

Next, we test non-integrable case of equation (1), $\sigma = 3$. Let us assume $\lambda = 1$ for simplicity, then equation (1) becomes

$$x_{n+1}x_n^3x_{n-1} = \alpha x_n + 1, \tag{11}$$

and we get a singularity pattern

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
f	δ	$\frac{1}{f}\delta^{-3}$	$\alpha f^2\delta^5$	$\frac{1}{\alpha^3 f^5}\delta^{-12}$	$\alpha^6 f^8\delta^{19}$	$\frac{1}{\alpha^{15}f^{19}}\delta^{-45}$	$\alpha^{25}f^{30}\delta^{71}$

(12)

with $x_0 = f$ and $x_1 = \delta$. We can easily prove that $x_n = O(\delta^{p_n})$ and $p_{2n} \sim -\frac{\sqrt{3}}{2}((2+\sqrt{3})^n - (2-\sqrt{3})^n)$ and $p_{2n+1} \sim \frac{\sqrt{3}+1}{2}(2+\sqrt{3})^n - \frac{\sqrt{3}-1}{2}(2-\sqrt{3})^n$. Singularities are not confined and information on initial data is lost for large n when $\delta \to 0$. Following to SC test, this strongly implies that equation (11) is not integrable.

The corresponding ultradiscrete equation is

$$X_{n+1} + 3\,X_n + X_{n-1} = \max(0,\, X_n). \tag{13}$$

and a pattern becomes

X_0	X_1	X_2	X_3	X_4	X_5
ρ	$-K$	$3K-\rho$	$-5K+2\rho$	$12K-5\rho$	$-19K+8\rho$

X_6	X_7	X_8	X_9
$45K-19\rho$	$71K+30\rho$	$-187K-41\rho$	$490K+93\rho$

(14)

In this pattern, the first term including K in every X_n $(n \geq 1)$ becomes dominant and information on ρ becomes negligible.

In the above examples, we have shown singularity pattern of difference equation (1) for finite α with $\lambda = 1$ and corresponding pattern of ultradiscrete equation (6) simultaneously. The values f (finite) and $1/\delta$ (singular) in equation (1) corresponds to ρ and K ($|\rho| \ll K$) in equation (6), respectively. When equation (1) is integrable ($\sigma = 0, 1, 2$), information of ρ survives for large n in corresponding ultradiscrete equations. Confined singularity in the SC test on difference equation (1) corresponds to separation of positive values including K by small ρ.

On the other hand, when equation (1) is non-integrable ($\sigma = 3$), information on ρ is negligible and K becomes dominant for large n in corresponding ultradiscrete equation, and separation of positive values by ρ does not occur.

We note that the above singular solution patterns for ultradiscrete equation are obtained without inconsistency. Namely, the pattern obtained by taking ultradiscrete limit on the singularity pattern of difference equation always coincides with that obtained by successive iteration of ultradiscrete equation directly.

Therefore, we could obtain an interpretation of the SC test on the ultradisctete equation (6) with $L = A = 0$, and observe a critical difference between the case of $\sigma = 0, 1, 2$ and other cases.

3. Lyapunov exponent of perturbed ultradiscrete solution

So far, we have considered the ultradiscrete equation (6) with $L = A = 0$. However, the parameters L and A can effect a solution drastically. For example, let us take $\sigma = 2$ case in equation (6) again,

$$X_{n+1} + 2\,X_n + X_{n-1} = \max(0,\, X_n + n\,L + A), \tag{15}$$

and put $L = 0$ and $A = 1$. Let us start with the initial data $X_0 = 0$ and $X_1 = -1$. Then we see that this solution is periodic with period 3. However, starting with $X_0 = 0$

and $X_1 = -2$, we see that the period is now 20. Compare this phenomenon with a fact that the period for $L = A = 0$ was 8 for any $X_0 = \rho$ and $X_1 = -K$. Therefore, the period of solution can change by the combination of parameter A and initial data X_0, X_1. Moreover, in a case of $L \neq 0$, equation (15) becomes non-autonomous equation. Therefore, since the values of X_n's grow due to the term nL, both criterions obtained in the previous section do not work as they are. That is, information on initial value becomes negligible and separation of positive values by ρ does not occur. A solution to equation (15) in the case of $A = 3$, $L = 2$, $X_0 = 0$, $X_1 = 1$ is shown in Fig. 1.

These observations imply that we cannot simply transform a singularity pattern of difference equation to ultradiscrete one for $L \neq 0$ and $A \neq 0$. This is because taking the parameters L and A to be non-zero means that we have introduced singularities on parameter λ and α in the difference equation (1), respectively. Therefore, we must improve an integrability test for ultradiscrete equation.

Let us look back the singular solution patterns (10) and (14) of the ultradiscrete equations (8) and (13), respectively. We can give another interpretation of ρ and K. K is a parameter giving a solution orbit with initial data $X_0 = 0$, $X_1 = -K$ and ρ is a perturbation to the orbit. From this viewpoint, growth rates of perturbation in the patterns (10) and (14) are clearly different. Therefore, it might be possible to regard SC analysis on ultradiscrete equations is nothing but analysis on a growth rate of perturbation.

Let us assume that X_n is a solution to equation (15) and X'_n is one perturbed by ρ. Then, amplitude of perturbation is defined by a_n where $a_n \equiv |\frac{X_n - X'_n}{\rho}|$. Figure 2 shows a_n for $A = 3$, $L = 2$, $X_0 = 0$, $X'_0 = \rho$ and $X_1 = X'_1 = 1$. We can see that a_n grows linearly for $n < 0$ and does not grow for $n > 0$. Therefore, the solution is not chaotic and we can easily estimate a global behavior of the solution.

A quantitative index to check integrability of a system is Lyapunov exponent which is a mean growth rate of perturbation. [15] If we define λ_n by $\frac{1}{n} \log a_n$, λ_∞ ($\lambda_{-\infty}$) is the exponent. When $\lambda_\infty > 0$, the perturbation grows exponentially and the system becomes chaotic. As for equation (13), λ_{10n} for $X_0 = 0$, $X'_0 = \rho$ and $X_1 = X'_1 = 1$ is plotted in Fig. 3. λ_n is asymptotically about 0.66 for large n. Therefore, we can conclude that equation (13) is chaotic and this result is compatible with that of the previous section.

Next we try another example, the cutoff Toda equation [16]

$$\begin{cases} x_{n+1} &= x_n / y_n^\sigma \\ y_{n+1} &= y_n \, (h + x_{n+1})/(h + 1/x_{n+1}) \end{cases}, \tag{16}$$

where σ and h are constant parameters. Eliminating y, this equation reduces to the following ordinary difference equation on x_n,

$$x_{n+1} x_{n-1} = x_n^{2-\sigma} (1 + h\, x_n)^\sigma / (h + x_n)^\sigma. \tag{17}$$

Equation (17) with $\sigma = 1$ and that with $\sigma = 2$ are nothing but the autonomous discrete Painlevé II and III equations, respectively. [5] According to SC test, only the cases of $\sigma = 1$ and 2 are integrable and not otherwise. Here we discuss details of perturbed solutions of ultradiscrete analogue to (16) in the cases of $\sigma = 2$ (integrable) and 3 (non-integrable).

If we introduce $x_n = \exp(X_n/\varepsilon)$, $h = \exp(H/\varepsilon)$ and take the limit $\varepsilon \to +0$, we obtain

$$X_{n+1} + X_{n-1} = (2-\sigma) X_n + \sigma (\max(0, X_n + H) - \max(H, X_n)), \qquad (18)$$

from equation (17). If $H \neq 0$, this means that equation (17) includes singular parameter h. In the case of $\sigma = 2$, we see that the difference equation (17) has the following conserved quantity,

$$x_{n+1}x_n + 2h(x_{n+1}+x_n) + h^2(\frac{x_{n+1}}{x_n} + \frac{x_n}{x_{n+1}}) + 2h(\frac{1}{x_{n+1}} + \frac{1}{x_n}) + \frac{1}{x_{n+1}x_n}. \qquad (19)$$

Then, we obtain a conserved quantity of ultradiscrete equation (18) with $\sigma = 2$,

$$\max(X_n + X_{n+1},\ -X_n - X_{n+1},\ 2H - X_n + X_{n+1},\ 2H + X_n - X_{n+1}), \qquad (20)$$

through the above ultradiscrete limit. Define K by $\max(X_0+X_1,\ -X_0-X_1,\ 2H-X_0+X_1,\ 2H+X_0-X_1)$, then any point $(X_n,\ X_{n+1})$ $(n \geq 0)$ in phase space always exists on sides of a rectangle with vertices $(H-K,\ -H)$, $(H,\ K-H)$, $(-H,\ H-K)$, $(K-H,\ H)$. If we use perturbed initial data $(X_0 + \rho_1,\ X_1 + \rho_2)$ where $\rho_1 \sim 0$ and $\rho_2 \sim 0$, then an orbit of $(X_n,\ X_{n+1})$ shifts infinitesimally but it is still stable.

To estimate a growth of perturbation, let us calculate amplitude of perturbation a_n by solutions X_n and X'_n which are the solutions from given initial data $(X_0,\ X_1)$ and $(X_0 + \rho,\ X_1)$, respectively. If we choose $H = 5$, $X_0 = 0$ and $X_1 = 1$, then X_n becomes periodic with period 24 and $a_{24n} = 20n - 1$. If we take $H = 10$, $X_0 = 11$ and $X_1 = 17$, period is 9 and $a_{9n} = 5n + 1$. In both cases, a_n grows linearly on n. These observations suggest that equation (18) with $\sigma = 2$ is integrable.

On the other hand, in the case of $\sigma = 3$, the growth of perturbation changes drastically. Choosing $H = 5$, $X_0 = 0$ and $X_1 = 1$, X_n becomes periodic with period 18 and $a_{18n} = 1,\ 901,\ 326101,\ 118047661,\ 42732927181, \cdots$. Clearly a_n shows an exponential growth and does not grow linearly on n. If we take $H = 10$, $X_0 = 11$ and $X_1 = 17$, period is now 132 and $a_{132n} = 1,\ 84020277977,\ 62976889591510213505 15,$ $4720394550119148871383960629622 53, \cdots$. Fig. 4 shows a figure of the solution orbit and Fig. 5 shows a relation between n and $\log a_{132n}$, both in the latter case. From Fig. 4, we can see that the orbit is more complicated than that in $\sigma = 2$ case. From Fig. 5, $a_{132n} \sim e^{25n}$ and $\lambda \sim 25/132$. Therefore, we can consider that solutions to equation (18) with $\sigma = 3$ become chaotic and thus equation (18) is not integrable.

4. Concluding remarks

In this letter, we obtained the following results.

(a) If parameters of ultradiscrete equation are all 0, that is, if corresponding parameters of difference equation are not singular, we can consistently transform singularity pattern of difference equation into ultradiscrete one. Finite value and singularity of solution to difference equation correspond to perturbation and finite value of ultradiscrete solution respectively. From other point of view, the SC analysis on difference equation can be regarded as perturbation analysis of solution on ultradiscrete equation.

(b) If parameters of ultradiscrete equation are not 0, that is, if corresponding parameters of difference equation are singular, we cannot consistently transform singularity pattern of difference equation into ultradiscrete one. However, we can observe a growth rate of perturbation, that is, Lyapunov exponent and check integrability by the exponent.

Moreover, we point out an advantage of the above test. In SC test on difference equation, we need asymptotic evaluation of a solution including small parameters. It often needs a large amount of symbolic manipulation and becomes hard even with help of a computer. If we can transform such a difference equation into ultradiscrete one, we easily calculate a solution pattern. Therefore, when we test integrability of difference equation, it is much easier to test a corresponding ultradiscrete equation instead.

Finally, we give future problems.

(i) Since integrability test described in (b) is generic, we can use it to any ultradiscrete equation. It is easy to conclude a given equation is chaotic if Lyapunov exponent of a solution from a particular initial data is positive. However, it is difficult to judge a given equation is integrable. Because we must check Lyapunov exponent is not positive for ANY initial data. To avoid this, we need a rule to select initial data to make the test finite.

(ii) As mentioned in the introduction, we get some cellular automata using ultradiscretization when we can restrict values of dependent variables in a range of integers. [13] We can test integrability of the cellular automata using perturbation. Therefore, intermediate values as well as discrete values are significant. However, cellular automata are normally defined by binary operations irrelevant to difference equations and we cannot obtain information on intermediate values. Thus, further considerations are required to study cellular automata using our approach.

(iii) It is remarkable that Hietarinta and Viallet reported that there are some difference equations that passes the SC test but not integrable. [17] Therefore, it is not necessarily true that given equation is integrable even if it passes the SC test. (Note that there is no example of integrable difference equation that does not pass the SC test.) They have proposed to use the criterion of algebraic entropy instead. Developing the notion of algebraic entropy for ultradiscrete equations, together with the development of symbolic manipulation package to deal with max-plus algebra, may be quite interesting and important.

Acknowledgement

The authors are grateful to Prof. Ryogo Hirota and Mr. Kinji Kimura for fruitful discussions and helpful comments. This work is partially supported by Grant-in-Aid from the Ministry of Education, Science and Culture (No. 09750087 and No. 09740164).

[1] See for example, *Painlevé Transcendents: Their Asymptotics and Physical Applications eds. by D. Leve and P. Winternitz* (NATO ASI series B278, Prenum, New York, 1992).

[2] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825

[3] Quispel G R W, Roberts J A G and Thompson C J 1988 *Phys. Lett.* **A126** 419

[4] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* **D34** 183

[5] Grammaticos B, Ohta Y, Ramani A, Takahashi D and Tamizhmani K M 1997 *Phys. Lett.* **A226** 53

[6] Takahashi D and Satsuma J 1990 *J. Phys. Soc. Jpn.* **59** 3514

[7] Takahashi D and Matsukidaira J 1995 *Phys. Lett.* **A209** 184

[8] Matsukidaira J, Satsuma J, Takahashi D, Tokihiro T and Torii M 1997 *Phys. Lett.* **A225** 287

[9] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 *Phys. Rev. Lett.* **76** 3247

[10] Takahashi D, Tokihiro T, Grammaticos B, Ohta Y and Ramani A 1997 *J. Phys.* **A30** 7953

[11] Takahashi D and Matsukidaira J 1997 *J. Phys.* **A30** L733

[12] Hirota R, Iwao R, Ramani A, Takahashi D, Grammaticos B and Ohta Y 1997 *Phys. Lett.* **A236** 39

[13] Nishinari K and Takahashi D 1998 *J. Phys.* **A31** 5439

[14] Ramani A, Takahashi D, Grammaticos B and Ohta Y 1988 *Physica* **D114** 185

[15] See for example, Gulick D 1992 *Encounters with Chaos* (McGraw-Hill, New York)

[16] Private communication with Hirota R and Kimura K

[17] Hietarinta J and Viallet C 1998 *Phys. Rev. Lett.* **81** 325

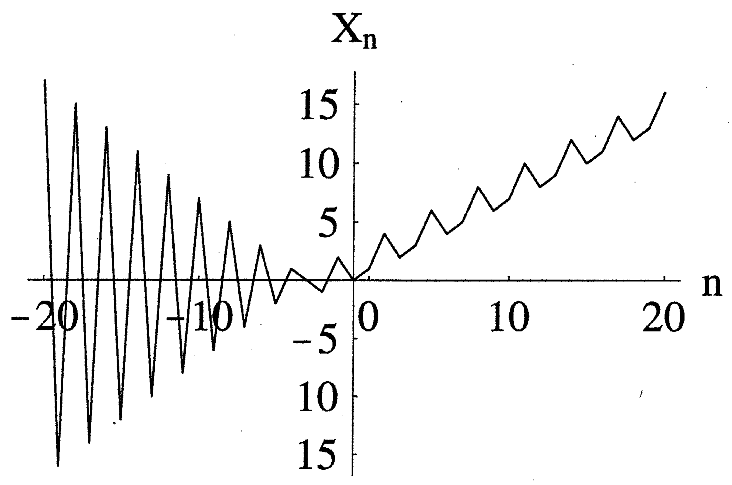

Figure 1. Plot of X_n to (15) for $A = 3$, $L = 2$, $X_0 = 0$ and $X_1 = 1$.

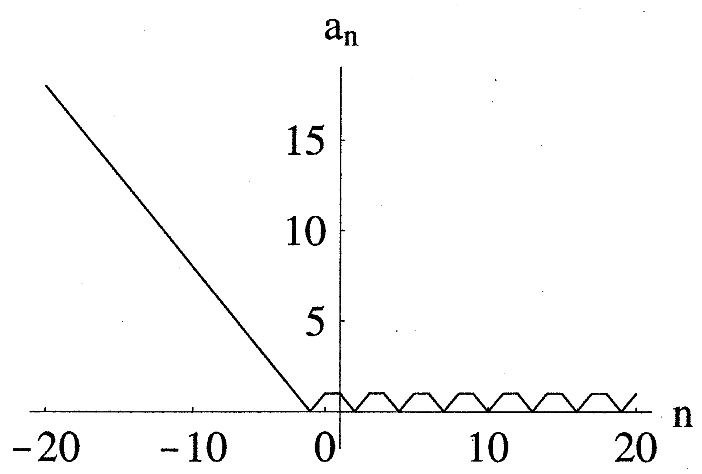

Figure 2. Plot of a_n to (15) for $A = 3$, $L = 2$, $X_0 = 0$, $X_0' = \rho$, and $X_1 = X_1' = 1$.

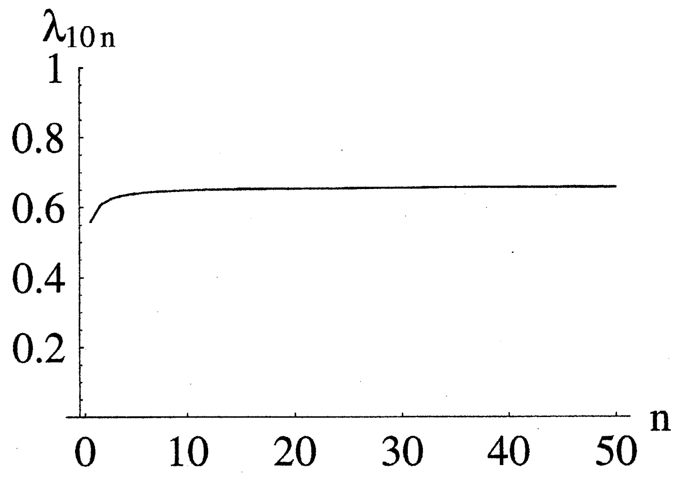

Figure 3. Plot of λ_{10n} to (13) for $X_0 = 0$, $X_0' = \rho$, and $X_1 = X_1' = 1$.

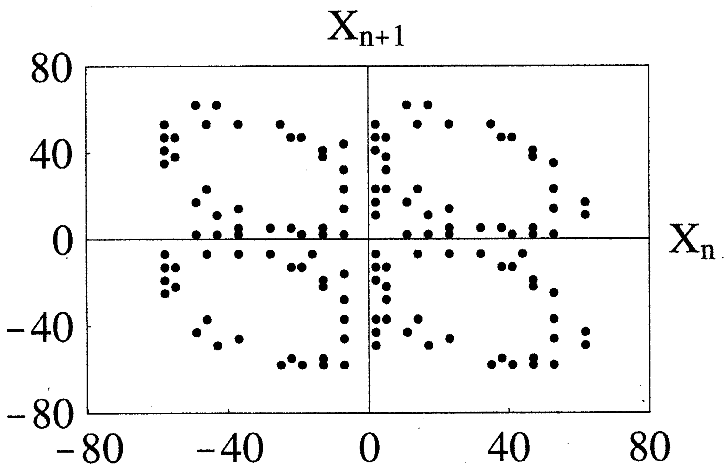

Figure 4. An orbit of (18) for $\sigma = 3$, $H = 10$, $X_0 = 11$, $X_1 = 17$. Period is 132.

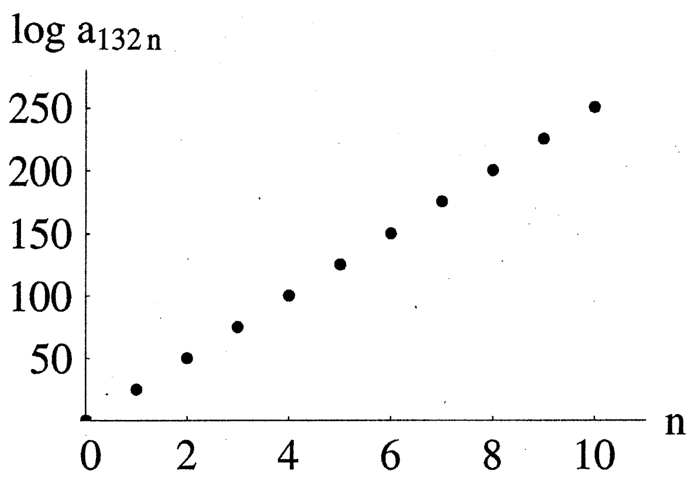

Figure 5. A logarithmic plot of a_{132n} to (18) for $\sigma = 3$, $H = 10$, $X_0 = 11$, $X'_0 = 11+\rho$, $X_1 = X'_1 = 17$. Since slope is about 25, $a_{132n} \sim e^{25n}$.