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# Shift Automorphisms of the Reduced Free Products of Infinitely Many Algebras

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## §0. Introduction.

In the class of $C^*$-algebras constructed by the reduced free product, the class of the reduced free products of infinitely many copies indexed by the integers $\mathbb{Z}$ is the smallest in the sense of Theorem 1 below.

For such the reduced free product $C^*$-algebras, the shift automorphism coming from the shift on integers are investigated.

## §1. Reduced free products.

The reduced free product construction for $C^*$-algebras was introduced independently by Avitzour [A] and Voiculescu [V].

By a $C^*$-probability space $(A, \phi)$ , we means that $A$ is a unital $C^*$-algebra (operator norm closed *-algebra of bounded operators on a Hilbert space) and $\phi$ is a state of $A$.

Given a family of $C^*$-probability space $(A_i, \phi_i)_{i\in I}$, where the GNS representation by each $\phi_i$ is assumed to be faithful, the reduced free product $(A, \phi)$ of $(A_i, \phi_i)_{i\in I}$ is denoted by

$$(A, \phi) = \underset{i\in I}{*}(A_i, \phi_i).$$

Let $H_i = L^2(A_i, \phi_i)$, and $\xi_i$ is the image vector of the unit of $A_i$ in $H_i$. Let $H_i^\circ = H_i \ominus \mathbb{C}\,\xi_i$. The free product Hilbert space $H$ of $(H_i, \xi_i)_{i\in I}$ is defined for the distinguished vector $\xi$ (which is called the vacuum vector) by the form :

$$H = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1}\Big(\bigoplus_{i_1\neq\cdots\neq i_n} H_{i_1}^\circ \otimes \cdots \otimes H_{i_n}^\circ\Big), \quad i_j \in I$$

and it is denoted by $(H, \xi) = \underset{i\in I}{*}(H_i, \xi_1)$.

Each $A_i$ acts faithfully on $H_i$ as the left multiplication operators. Let $\overset{\circ}{A}_i = \{x \in A_i; \phi_i(x) = 0\}$, and let

$$red(A) = \{x_1x_2\cdots x_n; x_j \in \overset{\circ}{A}_{i_j}, (i_1 \neq i_2 \neq \cdots \neq i_n),\}.$$

The $A$ in the reduced free product $(A, \phi) = (A_i, \phi_i)$ is the $C^*$-algebra on the reduced free product Hilbert space $H$, and $A$ is generated by the identity operator on $H$ and $red(A)$. The $\phi$ is the state of $A$, which is called the vacuum state and defined by

$$\phi(a) = < a\xi, \xi > .$$

How to operate an $x \in red(A)$ on $H$ ?

Assume that $x \in red(A)$ has a form that $x = x_1 x_2 \cdots x_n$, for $x_j \in \overset{\circ}{A}_{i_j}, (i_1 \neq i_2 \neq \cdots \neq i_n)$. Then

$$x\xi = x_1\xi_{i_1} \otimes x_2\xi_{i_2} \otimes \cdots \otimes x_n\xi_{i_n},$$

and for the vector

$$\zeta = \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_m, \quad \zeta_i \in \overset{\circ}{H}_{j_i}, \quad (i = 1, \cdots, m),$$

$$x\zeta = x\xi \otimes \zeta, \quad \text{for} \quad i_n \neq j_1,$$

$$x\zeta = (x_1\zeta_1 - < x_1\zeta_1, \xi_{i_1} > \xi_{i_1}) + < x_1\zeta_1, \xi_{i_1} > \xi, \quad \text{for} \quad i_n = j_1, n = m = 1$$

$$\begin{aligned} x\zeta = (x_1\zeta_1 - < x_1\zeta_1, \xi_{i_1} > \xi_{i_1}) \otimes \zeta_2 \otimes \cdots \otimes \zeta_m \\ + < x_1\zeta_1, \xi_{i_1} > \zeta_2 \otimes \cdots \otimes \zeta_m, \quad \text{for} \quad i_n = j_1, n = 1, m \geq 2 \end{aligned}$$

$$\begin{aligned} x\zeta = x_1\xi_{i_1} \otimes \cdots \otimes x_{n-1}\xi_{i_{n-1}} \otimes (x_n\zeta_1 - < x_n\zeta_1, \xi_{i_n} > \xi_{i_n}) \\ + < x_n\zeta_1, \xi_{i_n} > x_1\xi_{i_1} \otimes \cdots \otimes x_{i_{n-1}}\xi_{i_{n-1}}, \quad \text{for} \quad i_n = j_1, n \geq 2, m = 1 \end{aligned}$$

and

$$\begin{aligned} x\zeta = x_1\xi_{i_1} \otimes \cdots \otimes x_{i_{n-1}}\xi_{i_{n-1}} \otimes (x_n\zeta_1 - < x_n\zeta_1, \xi_{i_n} > \xi_{i_n}) \otimes \zeta_2 \otimes \cdots \otimes \zeta_m \\ + < x_n\zeta_1, \xi_{i_n} > x_1 \cdots x_{n-1}(\zeta_{j_2} \cdots \otimes \zeta_{j_m}), \quad \text{for} \quad i_n = j_1, n, m \geq 2. \end{aligned}$$

The characterization for the reduced free product of $C^*$-algebras is given in [V1] as follows :

A given $(A, \phi)$ is isomorphic to $\underset{i \in I}{*}(A_i, \phi_i)$ if and only if there exists an isomorphism $\sigma_i : A_i \to A$ such that

(i) $A$ is generated by $\{\sigma_i(A_i)\}_{i \in I}$,

(ii) $\phi \cdot \sigma_i = \phi_i$,

(iii) $\phi(\sigma_{i_1}(a_{i_1}) \cdots \sigma_{i_n}(a_{i_n})) = 0$ for $a_{i_1} \in A^{\circ}_{i_j}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$,

and the GNS construction applied to $(A, \phi)$ yields a faithful representation of $A$.

## §2. Free products of infinitely many $C^*$-algebras.

The most typical example of reduced free product $C^*$-algebra appears as the $C^*$-algebra generated by the left regular representation for the free product of some groups.

Remember that the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ of the group $\mathbb{Z}_2$ is amenable and not a free group. To get a $C^*$-algebra belonging to a "new class" from the reduced free product construction $(A_1, \phi_1) * (A_2, \phi_2)$, it is offen used so called *Arvitzour's unitary condition* :

$$\exists \quad \text{unitaries} \quad a, b, c \quad \text{with} \quad a \in \overset{\circ}{A}_1, \quad b, \ c, \ b^*c \in \overset{\circ}{A}_2.$$

In the class of $C^*$-algebras constructed by the reduced free product, the class of the reduced free products of infinitely many copies is the smallest in the following sense :

**Theorem 1.** *Assume that the pair* $\{(A_1,\phi_1),(A_2,\phi_2)\}$ *satisfies Arvitzour's unitary condition. Then*

$$(A_1,\phi_1)*(A_2,\phi_2)\supset \mathop{*}_{i\in\mathbb{Z}}(B_i,\psi_i),$$

*where*

$$(B_i,\psi_i)=(A_1,\phi_1)*(A_2,\phi_2),\quad \textit{for all}\quad i\in\mathbb{Z}.$$

Let $(A_i,\phi_i)$ be the copy of a $C^*$-dynamical system $(A_0,\phi_0)$ for all $i\in\mathbb{Z}$. The reduced free product $(A,\phi)=\mathop{*}_{i\in\mathbb{Z}}(A_i,\psi_i)$ has the automorphism $\alpha$ induced by the shift $i\to i+1$ on the integers $\mathbb{Z}$. We call such the automorphism $\alpha$ on $A$ *free shift*.

As an example of a more precise inclusion of Theorem 1, we have the following :

**Example** (cf. [CD]). Let $(A_1,\phi_1)$ be a non-trivial $C^*$-dynamical sysyem, that is, $A_1\neq\mathbb{C}$. Then

$$(A_1,\phi_1)*(C(\mathbb{T}),\int\cdot dt)=\mathop{*}_{i\in\mathbb{Z}}(B_i,\psi_i)\rtimes_\alpha\mathbb{Z}\ \text{the crossed product},$$

for the copy $(B_i,\psi_i)$ of $(A_1,\phi_1)*(C(\mathbb{T}),\int\cdot dt)$, $(i\in\mathbb{Z})$. Here $C(\mathbb{T})$ is the continuous functions on the Torus $\mathbb{T}$, $\int\cdot dt$ is the state given by the integral, and $\alpha$ is the free shift.

**Proposition 2.** *Let* $(A,\phi)=\mathop{*}_{i\in\mathbb{Z}}(A_i,\psi_i)$, *where* $(A_i,\phi_i)=(A_0,\phi_0)$ *for all* $i\in\mathbb{Z}$. *Then the free shift* $\alpha$ *on* $A$ *gives an algebraic K-system in the sense of Narnhofer-Thirring [NT], that is, there exists a* $C^*$*-subalgera* $B\subset A$ *with* $B\subset\alpha(B)$, $A$ *is generated by the family* $\{\alpha^n(B)\}_{n\in BZ}$, *and* $\cap_n\alpha^n(B)=\mathbb{C}1$.

## §3. Entropies for the free shift.

By a $C^*$-dynamical system $(A,\alpha,\phi)$, we mean that $A$ is a unital $C^*$-algebra, $\alpha$ is an automorphism, and $\phi$ is an $\alpha$-invariant state of $A$. For a given $C^*$-dynamical system $(A,\alpha,\phi)$, we denote by $h_\phi(\alpha)$ the Connes-Narnhofer-Thirring entropy ([CNT]), by $H_\phi(\alpha)$ the Sauvageot-Thouveneot entropy ([ST]), and by $H_{(\phi,\Phi,\mathcal{A}_0)}$ Alicki-Fannes entropy ([Af]) which is defined with respect to a glovally $\alpha$-invariant *-subalgebra $\mathcal{A}_0$ of $A$.

**Theorem 3.** *Let* $(A,\phi)=\mathop{*}_{i\in\mathbb{Z}}(A_i,\psi_i)$, *where* $(A_i,\phi_i)=(A_0,\phi_0)$ *for all* $i\in\mathbb{Z}$, *and let* $\alpha$ *be the free shift on* $A$.

*(i) ([C1]) Sauvageot-Thouveneot entropy has the relation*

$$H_{\phi*\psi}(\alpha*\beta)=H_\psi(\beta)=H_{\phi\otimes\psi}(\alpha\otimes\beta),$$

*for any* $C^*$*-dynamical system* $(B,\psi,\beta)$. *In special,* $H_\phi(\alpha)=0$.

*(ii) ( [CT]) Let* $\mathcal{A}_0\subset A$ *be the natural* $\alpha$*-invariant *-subalgebra generated by the identity and* $red(A)$. *Then Alicki-Fannes entropy has*

$$H_{(\phi,\alpha,\mathcal{A}_0)}=+\infty.$$

In the case where the $C^*$-algebra is nuclear, Sauvageot-Thouveneot entropy coinsides with Connes-Narnhofer-Thirring entropy ([ST]).

The notion of "nuclear" for $C^*$-algebras is corresponds to that of "amenable" for groups. In general, the free product of groups is not amenable, and the reduced free product $C^*$-algebra is not nuclear.

However there exista a nuclear $C^*$-algebra which is given as the reduced free product of $\mathbb{Z}$ copies of an algebra. As an example, we have the Cuntz algebra $\mathcal{O}_\infty$.

**Remark.** Stormer([S]) showed that $h_\phi(\alpha) = 0$ for any free shift $\alpha$.

As another type of a $C^*$-dynamical entropy, we have a slight modification of Voiculescu's topological entropy $ht(\cdot)$ ([V2]). The definition is as follows :

**Definition** ([C2]). For a nuclear $C^*$-algebra $A$ with unity, let $CPA(A)$ be the set of tripple $(\varphi, \eta, C)$ such that $C$ is a finite dimensional $C^*$-algebra, and $\varphi : A \to C$ and $\eta : C \to A$ are unital completely positive maps.

Let $\Omega$ be the set of finite subsets of $A$ and let $\phi$ be a state of $A$. For an $\omega \in \Omega$, put

$$scp_\phi(\omega; \delta) = \inf\{S(\phi \cdot \eta) : (\varphi, \eta, C) \in CPA(A), \|\eta \cdot \varphi(a) - a\| < \delta, a \in \omega\}.$$

Here $S(\phi \cdot \eta)$ means the entropy of the state $\phi \cdot \eta$ of $C$. For a unital endomorphism $\rho$ of $A$ with $\phi \cdot \rho = \phi$, put

$$ht_\phi(\rho, \omega\ ; \delta) = \overline{\lim_{\mathrm{N}\to\infty}} \frac{1}{\mathrm{N}} scp_\phi(\bigcup_{i=0}^{N-1} \rho^i(\omega); \delta)$$

$$ht_\phi(\rho, \omega) = \sup_{\delta>0} ht_\phi(\rho, \omega\ ; \delta).$$

Then the entropy $ht_\phi(\rho)$ of $\rho$ is defined by

$$ht_\phi(\rho) = \sup_{\omega\in\Omega} ht_\phi(\rho, \omega).$$

**Proposition 4.** *([C2]) For a $\phi$-preserving automorphism $\alpha$ of $A$, Connes - Narnhofer - Thirring entropy $h_\phi(\alpha)$, the entropy $ht_\phi(\alpha)$ and Voiculescu's topological entropy $ht(\alpha)$ have in general the following relation :*

$$h_\phi(\alpha) \le ht_\phi(\alpha) \le ht(\alpha).$$

**Remark.** ([C2]) We have exmples that $h_\phi(\alpha) \neq ht_\phi(\alpha)$ or $ht_\phi(\alpha) \neq ht(\alpha)$.

Since $\mathcal{O}_\infty$ is nuclear, we can apply the entropy $ht_\phi(\alpha)$ for the free shift on $\mathcal{O}_\infty$.

**Proposition 5.** *Let $\alpha$ be the free shift of $\mathcal{O}_\infty$. Then*

$$ht_\phi(\alpha) = 0,$$

*for the vacuun state $\phi$ which is the unique $\alpha$-invariant state of $\mathcal{O}_\infty$.*

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