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A Class of Dirac-Type Operators on the Abstract Boson-Fermion Fock Space and Their Strong Anticommutativity

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1 Introduction

In a previous paper [4], we introduced a family $\{Q_S|S \in \mathcal{C}(\mathcal{H},\mathcal{K})\}$ of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space $\mathcal{F}(\mathcal{H},\mathcal{K})$ over the pair $\langle\mathcal{H},\mathcal{K}\rangle$ of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, where the index set $\mathcal{C}(\mathcal{H},\mathcal{K})$ of the family is the set of all densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$, and investigated fundamental properties of them. As is shown in [4], this class of Dirac-type operators has a connection with supersymmetric quantum field theory (SQFT) [19]. Namely Q_S gives an abstract form of free supercharges in some models of SQFT. Interacting models of SQFT can be constructed from perturbations of Q_S [4]. For related aspects and further developments, see, e.g., [1], [2], [3], [5], [6], [10], [14], [16], [17], [20], [21].

Generally speaking, Dirac-type operators have something to do with a notion of anticommutativity, because they are related to representations of Clifford algebras, and this aspect may be an essential feature of Dirac-type operators (cf. [7], [8], [9], [11], [12]). A proper notion of anticommutativity, i.e., *strong anticommutativity*, of (unbounded) self-adjoint operators was given in [27] and developed by some authors (e.g., [25], [22], [7], [9], [11], [12]). In a recent papr [15], a theorem on the strong anticommutativity of two Dirac operators Q_S and Q_T was established with application to constructing representations on $\mathcal{F}(\mathcal{H},\mathcal{K})$ of a supersymmetry algebra arising in a two-dimensional relativistic SQFT.

The aim of this note is to review fundamental aspects of the theory of infinite dimensional Dirac-type operators on the abstract Boson-Fermion Fock space and to present a summary of the results on their strong anticommutativity obtained in [15].

2 Dirac-type operators on the abstract Boson-Fermion Fock space—a brief review

Let $\mathcal{H}$ be a Hilbert space and $\otimes^n\mathcal{H}$ be the n-fold tensor product Hilbert space of $\mathcal{H}$ ($n = 0, 1, 2, \cdots$; $\otimes^0(\mathcal{H}) := \mathbb{C}$). We denote by S_n (resp. A_n) the symmetrizer (resp. the anti-symmetrizer) on $\otimes^n\mathcal{H}$ and by $S_n(\otimes^n\mathcal{H})$ (resp. $A_n(\otimes^n\mathcal{H})$) its range, which is called the n-fold symmetric (resp. anti-symmetric) tensor product of $\mathcal{H}$. The *Boson Fock space* $\mathcal{F}_{\rm b}(\mathcal{H})$ and the *Fermion Fock space* $\mathcal{F}_{\rm f}(\mathcal{H})$ over $\mathcal{H}$ are respectively defined by

$$\mathcal{F}_{\rm b}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} S_n(\otimes^n\mathcal{H}), \quad \mathcal{F}_{\rm f}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} A_n(\otimes^n\mathcal{H}) \tag{2.1}$$

(e.g., [23, §II.4], [18, §5.2]). Let $\mathcal{K}$ be a Hilbert space. Then the *Boson-Fermion Fock space* $\mathcal{F}(\mathcal{H},\mathcal{K})$ associated with the pair $\langle\mathcal{H},\mathcal{K}\rangle$ is defined by

$$\mathcal{F}(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\rm b}(\mathcal{H}) \otimes \mathcal{F}_{\rm f}(\mathcal{K}), \tag{2.2}$$

the tensor product Hilbert space of the Boson Fock space over $\mathcal{H}$ and the Fermion Fock space over $\mathcal{K}$. We denote by $\mathcal{C}(\mathcal{H},\mathcal{K})$ the set of densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$.

We first present the definitions of basics objects in the Boson Fock space and the Fermion Fock space. More detailed descriptions on Fock space objects can be found, e.g., in [23, §II.4, Example 2], [24, §X.7] and [18, §5.2].

For each vector $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm b}(\mathcal{H})$ ($\Psi^{(n)} \in S_n(\otimes^n\mathcal{H})$), we use the natural identification of $\Psi^{(n)}$ with $\{0,\cdots,0,\Psi^{(n)},0,\cdots\} \in \mathcal{F}_{\rm b}(\mathcal{H})$. The same applies to vectors in other inifinite direct sums of Hilbert spaces.

For a subset V of a Hilbert space, we denote by $\mathcal{L}V$ the subspace algebraically spanned by all the vectors of V.

Let $\Omega_{\rm b} := \{1,0,0,\cdots\} \in \mathcal{F}_{\rm b}(\mathcal{H})$, the *boson Fock vacuum* in $\mathcal{F}_{\rm b}(\mathcal{H})$. For a subspace $\mathcal{D}$ of $\mathcal{H}$, we define

$$\mathcal{F}_{\rm b,fin}(\mathcal{D}) := \mathcal{L}\left\{\Omega_{\rm b},\, S_n(f_1 \otimes \cdots \otimes f_n) | n \in \mathbb{N},\ f_j \in \mathcal{D},\ j = 1,\cdots,n\right\}. \tag{2.3}$$

If $\mathcal{D}$ is dense, then $\mathcal{F}_{\rm b,fin}(\mathcal{D})$ is dense in $\mathcal{F}_{\rm b}(\mathcal{H})$.

For each $f \in \mathcal{H}$, there exists a unique densely defined closed (unbounded) linear operator $a(f)$ on $\mathcal{F}_{\rm b}(\mathcal{H})$, called *boson annihilation operators* (its adjoint $a(f)^*$ is called a *boson creation operator*), such that (i) for all $f \in \mathcal{H}$, $a(f)\Omega_{\rm b} = 0$, (ii) for all $n \in \mathbb{N}$, $f_j \in \mathcal{H}$, $j = 1,\cdots,n$,

$$a(f)S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(f,f_j)_{\mathcal{H}} S_{n-1}(f_1 \otimes \cdots \otimes \hat{f}_j \otimes \cdots \otimes f_n),$$

where $\hat{f}_j$ indicates the omission of f_j, and (iii) $\mathcal{F}_{\rm b,fin}(\mathcal{H})$ is a core of $a(f)$. We have

$$S_n(\otimes^n\mathcal{H}) = \overline{\mathcal{L}(\{a(f_1)^*\cdots a(f_n)^*\Omega_{\rm b} | f_j \in \mathcal{H},\ j = 1,\cdots,n\})}, \tag{2.4}$$

where $\overline{\{\cdot\}}$ denotes the closure of the set $\{\cdot\}$. The set $\{a(f),\, a(f)^* | f \in \mathcal{H}\}$ satisfies the canonical commutation relations

$$[a(f), a(g)^*] = (f,g)_{\mathcal{H}}, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{H}$ on $\mathcal{F}_{\mathrm{b,fin}}(\mathcal{H})$.

A similar consideration can be done in the Fermion Fock space $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$. The *fermion Fock vacuum* Ω_{f} in $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ is defined by $\Omega_{\mathrm{f}} := \{1, 0, 0, \cdots\} \in \mathcal{F}_{\mathrm{b}}(\mathcal{K})$. For a subspace $\mathcal{D}$ of $\mathcal{K}$, we define

$$\mathcal{F}_{\mathrm{f,fin}}(\mathcal{D}) := \mathcal{L}\{\Omega_{\mathrm{f}},\ A_n(u_1 \otimes \cdots \otimes u_n) | n \geq 1,\ u_j \in \mathcal{D},\ j = 1, \cdots, n\}. \tag{2.5}$$

If $\mathcal{D}$ is dense, then $\mathcal{F}_{\mathrm{f,fin}}(\mathcal{D})$ is dense in $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$.

For each $u \in \mathcal{K}$, there exists a unique bounded linear operator $b(u)$ on $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$, called *fermion annihilation operators* on $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ ($b(u)^*$ is called a *fermion creation operator*), such that (i) for all $u \in \mathcal{K}$, $b(u)\Omega_{\mathrm{b}} = 0$, (ii) for all $n \in \mathbb{N}$, $u_j \in \mathcal{K}$, $j = 1, \cdots, n$

$$b(u)A_n(u_1 \otimes \cdots \otimes u_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j-1} (u, u_j)_{\mathcal{H}} S_{n-1}(u_1 \otimes \cdots \otimes \hat{u}_j \otimes \cdots \otimes u_n).$$

We have

$$A_n(\otimes^n \mathcal{K}) = \overline{\mathcal{L}\{b(u_1)^* \cdots b(u_n)^* \Omega_{\mathrm{f}} | u_j \in \mathcal{K},\ j = 1, \cdots, n\}}. \tag{2.6}$$

The set $\{b(u),\ b(u)^* | u \in \mathcal{K}\}$ satisfies tha canonical anti-commutation relations

$$\{b(u), b(v)^*\} = (u, v)_{\mathcal{K}}, \quad \{b(u), b(v)\} = 0, \quad \{b(u)^*, b(v)^*\} = 0$$

for all $u, v \in \mathcal{K}$, where $\{A, B\} := AB + BA$.

The *Fock vacuum in the Boson -Fermion Fock space* $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is defined by

$$\Omega := \Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}}. \tag{2.7}$$

The annihilation operators $a(f)$ and $b(u)$ are extended to operators on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ as

$$A(f) := a(f) \otimes I, \quad B(u) := I \otimes b(u), \tag{2.8}$$

where I denotes identity operator.

For a linear operator A, we denote by $D(A)$ its domain. Let $S \in \mathcal{C}(\mathcal{H}, \mathcal{K})$. Then we define

$$\mathcal{D}_S := \mathcal{L}\left\{A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega \,\middle|\, n, p \geq 0, f_j \in D(S), \right. \tag{2.9}$$
$$\left. j = 1, \cdots, n,\ u_k \in D(S^*), k = 1, \cdots, p\right\},$$
$$= \mathcal{F}_{\mathrm{b,fin}}(D(S)) \otimes_{\mathrm{alg}} \mathcal{F}_{\mathrm{f,fin}}(D(S^*)), \tag{2.10}$$

where $\otimes_{\mathrm{alg}}$ denotes algebraic tensor product. It follows that $\mathcal{D}_S$ is dense in $\mathcal{F}$. The following proposition is proved in [4].

Proposition 2.1 *There exists a unique densely defined closed linear operator d_S on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ with the following properties: (i) $\mathcal{D}_S$ is a core of d_S; (ii) for each vector $\Psi \in \mathcal{D}_S$ of the form*

$$\Psi = A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega, \tag{2.11}$$

d_S acts as

$$d_S\Psi = 0 \quad \text{for } n = 0,$$
$$d_S\Psi = \sum_{j=1}^{n} A(f_1)^* \cdots \widehat{A(f_j)}^* \cdots A(f_n)^* B(Sf_j)^* B(u_1)^* \cdots B(u_p)^*\Omega \quad \text{for } n \geq 1,$$

where $\widehat{A(f_j)}^*$ indicates the omission of $A(f_j)^*$. Moreover the following (a)-(d) *hold:*

(a) $d_S^2 = 0$.

(b) *For each complete orthonormal system (CONS) $\{e_n\}_{n=1}^\infty$ of $\mathcal{K}$ with $e_n \in D(S^*)$,*

$$d_S\Psi = \sum_{n=1}^{\infty} A(S^*e_n)B(e_n)^*\Psi, \quad \Psi \in \mathcal{D}_S,$$

where the convergence is taken in the strong topology of $\mathcal{F}(\mathcal{H}, \mathcal{K})$.

(c) *For each CONS $\{\phi_n\}_{n=1}^\infty$ of $\mathcal{H}$ with $\phi_n \in D(S)$, we have*

$$(\Phi, d_S\Psi)_{\mathcal{F}(\mathcal{H},\mathcal{K})} = \lim_{N\to\infty} \left(\Phi, \sum_{n=1}^{N} A(\phi_n)B(S\phi_n)^*\Psi\right)_{\mathcal{F}(\mathcal{H},\mathcal{K})}, \quad \Phi, \Psi \in \mathcal{D}_S.$$

(d) *$\mathcal{D}_S \subset D(d_S^*)$ and*

$$d_S^*\Psi = \sum_{k=1}^{p} (-1)^{k-1} A(S^*u_k)^* A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots \widehat{B(u_k)}^* \cdots B(u_p)^*\Omega$$

for vectors Ψ of the form (2.11) with $p \geq 1$. In the case $p = 0$, we have $d_S^*\Psi = 0$.

A Dirac-type operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ is defined by

$$Q_S = d_S + d_S^* \tag{2.12}$$

with $D(Q_S) = D(d_S) \cap D(d_S^*)$.

Let A be a self-adjoint operator on a Hilbert space $\mathcal{X}$. Then there is a unique self-adjoint operator A_n on $\otimes^n\mathcal{X}$ such that $\otimes_{\text{alg}}^n D(A)$ is a core of $D(A_n)$ and, for all $f_j \in D(A)$, $j = 1, \cdots, n$, $A_n(f_1 \otimes \cdots \otimes f_n) = \sum_{j=1}^n f_1 \otimes \cdots \otimes f_{j-1} \otimes Af_j \otimes f_{j+1} \otimes \cdots \otimes f_n$ ([23, §VIII.10, Corollary]). Putting $A_0 = 0$, one can define a self-adjoint operator

$$d\Gamma(A) := \bigoplus_{n=0}^{\infty} A_n \tag{2.13}$$

on $\bigoplus_{n=0}^\infty \otimes^n\mathcal{X}$, called the *second quantization* of A ([23, §VIII. 10, Example 2], [18, §5.2]). It is easy to show that $d\Gamma(A)$ is reduced by $\mathcal{F}_\#(\mathcal{X})$ ($\# = \text{b}, \text{f}$). We denote the reduced part of $d\Gamma(A)$ to $\mathcal{F}_\#(\mathcal{X})$ by $d\Gamma_\#(A)$. We put

$$N_\# := d\Gamma_\#(I), \tag{2.14}$$

called the *number operator* on $\mathcal{F}_{\#}(\mathcal{X})$.

Let

$$\Gamma_{\#} = (-1)^{I \otimes N_{\#}}. \tag{2.15}$$

We introduce an operator

$$\Delta_S := d\Gamma_{\rm b}(S^*S) \otimes I + I \otimes d\Gamma_{\rm f}(SS^*) \tag{2.16}$$

acting in $\mathcal{F}(\mathcal{H},\mathcal{K})$, which is nonegative and self-adjoint (cf. [23, §VIII.10, Corollary]). For a linear operator A on a Hilbert space, we set

$$C^\infty(A) := \cap_{n=1}^\infty D(A^n).$$

Let

$$\begin{aligned} &\mathcal{D}_S^\infty \\ &= \mathcal{L}\Big\{ A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_p)^* \Omega \Big| n, p \geq 0,\ f_j \in C^\infty(S^*S), \\ &\qquad j = 1, \cdots, n,\ u_k \in C^\infty(SS^*),\ k = 1, \cdots, p \Big\}. \end{aligned} \tag{2.17}$$

Theorem 2.2 [4]

(i) *The operator Q_S is self-adjoint, and essentially self-adjoint on every core of Δ_S. In particular, Q_S is essentially self-adjoint on $\mathcal{D}_S^\infty$.*

(ii) *The operator $\Gamma_{\#}$ leaves $D(Q_S)$ invariant and*

$$\Gamma_{\#} Q_S + Q_S \Gamma_{\#} = 0$$

on $D(Q_S)$.

(iii) *The following operator equations hold :*

$$\Delta_S = Q_S^2 = d_S^* d_S + d_S d_S^*.$$

Remark 2.1 *The operators d_S and d_S^* leave $\mathcal{D}_S^\infty$ invariant and so does Q_S.*

Because of part (iii) of Theorem 2.2, we call the operator Δ_S the *Laplacian* associated with the Dirac-type operator Q_S.

3 Strong anticommutativity of the Dirac-type operators

Let A and B be self-adjoint operators on a Hilbert space. We say that A and B *strongly commute* if their spectral measures commute. On the other hand, A and B are said to *strongly anticommute* if $e^{itB}A \subset Ae^{-itB}$ for all $t \in \mathbb{R}$ ([27], [22])[1]. It turns out that this definition is symmetric in A and B [22].

For various Dirac-type operators, the notion of strong anticommutativity plays an important role ([7], [8], [10], [11]).

For each $S \in \mathcal{C}(\mathcal{H},\mathcal{K})$, the operator

$$L_S := \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \tag{3.1}$$

acting in $\mathcal{H} \bigoplus \mathcal{K}$ is self-adjoint. This operator is an abstract Dirac operator on the Hilbert space $\mathcal{H} \bigoplus \mathcal{K}$[26, Chapter 5].

The strong anticommutativity of Q_S and Q_T ($S, T \in \mathcal{C}(\mathcal{H},\mathcal{K})$) is characterized as follows.

Theorem 3.1 *Let $S,T \in \mathcal{C}(\mathcal{H},\mathcal{K})$. Then Q_S and Q_T strongly anticommute if and only if L_S and L_T strongly anticommute. In that case, $S \pm T \in \mathcal{C}(\mathcal{H},\mathcal{K})$ and $Q_{S\pm T} = Q_S \pm Q_T$.*

This theorem is one of the main results of the paper [15], which establishes a beautiful correspondence between the strong anticommutativity of L_S and L_T and that of Q_S and Q_T.

To prove Theorem 3.1, we need some fundamental facts in the theory of strongly anticommuting self-adjoint operators [27, 22] as well as its applications, together with the following lemma. For the details, see [15].

Lemma 3.2 *Let $S,T \in \mathcal{C}(\mathcal{H},\mathcal{K})$. Suppose that L_S and L_T strongly anticommute. Then the following* (i)-(v) *hold:*

(i) $S \pm T \in \mathcal{C}(\mathcal{H},\mathcal{K})$ *and* $(S \pm T)^* = S^* \pm T^*$.

(ii) $|S|$ *and* $|T|$ *strongly commute.*

(iii) $|S^*|$ *and* $|T^*|$ *strongly commute.*

(iv) $D(S^*S) \cap D(T^*T) \subset D(T^*S) \cap D(S^*T)$ *and, for all* $f \in D(S^*S) \cap D(T^*T)$,

$$(T^*S + S^*T)f = 0.$$

(v) $D(SS^*) \cap D(TT^*) \subset D(TS^*) \cap D(ST^*)$ *and, for all* $u \in D(SS^*) \cap D(TT^*)$,

$$(TS^* + ST^*)u = 0.$$

[1]The authors of [27] and [22] call this notion simply anticommutativity, but, to be definite, we call it *strong* anticommutativity.

In terms of S and T, a necessary and sufficient condtion for L_S and L_T to strongly anticommute is given as follows.

Proposition 3.3 *Let $S,T \in \mathcal{C}(\mathcal{H},\mathcal{K})$. Then L_S and L_T strongly anticommute if and only if the folloiwng* (i) *and* (ii) *hold:*

(i) $S \pm T \in \mathcal{C}(\mathcal{H},\mathcal{K})$ *and* $(S \pm T)^* = S^* \pm T^*$.

(ii) *For all* $f,g \in D(S) \cap D(T)$ *and* $u,v \in D(S^*) \cap D(T^*)$,

$$(Sf,Tg) + (Tf,Sg) = 0, \quad (S^*u,T^*v) + (T^*u,S^*v) = 0.$$

4 Application to constructing representations of a supersymmetry algebra

We consider Fock space representations of the algebra $\mathcal{A}_{\mathrm{SUSY}}$ generated by four elements Q_1, Q_2, H, P with defining relations

$$Q_1^2 = H + P, \quad Q_2^2 = H - P, \quad Q_1Q_2 + Q_2Q_1 = 0. \tag{4.1}$$

This algebra is called a *supersymmetry algebra*, which arises in a relativistic SQFT in the two-dimensional space-time ([19], [13]). The elements H, P and Q_j ($j = 1,2$) are called the *Hamiltonian*, the *momentum operator* and the *supercharge*, respectively.

We recall a definition from [13]. Let $\mathcal{F}$ be a Hilbert space, $\mathcal{D}$ a dense subspace of $\mathcal{F}$, and H,P,Q_1,Q_2 be linear operators on $\mathcal{F}$. We say that $\{\mathcal{F},\mathcal{D},H,P,Q_1,Q_2\}$ is a *symmetric representation* of $\mathcal{A}_{\mathrm{SUSY}}$ if H,P,Q_1 and Q_2 are symmetric and leave $\mathcal{D}$ invariant satisfying (4.1) on $\mathcal{D}$. A symmetric representation $\{\mathcal{F},\mathcal{D},H,P,Q_1,Q_2\}$ of $\mathcal{A}_{\mathrm{SUSY}}$ is said to be *integrable* if (i) H,P,Q_1 and Q_2 are essentially self-adjoint (denote their closures by $\bar{H},\bar{P},\bar{Q}_1$ and $\bar{Q}_2$, respectively); (ii) $\{\bar{H},\bar{P},\bar{Q}_1\}$ and $\{\bar{H},\bar{P},\bar{Q}_2\}$ are families of strongly commuting self-adjoint operators, respectively; (iii) $\bar{H}$ and $\bar{P}$ satisfy the *relativistic spectral condition*

$$\pm \bar{P} \le \bar{H}. \tag{4.2}$$

Suppose that L_S and L_T strongly anticommute. Then, by Lemma 3.3(ii) and (iii), S^*S and T^*T strongly commute, and SS^* and TT^* strongly commute. Hence $S^*S + T^*T$ and $SS^* + TT^*$ are nonnegative, self-adjoint, and $S^*S - T^*T$ and $SS^* - TT^*$ are essentially self-adjoint. Therefore we can define self-adjoint operators

$$H_{S,T} := \frac{1}{2}\{d\Gamma_{\mathrm{b}}(S^*S + T^*T) \otimes I + I \otimes d\Gamma_{\mathrm{f}} SS^* + TT^*)\}, \tag{4.3}$$

$$P_{S,T} := \frac{1}{2}\left\{d\Gamma_{\mathrm{b}}(\overline{S^*S - T^*T}) \otimes I + I \otimes d\Gamma_{\mathrm{f}}(\overline{SS^* - TT^*})\right\}^{-} \tag{4.4}$$

where for a closable linear operator A, $\bar{A}$ (or A^-) denotes its closure. Note that $H_{S,T}$ is nonnegative, but, $P_{S,T}$ may be neither bounded below nor bounded above.

For a self-adjoint operator A, we denote by E_A its spectral measure. Let

$$\mathcal{D}_{S,T} := \mathcal{L}\{E_{|Q_S|}([a,b])E_{|Q_T|}([c,d])\Psi | \Psi \in \mathcal{F}(\mathcal{H},\mathcal{K}), 0 \le a < b < \infty, 0 \le c < d < \infty\}. \tag{4.5}$$

We can prove the following theorem (for the proof, see [15]).

Theorem 4.1 *Let $S,T \in \mathcal{C}(\mathcal{H},\mathcal{K})$ and suppose that L_S and L_T strongly anticommute. Then $\{\mathcal{F}(\mathcal{H},\mathcal{K}),\mathcal{D}_{S,T},H_{S,T},P_{S,T},Q_S,Q_T\}$ is an integrable representation of $\mathcal{A}_{\mathrm{SUSY}}$.*

We give only one basic example from SQFT (for other examples, see [19], [4]).

Example Let $\mathcal{H} = \mathcal{K} = L^2(\mathbb{R})$ and $\mathbb{R} \ni p \to \omega(p)$ be a nonnegative function on $\mathbb{R}$ which is Borel measurable, almost everywhere (a.e.) finite with respect to the Lebesgue measure on $\mathbb{R}$, and satisfies

$$|p| \leq \omega(p), \quad \text{a.e.} p \in \mathbb{R}.$$

Let

$$\nu(p) = \sqrt{\lambda p + \omega(p)}$$

with $\lambda \in [0,1]$ (a constant parameter) and $\theta(p)$ be an a.e. finite real-valued Borel measurable function on $\mathbb{R}$. Define the operators S and T on $L^2(\mathbb{R})$ to be the multiplication operators by the functions

$$S(p) := i\nu(p)e^{i\theta(p)}, \quad T(p) := \nu(-p)e^{i\theta(p)},$$

respectively. Then it is easy to see that S and T satisfy the conditions (i) and (ii) in Proposition 3.3 with $D(T) = D(S) = D(S^*) = D(T^*)$ and

$$\begin{aligned} S^*S &= SS^* = \lambda p + \omega, \quad T^*T = TT^* = -\lambda p + \omega, \\ S^*T &= TS^* = -i\sqrt{\omega^2 - \lambda^2 p^2}, \quad T^*S = ST^* = i\sqrt{\omega^2 - \lambda^2 p^2}. \end{aligned}$$

Hence, by Proposition 3.3, L_S and L_T strongly anticommute. Therefore, by Theorem 4.1, $\{\mathcal{F}(L^2(\mathbb{R}),L^2(\mathbb{R})),\mathcal{D}_{S,T},H_{S,T},P_{S,T},Q_S,Q_T\}$ with these S and T is an integrable representation of $\mathcal{A}_{\mathrm{SUSY}}$. We have

$$\begin{aligned} H_{S,T} &= d\Gamma_{\mathrm{b}}(\omega) \otimes I + I \otimes d\Gamma_{\mathrm{f}}(\omega), \\ P_{S,T} &= \lambda\overline{\{d\Gamma_{\mathrm{b}}(p) \otimes I + I \otimes d\Gamma_{\mathrm{f}}(p)\}}. \end{aligned}$$

Note that $H_{S,T}$ and $P_{S,T}$ are independent of θ.

If $\omega(p) = \sqrt{p^2 + m^2}$ with a constant $m \geq 0$, $\lambda = 1$ and $\theta = 0$, then $H_{S,T}$ and $P_{S,T}$ are respectively the Hamiltonian and the momentum operator of a free relativistic SQFT in the two-dimensional space-time, called the $N = 1$ *Wess-Zumino model* (cf. [19]).

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