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Geometrical View of the Furuta Inequality

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1. Introduction. Throughout this note, we use a capital letter as an operator on a Hilbert space H. An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible.

The original form of the Furuta inequality [5] given by Furuta himself is the following(cf.[6],[17]).

Furuta inequality: *If* $A \geq B \geq 0$, *then for each* $r \geq 0$,

$$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for p *and* q *such that* $p \geq 0$ *and* $q \geq 1$ *with*

$$(1+r)q \geq p+r$$

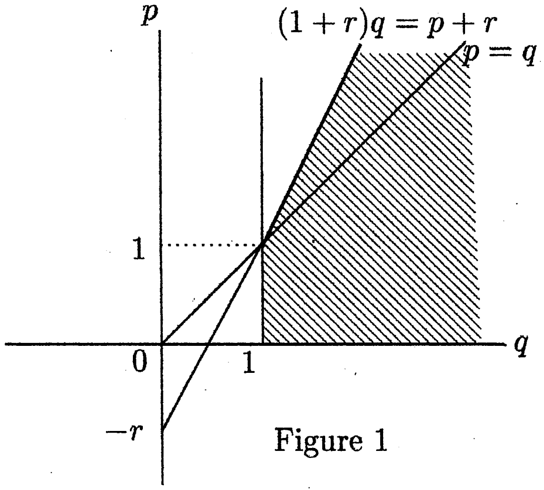

Figure 1

The case of $r = 0$ in this inequality is the Löwner-Heinz inequality:

$$(LH) \qquad A^{\alpha} \geq B^{\alpha} \quad \textit{for} \;\; A \geq B \geq 0 \;\; \textit{and} \;\; 0 \leq \alpha \leq 1.$$

From the viewpoint of operator mean ([2],[3],[10],[11] etc.), the Furuta inequality is rewritten as follows;

$$A^{u} \sharp_{\frac{1-u}{p-u}} B^{p} \leq A \quad \textit{and} \quad B \leq B^{u} \sharp_{\frac{1-u}{p-u}} A^{p}$$

for $p \geq 1$ and $u \leq 0$. The notations $\sharp_{\alpha}$ and $\natural_{\alpha}$ are defined for positive operators A and B by

$$A \natural_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}, \quad \textit{for} \; \alpha \in \mathbf{R}$$

and $\sharp_{\alpha} = \natural_{\alpha}$ when $\alpha \in [0,1]$. Note that $\sharp_{\alpha}$ is an operator mean in the sense of Kubo-Ando [16] which corresponds to the operator monotone function x^{α} in the Löwner theory.

As shown in [11], we had arranged these inequalities in one line by using the operator mean $\natural_\alpha$ as follows:

Satellite theorem of the Furuta inequality: *If $A \geq B \geq 0$, then*

$$A^u \natural_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \natural_{\frac{1-u}{p-u}} A^p$$

for all $p \geq 1$ and $u \leq 0$.

We can generalize this inequality as follows, in which the case of $\delta = 1$ is the satellite theorem([13], [14]).

Theorem A. *If $A \geq B > 0$, then for $0 \leq \delta \leq 1$, $\delta \leq p$ and $u \leq 0$*

$$A^u \natural_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \leq A^\delta \leq B^u \natural_{\frac{\delta-u}{p-u}} A^p,$$

and for $-1 \leq \gamma \leq 0$, $u \leq \gamma$ and $p \geq 0$

$$A^u \natural_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma \leq B^\gamma \leq B^u \natural_{\frac{\gamma-u}{p-u}} A^p.$$

More generally we have the following and called it a paramertrization of the Furuta inequality ([13], [14]).

Theorem A'. *If $A \geq B > 0$, then for $0 \leq \delta \leq p$ and $u \leq 0$*

$$A^u \natural_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \text{ and } B^u \natural_{\frac{\delta-u}{p-u}} A^p \geq A^\delta,$$

and for $u \leq \gamma \leq 0$ and $p \geq 0$

$$A^u \natural_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma \text{ and } B^u \natural_{\frac{\gamma-u}{p-u}} A^p \geq B^\gamma.$$

We can explain these relations by the following Figure 2.

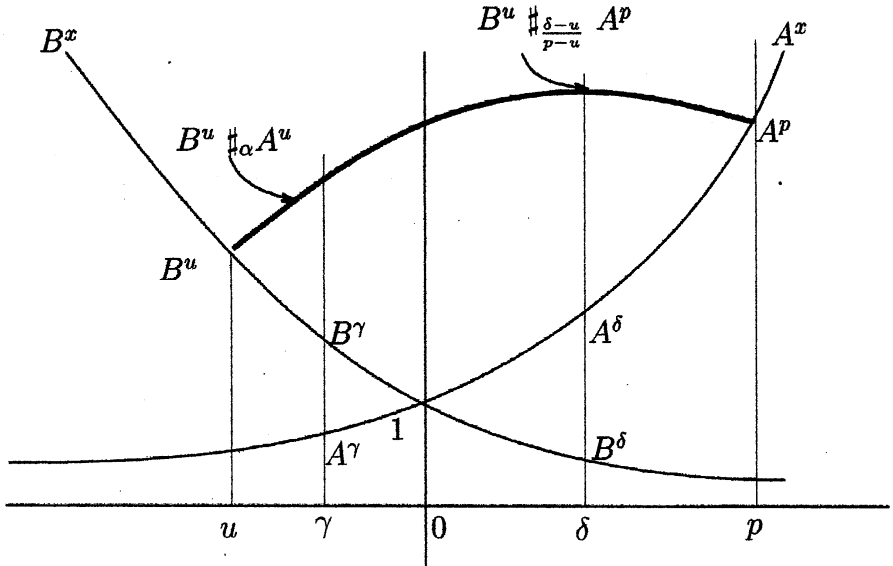

Figure 2

As a generalization of the Furuta inequality, Furuta [7] had given an inequality which we called the grand Furuta inequality. It interpolates the Furuta inequality and the Ando-Hiai inequality [1] equivalent to the main result of log majorization. We cite here it in terms of operator mean ([3]):

The grand Furuta inequality: *If $A \geq B \geq 0$ and A is invertible, then for each $p \geq 1$ and $0 \leq t \leq 1$,*

$$A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A$$

holds for $r \geq t$ and $s \geq 1$.

The best possibility of $\frac{1-t+r}{(p-t)s+r}$ is shown in [18]. We can state this theorem also by the satellte form as follows [14];

Theorem B. *If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$, $0 \leq \delta \leq 1$ and $\delta \leq \beta$*

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^\delta \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}} \leq B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

More generally we have shown the next theorem as a parametrized form of the grand Furuta inequality [15].

Theorem B'. *If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$ and $0 \leq \delta \leq \beta$*

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}}$$

and

$$B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}}.$$

On the complementary domain of the Furuta inequality, that is, $0 \leq t < p \leq 1$, the following inequality holds ([12],[15]).

Theorem C. *If $A \geq B > 0$, then for $0 \leq t < p \leq 1$, $p \leq \delta \leq \min\{1, 2p\}$ and $\beta \geq \delta$*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^\delta \leq A^\delta \leq B^t \natural_{\frac{\delta-t}{p-t}} A^p \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}}.$$

If $A \geq B > 0$, then for $0 \leq t \leq 1 \leq p$, $p \neq t$ and $\beta \geq p$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}}.$$

2. Results and Proofs. At the begining, we give a generalization of Theorem C.

Theorem 1. *If $A \geq B > 0$, then for $0 \leq t \leq 1$ and $0 \leq t < p \leq \beta$*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p \quad and \quad (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{p}{\beta}} \geq A^p.$$

We prepare the next lemma to prove this theorem.

Lemma. *If $A \geq B > 0$, then*

$$A^t \natural_{\frac{q-t}{p-t}} B^p \leq B^q \quad and \quad B^t \natural_{\frac{q-t}{p-t}} A^p \geq A^q$$

for $0 \leq t \leq 1$, $t < p$ and q such that $1 \leq \frac{q-t}{p-t} \leq 2$.

Proof. We have

$$\begin{aligned} A^t \natural_{\frac{q-t}{p-t}} B^p &= B^p \natural_{1-\frac{q-t}{p-t}} A^t = B^p \natural_{\frac{p-q}{p-t}} A^t \\ &= B^p(B^{-p} \sharp_{\frac{q-p}{p-t}} A^{-t})B^p \leq B^p(B^{-p} \sharp_{\frac{q-p}{p-t}} B^{-t})B^p = B^q. \end{aligned}$$

Proof of Theorem 1. In the case of $p \leq 1$, it is already shown in Theorem C. So we have only to see the case of $p \geq 1$. If $1 \leq \frac{\beta-t}{p-t} \leq 2$, then by the use of we Lemma and (LH) we have $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p$.
Secondly, if we choose β_1 such as $1 \leq \frac{\beta_1-t}{\beta-t} \leq 2$ for the above β, then we can use

Lemma for A and $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$ since $B_1 \le A$ by Theorem C. So we have $A^t \natural_{\frac{\beta_1-t}{\beta-t}} B_1^{\beta} \le B_1^{\beta_1}$. That is,

$$A^t \natural_{\frac{\beta_1-t}{p-t}} B^p = A^t \natural_{\frac{\beta_1-t}{\beta-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \le (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}}.$$

By (LH) we have

$$(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}} \le (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \le B^p.$$

Thirdly, we choose β_2 such that $1 \le \frac{\beta_2-t}{\beta_1-t} \le 2$ for the above β_1, then Lemma is usable for A and $B_2 = (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{1}{\beta_1}}$ by Theorem C. So $A^t \natural_{\frac{\beta_2-t}{\beta_1-t}} B_2^{\beta_1} \le B_2^{\beta_2}$ holds. That is,

$$A^t \natural_{\frac{\beta_2-t}{p-t}} B^p = A^t \natural_{\frac{\beta_2-t}{\beta_1-t}} (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p) \le (A^t \natural_{\frac{\beta_2-t}{p-t}} B^p)^{\frac{\beta_2}{\beta_1}}.$$

Hence by using (LH), we have

$$(A^t \natural_{\frac{\beta_2-t}{p-t}} B^p)^{\frac{p}{\beta_2}} \le (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}}.$$

Combining this with the above cases, we have

$$(A^t \natural_{\frac{\beta_2-t}{p-t}} B^p)^{\frac{p}{\beta_2}} \le (A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}} \le (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \le B^p.$$

Repeating this method, we can obtain the desired inequality.

In parallel with Theorem B, we have the following:

Theorem 2. *If $A \ge B > 0$ and $0 \le t \le 1$, $0 \le t < p \le \beta$, $u \le 0$, then*

(1) $0 \le \delta \le 1$ *and* $\delta \le p$,

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \le A^u \sharp_{\frac{\delta-u}{p-u}} B^p \le B^\delta \le A^\delta \le B^u \sharp_{\frac{\delta-u}{p-u}} A^p \le B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

(2) $-1 \le \gamma \le 0$ *and* $u \le \gamma$,

$$A^u \sharp_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \le A^u \sharp_{\frac{\gamma-u}{p-u}} B^p \le A^\gamma \le B^\gamma \le B^u \sharp_{\frac{\gamma-u}{p-u}} A^p \le B^u \sharp_{\frac{\gamma-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The case of $\delta = 1$ in (1) shows the order between the Furuta inequality and grand Furuta inequality. We can more loosen the condition on δ and γ as follows:

Theorem 3 *If $A \ge B > 0$ and $0 \le t \le 1$, $0 \le t < p \le \beta$, $u \le 0$, then*

(1) $0 \le \delta \le p$,

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \le A^u \sharp_{\frac{\delta-u}{p-u}} B^p \le B^\delta$$

and

$$B^u \natural_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^u \natural_{\frac{\delta-u}{p-u}} A^p \geq A^\delta.$$

(2) $u \leq \gamma \leq 0$

$$A^u \natural_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \natural_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma$$

and

$$B^u \natural_{\frac{\gamma-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^u \natural_{\frac{\gamma-u}{p-u}} A^p \geq B^\gamma.$$

Proof. (1) It follows from Theorem B and Theorem 1 that

$$\begin{aligned} & A^u \natural_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \\ = \ & A^u \natural_{\frac{\delta-u}{p-u}} (A^u \natural_{\frac{p-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)) \\ \leq \ & A^u \natural_{\frac{\delta-u}{p-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq A^u \natural_{\frac{\delta-u}{p-u}} B^p. \end{aligned}$$

The case of (2) is also obtained as follows:

$$\begin{aligned} & A^u \natural_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \\ = \ & A^u \natural_{\frac{\gamma-u}{p-u}} (A^u \natural_{\frac{p-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)) \\ \leq \ & A^u \natural_{\frac{\gamma-u}{p-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq A^u \natural_{\frac{\gamma-u}{p-u}} B^p. \end{aligned}$$

In Theorem B', if we restrict the condition on δ to $0 \leq \delta \leq p$ the same as Theorem 3, we have the following parallel formulas to Theorem 3 (1).

$$A^u \natural_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta$$

and

$$B^u \natural_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}} \geq A^\delta.$$

We can explain the relations in Theorem 2 and Theorem 3 in the folloing Figure 3.

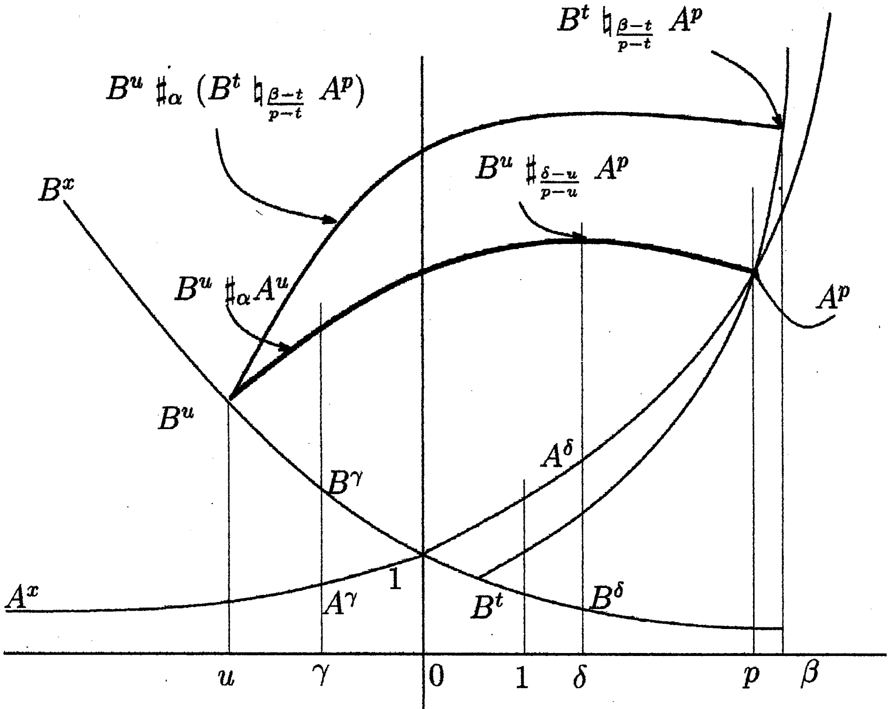

Figure 3

3. Remark. The results of Theorem 2 and Theorem 4 can be led by using some monotone property of operator function on the grand Furuta inequality. This viewpoint is an indication of Professor Furuta. In the papers [8] and [9], Furuta et al., the grand Furuta inequalty is treated as an operator function with monotone properties with respect to two variables. We also translated their assertions into our terms in [15] and showed this property being the structual necessity of the Furuta inequality in [4]. The representation of this function by us and our result [4] are the following.

Theorem D. *If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$ and $0 \leq \delta \leq \beta$*

$$H_{p,\delta,t}(A, B, u, \beta) = A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

is increasing for $u \leq 0$ and decreasing for $\beta \geq p$.

In this theorem, the case of $\beta = p$ is the form of the Furuta inequality and the monotone property of this function for β leads Theorem 2.

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