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Conditions for Choquet Integral Representation

(Choquet 積分表現のための条件)

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1 Introduction

The Choquet integral with respect to a fuzzy measure is a functional on the class B of measurable functions, that is comonotonically additive and monotone(for short c.m.). Sugeno et al. [15] proved that a c.m. functional I can be represented by a Choquet integral with respect to a regular fuzzy measure when the domain of I is the class K^+ of nonnegative continuous functions with compact support on a locally compact Hausdorff space. In [8, 9], it is proved that a c.m. functional is a rank- and sign-dependent functional, that is, the difference of two Choquet integrals. This functional is used in utility theory [5] and cumulative prospect theory [17, 18]. It is also proved that a rank- and sign-dependent functional is a c.m. functional if the universal set X is not compact.

In this paper, we discuss the conditions for which a c.m. functional can be represented by one Choquet integral. We define the conjugate conditions and show their basic properties in Section 4. The conjugate conditions are stronger than the boundedness and a c.m. functional I is represented by one Choquet integral when I satisfies one of the conjugate conditions. Conversely if a c.m. functional I is represented by one Choquet integral, I satisfies the conjugate condition when the universal set X is separable.

The proof of the main theorem is shown in Section 5 and the other proofs are omitted.

2 Preliminaries

In this section, we define the fuzzy measure, the Choquet integral and the rank- and sign-dependent functional, and show their basic properties.

Throughout the paper we assume that X is a locally compact Hausdorff space, $\mathcal{B}$ is the class of Borel subsets, $\mathcal{O}$ is the class of open subsets and $\mathcal{C}$ is the class of compact subsets.

Definition 2.1. [14] A *fuzzy measure* μ is an extended real valued set function,

$$\mu : \mathcal{B} \longrightarrow \overline{R^+}$$

with the following properties,

(1) $\mu(\emptyset) = 0$

(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B,\ A, B \in \mathcal{B}$

where $\overline{R^+} = [0, \infty]$ is the set of extended nonnegative real numbers.

When $\mu(X) < \infty$, we define *the conjugate* μ^c *of* μ by

$$\mu^c(A) = \mu(X) - \mu(A^C)$$

for $A \in \mathcal{B}$.

Definition 2.2. Let μ be a fuzzy measure on measurable space $(X, \mathcal{B})$. μ is said to be *outer regular* if

$$\mu(B) = \inf\{\mu(O) | O \in \mathcal{O}, O \supset B\}$$

for all $B \in \mathcal{B}$.

The outer regular fuzzy measure μ is said to be *regular*, if for all $O \in \mathcal{O}$

$$\mu(O) = \sup\{\mu(C) | C \in \mathcal{C}, C \subset O\}.$$

We denote by K the class of continuous functions with compact support, by K^+ the class of nonnegative continuous functions with compact support and by K_1^+ the class of nonnegative continuous functions with compact support that satisfies $0 \leq f \leq 1$.

We denote $supp(f)$ the support of $f \in K$, that is,

$$supp(f) = cl\{x | f(x) \neq 0\}.$$

Definition 2.3. [1, 6] Let μ be a fuzzy measure on $(X, \mathcal{B})$.

(1) *The Choquet integral* of $f \in K^+$ with respect to μ is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{x | f(x) \geq r\})$.

(2) Suppose $\mu(X) < \infty$. The Choquet integral of $f \in K$ with respect to μ is defined by

$$(C) \int f d\mu = (C) \int f^+ d\mu - (C) \int f^- d\mu^c,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. When the right hand side is $\infty - \infty$, the Choquet integral is not defined.

Definition 2.4. [3] Let $f, g \in K$. We say that f and g are *comonotonic* if

$$f(x) < f(x') \Rightarrow g(x) \leq g(x')$$

for $x, x' \in X$. We denote $f \sim g$, when f and g are comonotonic.

Definition 2.5. Let I be a real valued functional on K.

We say I is *comonotonically additive* iff

$$f \sim g \Rightarrow I(f+g) = I(f) + I(g)$$

for $f, g \in K^+$, and I is *monotone* iff

$$f \leq g \Rightarrow I(f) \leq I(g)$$

for $f, g \in K^+$.

If a functional I is comonotonically additive and monotone, we say that I is *a c.m. functional.*

Suppose that I is a c.m. functional, then we have $I(af) = aI(f)$ for $a \geq 0$ and $f \in K^+$, that is, I is positive homogeneous.

3 Representation and Boundedness

Definition 3.1. Let I be a real valued functional on K. I is said to be *a rank- and sign-dependent functional* (for short *a r.s.d. functional*) on K, if there exist two fuzzy measures μ^+, μ^- such that for every $f \in K$

$$I(f) = (C)\int f^+ d\mu^+ - (C)\int f^- d\mu^-$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$.

When $\mu^+ = \mu^-$, we say that the r.s.d. functional is the Šipoš functional [13]. If the r.s.d. functional is the Šipoš functional, we have $I(-f) = -I(f)$.

If $\mu^+(X) < \infty$ and $\mu^- = (\mu^+)^c$, we say that the r.s.d. functional is the Choquet functional.

Theorem 3.2. [8, 9] *Let I be a c.m. functional on K.*

(1) *We put*

$$\mu_I^+(O) = \sup\{I(f)|f \in K_1^+, supp(f) \subset O\},$$

and

$$\mu_I^+(B) = \inf\{\mu_I^+(O)|O \in \mathcal{O}, O \supset B\}$$

for $O \in \mathcal{O}$ and $B \in \mathcal{B}$.

Then μ_I^+ is a regular fuzzy measure.

(2) *We put*

$$\mu_I^-(O) = \sup\{-I(-f)|f \in K_1^+, supp(f) \subset O\},$$

and

$$\mu_I^-(B) = \inf\{\mu_I^-(O)|O \in \mathcal{O}, O \supset B\}$$

for $O \in \mathcal{O}$ and $B \in \mathcal{B}$.

Then μ_I^- is a regular fuzzy measure.

(3) *A c.m. functional is a r.s.d functional, that is,*

$$I(f) = (C)\int (f \vee 0)d\mu_I^+ - (C)\int -(f \wedge 0)d\mu_I^-$$

for every $f \in K$.

(4) *If X is compact, then a c.m. functional can be represented by one Choquet integral.*

(5) *If X is locally compact but not compact, then a r.s.d functional is a c.m. functional.*

Definition 3.3. Let I be a c.m. functional on K. We say that μ_I^+ defined in Theorem 3.2 is *the regular fuzzy measure induced by the positive part of I* , and μ_I^- *the regular fuzzy measure induced by the negative part of I.*

Definition 3.4. Let I be a real valued functional on K.

(1) I is said to be *bounded above* if there exists $M > 0$ such that

$$I(f) \leq M\|f\|$$

for all $f \in K$.

(2) I is said to be *bounded below* if there exists $M > 0$ such that

$$-M\|f\| \leq I(f)$$

for all $f \in K$.

(3) I is said to be *bounded* if I is bounded above and below.

Proposition 3.5. [8, 11] *Let I be a c.m. functional on K and μ_I^+ and μ_I^- the regular fuzzy measure induced by I.*

(1) *I is bounded above iff $\mu_I^+(X) < \infty$.*

(2) *I is bounded below iff $\mu_I^-(X) < \infty$.*

Proposition 3.6. [10] *Let X be separable and I be a c.m. functional on K that is bounded, and μ_I^+ and μ_I^- the regular fuzzy measure induced by I.*

(1) *If* $(C)\int f d\mu_I^+ = (C)\int f d(\mu_I^-)^c$ *for all* $f \in K^+$, *then* $\mu_I^+(C) = (\mu_I^-)^c(C)$ *for all* $C \in \mathcal{C}$.

(2) *If* $(C)\int f d\mu_I^- = (C)\int f d(\mu_I^+)^c$ *for all* $f \in K$, *then* $\mu_I^-(C) = (\mu_I^+)^c(C)$ *for all* $C \in \mathcal{C}$.

Proposition 3.6 says that if a c.m. functional I is Choquet integral with respect to μ_I^+ then we have $\mu_I^-(C) = (\mu_I^+)^c(C)$ for every $C \in \mathcal{C}$. Since $(\mu_I^+)^c$ is not always regular, it is not always true that $\mu_I^- = (\mu_I^+)^c$. That is, I is not always Choquet functional. See the example in [8].

4 Conjugate condition for compact sets

Definition 4.1. Let I be a c.m. functional and $C \in \mathcal{C}$.

(1) We say that I satisfies the positive *conjugate condition for* C if there exists a positive real number M such that for any $\epsilon > 0$ there exist $f_1, f_2 \in K_1$ satisfying the next condition.

$1_C \leq g_1 \leq f_1$ and $f_2 \leq g_2 \leq 1_{C^c}$ with $supp(f_2) \subset supp(g_2) \subset C^c$ imply

$$|I(-g_1) - I(g_2) + M| < \epsilon$$

for $g_1, g_2 \in K_1$.

(2) We say that I satisfies *the negative conjugate condition for* C if there exists a positive real number M such that for any $\epsilon > 0$ there exist $f_1, f_2 \in K_1$ satisfying the next condition.

$1_C \leq g_1 \leq f_1$ and $f_2 \leq g_2 \leq 1_{C^c}$ with $supp(f_2) \subset supp(g_2) \subset C^c$ imply

$$|-I(g_1) + I(-g_2) + M| < \epsilon.$$

for $g_1, g_2 \in K_1$.

Suppose that a c.m. functional I satisfies the positive conjugate condition for $\emptyset$. Let $g_1(x) = 0$ for all $x \in X$. Since $\emptyset \subset supp(g_1)$ and $I(g_1) = 0$, there exists $M > 0$ and for any $\epsilon > 0$ there exists $f_2 \in K_1^+$ such that $supp(f_2) \subset supp(g_2) \subset X$ implies

$$|-I(g_2) + M| < \epsilon.$$

Therefore we have the next proposition.

Proposition 4.2. *Let I be a c.m. functional.*

(1) *If I satisfies the positive conjugate condition for $\emptyset$, then I is bounded above.*

(2) *If I satisfies the negative conjugate condition for $\emptyset$, then I is bounded below.*

The next lemma follows from the definition of the induced regular fuzzy measure.

Lemma 4.3. *Let $A \in \mathcal{B}$ and $f \in K^+$. Suppose that $A \subset \{x|f \geq 1\}$, then we have*

$$\mu_I^+(A) \leq I(f) \text{ and } \mu_I^-(A) \leq -I(-f).$$

Applying this lemma, we have the next theorem. The detail of the proof is in Section 5.

Theorem 4.4. *Let $C \in \mathcal{C}$, I be a c.m. functional and μ_I^+ and μ_I^- the regular fuzzy measure induced by I.*

(1) *I satisfies the positive conjugate condition for every $C \in \mathcal{C}$ if and only if*

$$\mu_I^-(C) = (\mu_I^+)^c(C)$$

for every $C \in \mathcal{C}$.

(2) *I satisfies the negative conjugate condition for C if and only if*

$$\mu_I^+(C) = (\mu_I^-)^c(C)$$

for every $C \in \mathcal{C}$.

Suppose that a c.m. functional I satisfies the positive conjugate condition for all $C \in \mathcal{C}$. It follows from Theorem 4.4 that

$$\begin{aligned}\mu_I^-(X) &= \sup\{\mu_I^-(C) | C \subset X\} \\ &= \sup\{(\mu_I^+)^c(C) | C \subset X\} \\ &= \sup\{\mu_I^+(X) - \mu_I^+(C^c) | C \subset X\} \leq \mu_I^+(X).\end{aligned}$$

Therefore we have the next corollary.

Corollary 4.5. *If a c.m. functional I satisfies the positive or negative conjugate condition for all $C \in \mathcal{C}$, then I is bounded.*

It follow from Theorem 4.4 that

$$\mu_I^-(\{x | f(x) \geq r\}) = (\mu_I^+)^c(\{x | f(x) \geq r\})$$

for all $f \in K$ and $r > 0$. Therefore we have the next theorem.

Theorem 4.6. *Let I be a c.m. functional.*

(1) *If I satisfies the positive conjugate condition for all $C \in \mathcal{C}$, we have*

$$I(f) = (C) \int f d\mu_I^+$$

for all $f \in K$.

(2) *If I satisfies the negative conjugate condition for all $C \in \mathcal{C}$, we have*

$$I(f) = -(C)\int -f d\mu_I^-$$

for all $f \in K$.

The next theorem follows from Proposition 3.6

Theorem 4.7. *Let X be separable and I be a c.m. functional on K that is bounded, and μ_I^+ and μ_I^- the regular fuzzy measure induced by I.*

(1) *If $I(f) = (C)\int f d\mu_I^+$ for all $f \in K$, then I satisfies the positive conjugate condition for all $C \in \mathcal{C}$.*

(2) *If $I(f) = -(C)\int -f d\mu_I^-$ for all $f \in K$, then I satisfies the negative conjugate condition for all $C \in \mathcal{C}$.*

5 Proof of Theorem 4.4

In this secton, the proof of Theorem 4.4 (1) is shown. The proof of Theorem 4.4 (2) is much the same.

Let $\epsilon > 0$ and $C \in \mathcal{C}$.

First suppose that a c.m. functional I satisfies the positive conjugate condition for every compace set C. That is, there exists a positive real number M such that $\forall \epsilon > 0$, $\exists f_1, f_2 \in K_1$, $1_C \leq g_1 \leq f_1$ and $f_2 \leq g_2 \leq 1_{C^c}$ with $supp(f_2) \subset supp(g_2) \subset C^c$ imply

$$M - I(g_2) - \epsilon < -I(-g_1) < M - I(g_2) + \epsilon \qquad (1)$$

for $g_1, g_2 \in K_1$.

Since μ_I^- is regular, there exists an open set O such that $C \subset O$ and

$$\mu_I^-(C) + \epsilon \geq \mu_I^-(O). \tag{2}$$

Using Uryson's lemma, there exists $h_1 \in K_1^+$ such that $1_C \leq h_1 \leq 1_O$. Since $1_C \leq f_1$, we may suppose that $f_1 \geq h_1$. It follows from Lemma 4.3 that

$$\mu_I^-(C) \leq -I(-h_1). \tag{3}$$

Since $supp(h_1) \subset O$, we have

$$\mu_I^-(O) \geq -I(-h_1) \tag{4}$$

from the definition of μ_I^-. Then it follows from (2) and (4) that

$$\mu_I^-(C) + \epsilon \geq -I(-h_1). \tag{5}$$

Since C^c is an open set, it follows from the definition of the induced regular fuzzy measure μ_I^+ that there exists $h_2 \in K_1^+$ such that $supp(h_2) \subset C^c$ and

$$I(h_2) \geq \mu_I^+(C^c) - \epsilon. \tag{6}$$

We may suppose that $f_2 \leq h_2 \leq 1_{C^c}$. Then applying (5) and (6), we have

$$\mu_I^-(C) + \epsilon \geq M - I(h_2) - \epsilon. \tag{7}$$

Since we have $I(h_2) \leq \mu_I^+(C^c)$ from $supp(h_2) \subset C^c$, we have

$$\mu_I^-(C) + \epsilon \geq M - \mu_I^+(C^c) - \epsilon. \tag{8}$$

Since I satisfies the conjugate condition for $\emptyset$, we have $M = \mu_I^+(X)$. Therefore we have

$$2\epsilon \geq (\mu_I^+)^c(C) - \mu_I^-(C) \tag{9}$$

from (8).

On the other hand, it follows from (1),(2) and (6) that

$$\begin{aligned}-I(-h_1) &\le M - I(h_0) + \epsilon \\ &\le M - (\mu_I^+(C^c) - \epsilon) + \epsilon \\ &\le (\mu_I^+)^c(C) + 2\epsilon.\end{aligned}$$

Therefore we have

$$|\mu_I^-(C) - (\mu_I^+)(C)| \le 2\epsilon.$$

Since ϵ is an arbitrary, we have $\mu_I^-(C) = (\mu_I^+)^c(C)$.

Next suppose that $\mu_I^-(C) = (\mu_I^+)^c(C)$. Define $M = \mu_I^+(X)$. Then it follows from the definition of the conjugate of μ_I^- that

$$\mu_I^-(C) = M - \mu_I^-(C^c). \tag{10}$$

Since μ_I^- is regular, there exists an open set O such that $O \supset C$ and

$$\mu_I^-(C) + \varepsilon \ge \mu_I^-(O). \tag{11}$$

Using Uryson's lemma, there exists $f_1 \in K_1^+$ such that $1_C \le f_1 \le 1_O$. Then for every $g_1 \in K_1^+$ such that $1_C \le g_1 \le f_1$, we have

$$\mu_I^-(O) \ge -I(-g_1) \ge \mu_I^-(C) \tag{12}$$

from Lemma 4.3. It follows from the definition of the induced regular fuzzy measure μ_I^+ that there exists $f_2 \in K_1^+$ such that $supp(f_2) \subset C^c$ and

$$\mu_I^+(C^c) - \epsilon \le I(f_2). \tag{13}$$

Therefore for every $g_2 \in K_1^+$ such that $f_2 \leq g_2 \leq C^c$ and $supp(f_2) \subset supp(g_2) \subset C^c$, we have

$$\mu_I^+(C^c) - \epsilon \leq I(f_2) \leq I(g_2) \leq \mu_I^+(C^c). \tag{14}$$

It follows from (10),(11) and (12) that

$$M - \mu_I^+(C^c) + \epsilon \geq -(-g_2).$$

Then we have

$$\epsilon \geq -M - I(-g_1) + I(g_2) \tag{15}$$

from (14). On the other hand, it follows from (10) and (14) that

$$I(g_2) + \epsilon \geq M - \mu_I^-(C). \tag{16}$$

Then we have

$$\epsilon \geq M - I(g_2) + I(-g_1) \tag{17}$$

from (12). Therefore we have

$$|I(-g_1) - I(g_2) + M| < \epsilon$$

from (15) and (17). □

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