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Simultaneous confidence intervals for pairwise comparisons among mean vectors under nonnormality

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Abstract: Simultaneous confidence intervals for the pairwise multiple comparisons among mean vectors under the elliptical populations are considered. Simultaneous confidence intervals estimations are given by using approximate upper percentiles of the statistics based on Bonferroni's inequality. In order to achieve the purpose, an asymptotic expansion for the Hotelling's T^2-type statistic in the elliptical distributions is derived by the perturbation method. Simulation study is also given for some selected parameters.

1. Introduction

Consider the simultaneous confidence intervals for pairwise multiple comparisons among mean vectors under the elliptical populations. An elliptical distribution includes the multivariate normal, the multivariate t and the contaminated normal distributions and so on, which is referred to e.g., Muirhead [14], p.32, Fang, Kotz and Ng [3], ch.3. The probability density function is defined by $f(\boldsymbol{x}) = c_p|\boldsymbol{\Lambda}|^{-1/2}g((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Lambda}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))$, for some nonnegative function g, where c_p is the normalizing constant and $\boldsymbol{\Lambda}$ is positive definite. The characteristic function of the vector $\boldsymbol{x}$ is the form $\phi(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\boldsymbol{\Lambda}\boldsymbol{t})$ for some function ψ, where $i = \sqrt{-1}$. Note that $\boldsymbol{\Sigma} = -2\psi'(0)\boldsymbol{\Lambda}$. We also define the kurtosis parameter by $\kappa = \{\psi''(0)/(\psi'(0))^2\} - 1$. We discussed the simultaneous confidence intervals for pairwise multiple comparisons among mean vectors under the elliptical populations.

Let $\boldsymbol{x}_1^{(i)}, \cdots, \boldsymbol{x}_{N_i}^{(i)}$ $(i = 1, \cdots, k)$ be N_i independent observations on $\boldsymbol{x}^{(i)}$ having the elliptical distribution with a mean vector $\boldsymbol{\mu}^{(i)}$ and a common covariance matrix $\boldsymbol{\Sigma}$. Let the i-th sample mean vector and the sample covariance matrix be

$$\overline{\boldsymbol{x}}^{(i)} = \frac{1}{N_i}\sum_{j=1}^{N_i} \boldsymbol{x}_j^{(i)},$$
$$\boldsymbol{S}^{(i)} = \frac{1}{N_i - 1}\sum_{j=1}^{N_i}(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})',$$

respectively.

The usual simultaneous confidence intervals for pairwise comparisons among mean vectors are given as the form

$$\boldsymbol{a}'(\boldsymbol{\mu}^{(\ell)} - \boldsymbol{\mu}^{(m)}) \in \left[\ \boldsymbol{a}'(\overline{\boldsymbol{x}}^{(\ell)} - \overline{\boldsymbol{x}}^{(m)}) \pm q\sqrt{d_{\ell m}\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}}\ \right], \quad \forall \boldsymbol{a} \in \mathrm{R}^p - \{\boldsymbol{0}\},\ 1 \le \ell < m \le k, \tag{1}$$

where

$$\boldsymbol{S} = \frac{1}{\nu}\sum_{i=1}^{k}(N_i - 1)\boldsymbol{S}^{(i)}, \quad \nu = \sum_{i=1}^{k} N_i - k, \quad d_{\ell m} = \frac{1}{N_\ell} + \frac{1}{N_m},$$

$\mathrm{R}^p - \{\boldsymbol{0}\}$ is the set of any nonnull real p-dimensional vectors and q is some positive number. In order to obtain the simultaneous confidence intervals (1) with the given confidence level $1 - \alpha$, it is necessary to give the value $q(> 0)$ such that

$$\Pr\{T_{\max}^2 > q^2\} = \alpha,$$

where

$$T_{\max}^2 = \max_{1 \le \ell < m \le k}\left\{d_{\ell m}^{-1}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})'\boldsymbol{S}^{-1}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})\right\}, \quad \boldsymbol{y}^{(\ell)} = \overline{\boldsymbol{x}}^{(\ell)} - \boldsymbol{\mu}^{(\ell)}. \tag{2}$$

In the univariate and equal sample sizes case under normality, $T_{\max}^2/2$ reduces to the usual studentized range statistic whose upper percentiles have been given by Harter [5]. Under the unequal sample sizes, Tukey-Kramer procedure was proposed as approximate procedure by Tukey [24], Kramer [11, 12] and its conservativeness was discussed by Dunnett [2], Brown [1], Hayter [6, 7], Lin, Seppänen and Uusipaikka [13] and Somerville [23] and so on. The related discussions are summarized in e.g., Hochberg and Tamhane [8]. In the multivariate and equal sample sizes case under normality, $T^2_{\max}/2$ reduces to the multivariate studentized range statistic which first appeared in Roy and Bose [16] based on Roy's [15] union-intersection principle(see, e.g., Siotani, Hayakawa and Fujikoshi [22], p.227)). We note that, in multivariate setting, it is difficult to find the exact percentiles of the $T^2_{\max}$ statistic even if the populations are multivariate normal distributions. Approximations to the upper percentiles have been discussed by using modified Bonferroni's inequalities and asymptotic expansion method(see, Siotani [21], Seo and Siotani [20]). Seo [17] also has discussed $T^2_{\max}$ statistic for the case of correlated mean vectors including unequal sample sizes under normality. In addition, Seo, Mano and Fujikoshi [19] proposed a multivariate version of Tukey-Kramer procedure and they proved that the procedure has conservative simultaneous confidence intervals for the three correlated mean vectors. Recently, the evaluation for the bound of coverage probability for the multivariate Tukey-Kramer procedure is discussed by Seo [18].

In this paper, we discuss the upper percentiles of (2) in the elliptical distributions including the multivariate normal distribution. The approximate procedure based on Bonferroni's inequality is adopted in order to obtain a conservative simultaneous confidence intervals estimation. By the first order Bonferroni's inequality for $\Pr\{T^2_{\max} > q^2\}$;

$$\Pr\{T^2_{\max} > q^2\} < \sum_{\ell < m}\sum \Pr\{T^2_{\ell m} > q^2\},$$

where $T^2_{\ell m} = d^{-1}_{\ell m}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})'\boldsymbol{S}^{-1}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})$, an approximate upper percentile of $T^2_{\max}$ is given by $\sum\sum_{\ell<m}\Pr\{T^2_{\ell m} > q^2_B\} = \alpha$. Note that this approximation is essentially evaluating the upper percentiles of the Hotelling's T^2-type statistic in the elliptical distributions. An asymptotic expansion of the distribution of Hotelling's T^2-statistic in the elliptical distribution has been given by Iwashita [9]. Also, Kano [10] and Fujikoshi [4] have obtained the extensive result under the general distributions. However, these results of asymptotic expansions cannot be applied for the Hotelling's T^2-type statistic discussed in our paper, since the statistic in our paper is not exactly Hotelling's T^2 statistic under the elliptical populations. In order to find approximate simultaneous confidence intervals, in section 2, an asymptotic expansion for the upper percentiles of the Hotelling's T^2-type which is an extension of Hotelling's T^2 statistic is derived by the perturbation method. In section 3, the accuracy of approximate values for the upper percentiles of the $T^2_{\max}$ statistic in the elliptical distribution is also discussed by Monte Carlo simulation for the selected parameters.

2. Upper percentiles of the Hotelling's T^2-type statistic

In this section, we give asymptotic expansions for the Hotelling's T^2-type statistic $T^2_{\ell m}$ by perturbation method.

Note that

$$(N_i - 1)\boldsymbol{S}^{(i)} = N_i \boldsymbol{W}^{(i)} - N_i(\overline{\boldsymbol{x}}^{(i)} - \boldsymbol{\mu}^{(i)})(\overline{\boldsymbol{x}}^{(i)} - \boldsymbol{\mu}^{(i)})',$$

where

$$\boldsymbol{W}^{(i)} = \frac{1}{N_i}\sum_{j=1}^{N_i}(\boldsymbol{x}_j^{(i)} - \boldsymbol{\mu}^{(i)})(\boldsymbol{x}_j^{(i)} - \boldsymbol{\mu}^{(i)})'.$$

Without loss of generality, we can assume $\boldsymbol{\Sigma} = \boldsymbol{I}$ and $N_j \leq N_1 = N$. We put $r_j = N_j/N$ for $j = 1, \cdots, k$, $w = (\sum_{j=1}^k r_j)^{-1}$ and $a_{\ell m} = (1 + r_m/r_\ell)^{-1/2}$. Let

$$\overline{\boldsymbol{x}}^{(j)} = \boldsymbol{\mu}^{(j)} + \frac{1}{\sqrt{r_j N}}\boldsymbol{z}^{(j)}, \qquad \boldsymbol{W}^{(j)} = \boldsymbol{I}_p + \frac{1}{\sqrt{r_j N}}\boldsymbol{Z}^{(j)}$$

for $j = 1, \cdots, k$. Then we can write

$$\boldsymbol{S} = \frac{N}{N - wk}\Big(\boldsymbol{I}_p + \frac{1}{\sqrt{N}} w \sum_{i=1}^k \sqrt{r_i}\boldsymbol{Z}^{(i)} - \frac{1}{N} w \sum_{i=1}^k \boldsymbol{z}^{(i)}\boldsymbol{z}^{(i)\prime}\Big),$$

and hence

$$S^{-1} = I_p - \frac{1}{\sqrt{N}}A_0 + \frac{1}{N}A_1 + o_p(N^{-1}),$$

where

$$\begin{aligned} A_0 =& w\sum_{i=1}^{k}\sqrt{r_i}Z^{(i)}, \\ A_1 =& w\sum_{i=1}^{k} z^{(i)}z^{(i)\prime} + w^2\sum_{i=1}^{k}\sum_{j=1}^{k}\sqrt{r_i r_j}Z^{(i)}Z^{(j)} - wkI_p. \end{aligned}$$

Note that $T_{\ell m}^2 = \tau' S^{-1}\tau$, where $\tau = a_{m\ell}z^{(\ell)} - a_{\ell m}z^{(m)}$. Therefore the characteristic function of $T_{\ell m}^2$ can be written as

$$\begin{aligned} E\left[\exp(itT_{\ell m}^2)\right] =& E\Big[\exp\{it\tau'\tau\}\Big[1 + \frac{1}{\sqrt{N}}(-it)\tau' A_0\tau \\ &+ \frac{1}{N}\Big\{(it)\tau' A_1\tau + \frac{1}{2}(it)^2(\tau' A_0 \tau)^2\Big\} + o_p(N^{-1})\Big]\Big]. \end{aligned}$$

The following is the joint density of $z^{(j)}$ and $Z^{(j)}$ given by Iwashita [9].

$$\begin{aligned} f(z^{(j)}, Z^{(j)}) =& (2\pi)^{-\frac{p(p+3)}{4}}|\Omega_1^{(j)}|^{-\frac{1}{2}}(\kappa^{(j)}+1)^{-\frac{p(p-1)}{4}} \\ &\times \exp\Big\{-\frac{1}{2}z^{(j)\prime}z^{(j)}\Big\}\exp\Big\{-\frac{1}{2}z_1^{(j)\prime}\Omega_1^{(j)^{-1}}z_1^{(j)} - \frac{1}{2}z_2^{(j)\prime}\Omega_2^{(j)^{-1}}z_2^{(j)}\Big\} \\ &\times\Big[1 + \frac{1}{\sqrt{N}}\Big\{(b_0^{(j)} - \frac{1}{2})\operatorname{tr}Z^{(j)} - b_1^{(j)}(\operatorname{tr}Z^{(j)})^3 - b_2^{(j)}\operatorname{tr}(Z^{(j)})^2\operatorname{tr}Z^{(j)} \\ &- b_3^{(j)}\operatorname{tr}(Z^{(j)})^3 + \frac{1}{2}z^{(j)\prime}Z^{(j)}z^{(j)}\Big\} + o(N^{-\frac{1}{2}})\Big] \end{aligned}$$

where

$$Z^{(j)} = [z_{st}^{(j)}],\quad \Omega_1^{(j)} = 2(\kappa^{(j)}+1)I_p + \kappa^{(j)}1_p1_p',\quad \Omega_2^{(j)} = (\kappa^{(j)}+1)I_{p(p-1)/2}$$

$$z_1^{(j)} = (z_{11}^{(j)},\cdots,z_{pp}^{(j)})',\quad z_2^{(j)} = (z_{12}^{(j)}, z_{13}^{(j)},\cdots,z_{p-1p}^{(j)})',$$

$$\begin{aligned} &u^{(j)} = \frac{1}{2(\kappa^{(j)}+1)},\quad v^{(j)} = \frac{1}{p}\left\{\frac{1}{2+(p+2)\kappa^{(j)}} - \frac{1}{2(\kappa^{(j)}+1)}\right\}, \\ &\beta_1 \equiv \beta_1^{(j)} = -\frac{1}{6}(\varphi^{(j)}+1),\quad \varphi^{(j)} = \{\psi_j'''(0)/(\psi_j'(0))^3\} - 1, \\ &\beta_2 \equiv \beta_2^{(j)} = (\kappa^{(j)}+1),\quad \beta_3 = -\frac{1}{3}, \end{aligned}$$

$$b_0^{(j)} = (u+pv)[3(p+2)(p+4)(u+v)\beta_1 + \{(p+2)^2u + 3p(p+2)v\}\beta_2$$
$$+ 3p(u+pv)\beta_3],$$
$$b_1^{(j)} = \{u^3 + 3(p+4)u^2v + 3(p+2)(p+4)uv^2 + p(p+2)(p+4)v^3\}\beta_1$$
$$+ \{u^3 + (3p+4)u^2v + 3p(p+2)uv^2 + p^2(p+2)v^3\}\beta_2 + (u+pv)^3\beta_3,$$
$$b_2^{(j)} = 6\{u^3 + (p+4)u^2v\}\beta_1 + 2u^2(u+pv)\beta_2,$$
$$b_3^{(j)} = 8u^3\beta_1.$$

Calculating the characteristic function of $T^2_{\ell m}$ with respect to $\boldsymbol{z}^{(j)}(=\sqrt{r_jN}(\overline{\boldsymbol{x}}^{(j)} - \boldsymbol{\mu}^{(j)}))$ and $\boldsymbol{Z}^{(j)}(=\sqrt{r_jN}(\boldsymbol{W}^{(j)} - \boldsymbol{I}_p))$ by using the above joint density, we obtain

$$E[e^{itT^2_{\ell m}}] = (1-2it)^{-\frac{p}{2}}\left\{1 + \frac{1}{4N}\{c^{(0)}_{\ell m} + c^{(1)}_{\ell m}(1-2it)^{-1} + c^{(2)}_{\ell m}(1-2it)^{-2}\}\right\} \qquad (3)$$
$$+\mathrm{o}(N^{-1}),$$

where

$$c^{(0)}_{\ell m} = -\,wp^2 + \frac{1}{2}p(p+2)\left[\{r_\ell^{-1}a^2_{\ell m} - 2w\}a^2_{\ell m}\kappa^{(\ell)} + \{r_m^{-1}a^2_{m\ell} - 2w\}a^2_{m\ell}\kappa^{(m)} - w^2\sum_{i=1}^{k} r_i\kappa^{(i)}\right],$$
$$c^{(1)}_{\ell m} = -\,2wp - p(p+2)\left[\{r_\ell^{-1}a^2_{\ell m} - 4w\}a^2_{\ell m}\kappa^{(\ell)} + \{r_m^{-1}a^2_{m\ell} - 4w\}a^2_{m\ell}\kappa^{(m)} + w^2\sum_{i=1}^{k} r_i\kappa^{(i)}\right],$$
$$c^{(2)}_{\ell m} = wp(p+2) + \frac{1}{2}p(p+2)\left[\{r_\ell^{-1}a^2_{\ell m} - 6w\}a^2_{\ell m}\kappa^{(\ell)} + \{r_m^{-1}a^2_{m\ell} - 6w\}a^2_{m\ell}\kappa^{(m)} + 3w^2\sum_{i=1}^{k} r_i\kappa^{(i)}\right].$$

Therefore, inverting the characteristic function (3), we have the following theorem.

Theorem 2.1. The distribution of $T^2_{\ell m} = d^{-1}_{\ell m}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})'\boldsymbol{S}^{-1}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})$ can be expanded as

$$\Pr(T^2_{\ell m} > t^2) = \Pr(\chi^2_p > t^2) + \frac{1}{4N}\sum_{j=0}^{2} c^{(j)}_{\ell m}\Pr(\chi^2_{p+2j} > t^2) + \mathrm{o}(N^{-1}), \qquad (4)$$

and its upper α percentiles can be expanded as

$$t^2 = \chi^2_p(\alpha) - \frac{1}{2N}\chi^2_p(\alpha)\left\{\frac{1}{p}c^{(0)}_{\ell m} - \frac{1}{p(p+2)}c^{(2)}_{\ell m}\chi^2_p(\alpha)\right\} + \mathrm{o}(N^{-1}), \qquad (5)$$

where $\chi_p^2(\alpha)$ is the upper α percentile of χ^2 distribution with p degrees of freedom.

Corollary 2.2. If the k populations have elliptical distributions with the equal sample sizes($N_j = N,\ j = 1, \cdots, k$), then the coefficients of (4) and (5) are given by

$$
\begin{aligned}
c_{\ell m}^{(0)} &= -\frac{1}{k}p^2 + \frac{1}{2}p(p+2)\left\{\left(\frac{1}{4} - \frac{1}{k}\right)(\kappa^{(\ell)} + \kappa^{(m)}) - \frac{1}{k^2}\sum_{j=1}^{k}\kappa^{(j)}\right\}, \\
c_{\ell m}^{(1)} &= -\frac{2}{k}p - p(p+2)\left\{\left(\frac{1}{4} - \frac{2}{k}\right)(\kappa^{(\ell)} + \kappa^{(m)}) + \frac{1}{k^2}\sum_{j=1}^{k}\kappa^{(j)}\right\}, \\
c_{\ell m}^{(2)} &= \frac{1}{k}p(p+2) + \frac{1}{2}p(p+2)\left\{\left(\frac{1}{4} - \frac{3}{k}\right)(\kappa^{(\ell)} + \kappa^{(m)}) + \frac{3}{k^2}\sum_{j=1}^{k}\kappa^{(j)}\right\}.
\end{aligned}
$$

Corollary 2.3. If the k populations have elliptical distributions with the same kurtosis parameter and the equal sample sizes, i.e., $\kappa_j = \kappa$ and $N_j = N,\ j = 1, \cdots, k$, then the coefficients of (4) and (5) are given by

$$
\begin{aligned}
c_{\ell m}^{(0)} &= -\frac{1}{k}p^2 + \frac{1}{4k}p(p+2)(k-6)\kappa, \\
c_{\ell m}^{(1)} &= -\frac{2}{k}p - \frac{1}{2k}p(p+2)(k-6)\kappa, \\
c_{\ell m}^{(2)} &= \frac{1}{k}p(p+2) + \frac{1}{4k}p(p+2)(k-6)\kappa.
\end{aligned}
$$

In addition, when $\kappa^{(j)} = 0$ for $j = 1, \cdots, k$, we can see that the coefficients $c_{\ell m}^{(0)}$, $c_{\ell m}^{(1)}$ and $c_{\ell m}^{(2)}$ coincide with the result of an asymptotic expansion for the upper percentile of the Hotelling's T^2-statistic under the normal assumption(see, e.g., Seo and Siotani [20]). From Corollary 2.3, we note that, when $k = 6$, the coefficients $c_{\ell m}^{(0)}$, $c_{\ell m}^{(1)}$ and $c_{\ell m}^{(2)}$ don't depend on κ, that is, the asymptotic expansion for $k = 6$ coincides with an asymptotic expansion for the Hotelling's T^2-statistic under normality. Further, since the Hotelling's T^2-statistic under normality is F-statistic, we can obtain the following results from the characteristic function.

Theorem 2.4. The upper α percentiles of $T^2_{\ell m} = d^{-1}_{\ell m}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})'\boldsymbol{S}^{-1}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})$ can be also expanded as

$$\begin{aligned} t^2 = & \frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}(\alpha) \\ & - \frac{1}{2N}\chi^2_p(\alpha)\left\{\left(\frac{1}{p}c^{(0)}_{\ell m} + \frac{1}{k}p\right) - \left(\frac{1}{p(p+2)}c^{(2)}_{\ell m} - \frac{1}{k}\right)\chi^2_p(\alpha)\right\} + o(N^{-1}), \end{aligned} \tag{6}$$

where $\nu = \sum_{i=1}^k N_i - k$, $F_{p,\nu-p+1}(\alpha)$ and $\chi^2_p(\alpha)$ are the upper α percentiles of F distribution with p and $\nu - p + 1$ degrees of freedoms and χ^2 distribution with p degrees of freedom, respectively.

We note that, under normality, t^2 in Theorem 2.4 is exactly given by the upper percentiles F distribution which is the first term of (6). From Theorem 2.4, an asymptotic expansion approximation for the upper percentile of the usual Hotelling's T^2 statistic $T^2 = N\boldsymbol{y}'\boldsymbol{S}^{-1}\boldsymbol{y}$ under the elliptical distributions, which is given by

$$t^2 = \chi^2_p(\alpha) + \frac{1}{2N}\chi^2_p(\alpha)\left\{(p + (p+2)\kappa) + (1-\kappa)\chi^2_p(\alpha)\right\} + o(N^{-1}), \tag{7}$$

can be also wriiten as

$$t^2 = \frac{(N-1)p}{N-p}F_{p,N-p}(\alpha) + \frac{1}{2N}\chi^2_p(\alpha)\left\{(p+2) - \chi^2_p(\alpha)\right\}\kappa + o(N^{-1}). \tag{8}$$

Applying in the case of the simultaneous confidence intervals estimation for pairwise comparisons among mean vectors, we have two approximate simultaneous confidence intervals given by

$$q^2_{B\cdot\chi^2} = \chi^2_p(\alpha^*) - \frac{1}{2Nk^*}\chi^2_p(\alpha^*)\sum_{\ell<m}\sum\left\{\frac{1}{p}c^{(0)}_{\ell m} - \frac{1}{p(p+2)}c^{(2)}_{\ell m}\chi^2_p(\alpha^*)\right\}, \tag{9}$$

and

$$\begin{aligned} q^2_{B\cdot F} = & \frac{\nu p}{\nu - p + 1}F_{p,\nu-p+1}(\alpha^*) \\ & - \frac{1}{2Nk^*}\chi^2_p(\alpha^*)\times\sum_{\ell<m}\sum\left\{\left(\frac{1}{p}c^{(0)}_{\ell m} + \frac{1}{k}p\right) - \left(\frac{1}{p(p+2)}c^{(2)}_{\ell m} - \frac{1}{k}\right)\chi^2_p(\alpha^*)\right\}, \end{aligned} \tag{10}$$

where $\alpha^* = \alpha/k^*$ and $k^* = k(k-1)/2$.

3. Accuracy of the approximations

In order to examine the accuracy of the obtained approximations, we give the simulation results of the upper percentiles of $T_{\max}$ statistics for selected values of parameters. Table 1 gives the simulated and approximate values of the upper percentiles of $T_{\max}$ for the following combinations of selected parameters: $p = 2, 3, 5$, $k = 3, 6, 10$, $N_j(= N) = 20, 40, 60$ and $\alpha = 0.1, 0.05, 0.01$. In this section, the equal sample sizes case is treated in the Monte Carlo simulations, which is based on 100 replications for 10,000 simulations. Also, two elliptical distributions: multivariate normal($\kappa = 0$), contaminated normal($\varepsilon = 0.1$, $\sigma = 3$: $\kappa = 1.78$) are treated, where all populations have the same distribution. Table 1 lists the Bonferroni approximate value based on asymptotic expansion and χ^2 approximation $q_{B\cdot\chi^2}$, the analogous F approximation $q_{B\cdot F}$and the simulated value q^*. Note that under the normality, we have the approximate values based on F approximation are exactly the upper percentiles of F distribution, and overestimate. Form the simulation results, the asymptotic expansions up to the order N^{-1} for the upper percentiles of Hotelling's T^2-type statistic have a tendency to be underestimate. As a result, under the most cases of the elliptical populations, it may be noted that the value of $q_{B\cdot\chi^2}$is overestimate and is more closely to the simulated values than the other approximate values. Further, as showing in the previous section, it can be seen from Table 1 that the simulated values for the case of $k = 6$ are very similar values without respect to the distributions of the populations. Thus, it may be noted that the approximate value $q_{B\cdot\chi^2}$is useful for the simultaneous confidence intervals estimation for the pairwise multiple comparisons among mean vectors under the elliptical populations.

Table 1. Simualted and Approximate Values for the Upper Percentiles of $T_{\max}$

$\alpha = 0.10$							
$p = 2$		M.N.($\kappa = 0$)			C.N.($\kappa = 1.78$)		
k	N	$q_{B \cdot \chi^2}$	$q_{B \cdot F}$	q^*	$q_{B \cdot \chi^2}$	$q_{B \cdot F}$	q^*
	10	2.793	2.842	2.745	2.716	2.766	2.693
3	20	2.702	2.713	2.639	2.663	2.674	2.610
	40	2.656	2.658	2.587	2.636	2.638	2.573
	10	3.320	3.353	3.213	3.320	3.353	3.206
6	20	3.244	3.251	3.133	3.244	3.251	3.135
	40	3.205	3.207	3.098	3.205	3.207	3.099
	10	3.618	3.639	3.495	3.739	3.760	3.558
10	20	3.557	3.562	3.436	3.619	3.624	3.476
	40	3.526	3.528	3.411	3.558	3.559	3.433
$p = 3$							
	10	3.227	3.318	3.223	3.114	3.208	3.156
3	20	3.093	3.113	3.040	3.034	3.055	3.001
	40	3.023	3.028	2.963	2.993	2.998	2.942
	10	3.711	3.762	3.632	3.711	3.762	3.621
6	20	3.605	3.617	3.508	3.605	3.617	3.508
	40	3.551	3.554	3.453	3.551	3.554	3.456
	10	3.981	4.014	3.882	4.134	4.165	3.954
10	20	3.900	3.908	3.793	3.979	3.986	3.843
	40	3.859	3.861	3.756	3.899	3.901	3.785
$p = 5$							
	10	3.945	4.165	4.062	3.766	3.996	3.975
3	20	3.719	3.767	3.697	3.626	3.675	3.636
	40	3.602	3.613	3.551	3.554	3.565	3.519
	10	4.345	4.449	4.317	4.345	4.449	4.299
6	20	4.180	4.203	4.101	4.180	4.203	4.097
	40	4.095	4.100	4.012	4.095	4.100	4.013
	10	4.568	4.627	4.504	4.775	4.831	4.587
10	20	4.446	4.459	4.358	4.553	4.566	4.422
	40	4.383	4.387	4.294	4.438	4.441	4.332

Table 1. Continued

$\alpha = 0.05$							
$p=2$		M.N.($\kappa = 0$)			C.N.($\kappa = 1.78$)		
k	N	$q_{B \cdot \chi^2}$	$q_{B \cdot F}$	q^*	$q_{B \cdot \chi^2}$	$q_{B \cdot F}$	q^*
3	10	3.095	3.162	3.092	2.969	3.039	3.010
	20	2.981	2.996	2.945	2.916	2.932	2.896
	40	2.922	2.925	2.877	2.889	2.893	2.854
6	10	3.561	3.601	3.500	3.561	3.601	3.486
	20	3.471	3.480	3.398	3.471	3.480	3.396
	40	3.424	3.427	3.353	3.424	3.427	3.351
10	10	3.830	3.855	3.752	3.978	4.003	3.823
	20	3.760	3.766	3.676	3.836	3.842	3.725
	40	3.724	3.726	3.645	3.763	3.765	3.671
$p=3$							
3	10	3.535	3.654	3.587	3.362	3.487	3.490
	20	3.371	3.398	3.345	3.282	3.309	3.290
	40	3.286	3.293	3.248	3.241	3.247	3.214
6	10	3.952	4.014	3.920	3.952	4.014	3.899
	20	3.829	3.843	3.769	3.829	3.843	3.762
	40	3.766	3.770	3.700	3.766	3.770	3.700
10	10	4.192	4.230	4.137	4.376	4.412	4.219
	20	4.100	4.108	4.026	4.195	4.203	4.089
	40	4.053	4.055	3.982	4.101	4.103	4.019
$p=5$							
3	10	4.264	4.540	4.465	4.010	4.303	4.355
	20	4.001	4.061	4.013	3.868	3.930	3.931
	40	3.863	3.877	3.838	3.795	3.809	3.788
6	10	4.589	4.711	4.616	4.589	4.711	4.586
	20	4.403	4.430	4.358	4.403	4.430	4.350
	40	4.307	4.313	4.254	4.307	4.313	4.251
10	10	4.778	4.845	4.759	5.022	5.086	4.852
	20	4.642	4.658	4.588	4.770	4.785	4.660
	40	4.573	4.577	4.514	4.638	4.642	4.557

Table 1. Continued

		$\alpha = 0.01$					
$p = 2$		M.N.($\kappa = 0$)			C.N.($\kappa = 1.78$)		
k	N	$q_{B\cdot\chi^2}$	$q_{B\cdot F}$	q^*	$q_{B\cdot\chi^2}$	$q_{B\cdot F}$	q^*
	10	3.736	3.856	3.819	3.475	3.605	3.673
3	20	3.561	3.589	3.558	3.427	3.455	3.471
	40	3.471	3.477	3.456	3.402	3.409	3.403
	10	4.081	4.143	4.087	4.081	4.143	4.053
6	20	3.955	3.969	3.923	3.955	3.969	3.919
	40	3.890	3.894	3.857	3.890	3.894	3.854
	10	4.290	4.327	4.275	4.508	4.544	4.364
10	20	4.197	4.206	4.159	4.310	4.318	4.231
	40	4.150	4.152	4.117	4.207	4.209	4.154
$p = 3$							
	10	3.736	3.856	3.819	3.475	3.605	3.673
3	20	3.561	3.589	3.558	3.427	3.455	3.471
	40	3.471	3.477	3.456	3.402	3.409	3.403
	10	4.081	4.143	4.087	4.081	4.143	4.053
6	20	3.955	3.969	3.923	3.955	3.969	3.919
	40	3.890	3.894	3.857	3.890	3.894	3.854
	10	4.290	4.327	4.275	4.508	4.544	4.364
10	20	4.197	4.206	4.159	4.310	4.318	4.231
	40	4.150	4.152	4.117	4.207	4.209	4.154
$p = 3$							
	10	3.736	3.856	3.819	3.475	3.605	3.673
3	20	3.561	3.589	3.558	3.427	3.455	3.471
	40	3.471	3.477	3.456	3.402	3.409	3.403
	10	4.081	4.143	4.087	4.081	4.143	4.053
6	20	3.955	3.969	3.923	3.955	3.969	3.919
	40	3.890	3.894	3.857	3.890	3.894	3.854
	10	4.290	4.327	4.275	4.508	4.544	4.364
10	20	4.197	4.206	4.159	4.310	4.318	4.231
	40	4.150	4.152	4.117	4.207	4.209	4.154

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