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A scale-invariant form of Trudinger-Moser inequality and its best exponent

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0. Introduction

In this note, we study the limit case of Sobolev's inequalities; suppose $N \geq 2$ and let $D \subset \mathbf{R}^N$ be an open set. We denote by $W_0^{1,N}(D)$ the usual Sobolev space with the norm $\|u\|_{W_0^{1,p}(D)} = \|\nabla u\|_p + \|u\|_p$. Here

$$\|u\|_p = \left(\int |u|^p \, dx\right)^{1/p}.$$

The case $p = N$ is the limit case of Sobolev imbeddings and it is known that

$$W_0^{1,N}(D) \subset L^q(D) \quad \text{for } N \leq q < \infty,$$
$$W_0^{1,N}(D) \not\subset L^\infty(D).$$

This case is studied by Trudinger [**8**] more precisely and he showed for bounded domains $D \subset \mathbf{R}^N$

$$\int_D \exp\left(\alpha \left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{N/(N-1)}\right) dx \leq C\,|D| \tag{0.1}$$

for $u \in W_0^{1,N}(D) \setminus \{0\}$, where the constants α, C are independent of u and D.

Trudinger's result is extended into two directions; the first one is to find the best exponents in (0.1). Moser [4] proved that (0.1) holds for $\alpha \leq \alpha_N$ but not for $\alpha > \alpha_N$, where

$$\alpha_N = N\omega_{N-1}^{1/(N-1)} \tag{0.2}$$

and ω_{N-1} is the surface area of the unit sphere in $\mathbf{R}^N$. See also D. R. Adams [**2**]. The second direction is to extend Trudinger's result for unbounded domains and for Sobolev spaces of higher order and fractional order. We refer to R. A. Adams [3], Ogawa [5], Ogawa-Ozawa [6], Ozawa [7].

Here, we study a version of Trudinger inequalities in $\mathbf{R}^N$ and their best exponents; we show

$$\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right) dx \le C\frac{\|u(x)\|_N^N}{\|\nabla u\|_N^N} \tag{0.3}$$

for $u \in W^{1,N}(\mathbf{R}^N) \setminus \{0\}$, where

$$\Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2} \frac{1}{j!}\xi^j$$

and α, $C > 0$ is independent of u. This type of inequality was first introduced in [5] for $N = 2$ and extended in [7] for $N \ge 3$ and for Sobolev spaces of fractional order. As to the proof of the inequality (0.3), following the original idea of Trudinger, [5, 6, 7] made use of a combination of the power series expansion of the exponential function and sharp multiplicative inequalities:

$$\|u\|_q \le C(N,q)\|u\|_N^{N/q}\|\nabla u\|_N^{1-N/q}.$$

Our aim is to give a simplified proof of (0.3) and the best exponents α for (0.3).

One of the virtue of the inequality (0.3) is its scale-invariance; for $u \in W^{1,N}(\mathbf{R}^N)\setminus\{0\}$ and $\lambda > 0$, we set

$$u_\lambda(x) = u(\lambda x). \tag{0.4}$$

We can easily see that

$$\|\nabla u_\lambda\|_N = \|\nabla u\|_N,$$
$$\|u_\lambda\|_N^N = \lambda^{-N}\|u\|_N^N,$$
$$\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u_\lambda(x)|}{\|\nabla u_\lambda\|_N}\right)^{\frac{N}{N-1}}\right) dx = \lambda^{-N}\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right) dx.$$

Thus (0.3) is invariant under the scaling (0.4).

Our main result is the following.

Theorem 0.1 ([1]). *Suppose $N \ge 2$. Then for any $\alpha \in (0, \alpha_N)$, where α_N is given in (0.2), there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right) dx \le C_\alpha\frac{\|u(x)\|_N^N}{\|\nabla u\|_N^N} \quad \textit{for } u \in W^{1,N}(\mathbf{R}^N) \setminus \{0\}. \tag{0.5}$$

Next we show that the restriction $\alpha < \alpha_N$ is optimal. The limit exponent α_N is excluded for (0.5). It is quite different from Moser's result for (0.1).

Theorem 0.2 ([1]). *For $\alpha \geq \alpha_N$, there exists a sequence $(u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbf{R}^N)$ such that*

$$\|\nabla u_k\|_N = 1 \tag{0.6}$$

and

$$\frac{1}{\|u_k\|_N^N}\int_{\mathbf{R}^N} \Phi_N\left(\alpha\,|u_k(x)|^{\frac{N}{N-1}}\right)dx \geq \frac{1}{\|u_k\|_N^N}\int_{\mathbf{R}^N} \Phi_N\left(\alpha_N\,|u_k(x)|^{\frac{N}{N-1}}\right)dx \\ \to \infty \tag{0.7}$$

as $k \to \infty$.

1. Proof of Theorem 0.1

To prove Theorem 0.1, we use an idea of Moser [4]. By means of symmetrization, it suffices to show the desired inequality (0.5) for functions $u(x) = u(|x|)$, which are non-negative, compactly supported, radially symmetric, and $u(|x|) : [0,\infty) \to \mathbf{R}$ are decreasing.

Following Moser's argument, we set

$$w(t) = N^{\frac{N-1}{N}}\omega_{N-1}^{\frac{1}{N}}u\left(e^{-\frac{t}{N}}\right), \qquad |x|^N = e^{-t}. \tag{1.1}$$

Then $w(t)$ is defined on $(-\infty,\infty)$ and satisfies

$$w(t) \geq 0 \quad \text{for } t \in \mathbf{R}, \tag{1.2}$$

$$\dot{w}(t) \geq 0 \quad \text{for } t \in \mathbf{R}, \tag{1.3}$$

$$w(t_0) = 0 \quad \text{for some } t_0 \in \mathbf{R}. \tag{1.4}$$

Moreover we have

$$\int_{\mathbf{R}^N} |\nabla u|^N\,dx = \int_{-\infty}^{\infty} |\dot{w}(t)|^N\,dt, \tag{1.5}$$

$$\int_{\mathbf{R}^N} \Phi_N\left(\alpha u^{\frac{N}{N-1}}\right)dx = \frac{\omega_{N-1}}{N}\int_{-\infty}^{\infty} \Phi_N\left(\frac{\alpha}{\alpha_N}w(t)^{\frac{N}{N-1}}\right)e^{-t}\,dt, \tag{1.6}$$

$$\int_{\mathbf{R}^N} |u(x)|^N\,dx = \frac{1}{N^N}\int_{-\infty}^{\infty} |w(t)|^N e^{-t}\,dt. \tag{1.7}$$

Thus, to prove Theorem 0.1, it suffices to show that for any $\beta \in (0,1)$ there exists a constant $C_\beta > 0$ such that

$$\int_{-\infty}^{\infty} \Phi_N\left(\beta w(t)^{\frac{N}{N-1}}\right)e^{-t}\,dt \leq C_\beta \int_{-\infty}^{\infty} |w(t)|^N e^{-t}\,dt \tag{1.8}$$

for all functions $w(t)$ satisfying (1.2)–(1.4) and

$$\int_{-\infty}^{\infty} |\dot{w}(t)|^N \, dt = 1. \tag{1.9}$$

Proof of Theorem 0.1. Let $w(t)$ be a function satisfying (1.2)–(1.4) and (1.9). We set

$$T_0 = \sup\{t \in \mathbf{R}; w(t) \le 1\} \in (-\infty, \infty].$$

We decompose the integral in the left hand side of (1.8) according to the decomposition $(-\infty, \infty) = (-\infty, T_0] \cup [T_0, \infty)$.

For $t \in (-\infty, T_0]$, we have $w(t) \in [0, 1]$. We can find a constant $m_N > 0$ such that

$$\Phi_N(\xi) \le m_N \xi^{N-1} \quad \text{for } \xi \in [0, 1].$$

Thus we have

$$\int_{-\infty}^{T_0} \Phi_N\left(\beta w(t)^{\frac{N}{N-1}}\right) e^{-t} \, dt \le m_N \int_{-\infty}^{T_0} w(t)^N e^{-t} \, dt. \tag{1.10}$$

Next we consider the integral over $[T_0, \infty)$. Since $w(T_0) = 1$, we have for $t \ge T_0$

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^{t} \dot{w}(\tau) d\tau \\ &\le w(T_0) + (t - T_0)^{\frac{N-1}{N}} \left(\int_{T_0}^{\infty} \dot{w}(\tau)^N d\tau\right)^{\frac{1}{N}} \\ &\le 1 + (t - T_0)^{\frac{N-1}{N}}. \end{aligned}$$

We remark that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$1 + s^{\frac{N-1}{N}} \le ((1 + \varepsilon)s + C_\varepsilon)^{\frac{N-1}{N}} \quad \text{for all } s \ge 0.$$

Thus, we have

$$|w(t)|^{\frac{N}{N-1}} \le (1 + \varepsilon)(t - T_0) + C_\varepsilon \quad \text{for } t \ge T_0.$$

Since $\beta \in (0, 1)$, we can choose $\varepsilon > 0$ small so that $\beta(1 + \varepsilon) < 1$. Thus we have

$$\begin{aligned} \int_{T_0}^{\infty} \Phi_N\left(\beta w(t)^{\frac{N}{N-1}}\right) e^{-t} \, dt &\le \int_{T_0}^{\infty} \exp\left(\beta w(t)^{\frac{N}{N-1}} - t\right) \, dt \\ &\le \int_{T_0}^{\infty} \exp\left((\beta(1 + \varepsilon) - 1)(t - T_0) + \beta C_\varepsilon - T_0\right) \, dt \\ &= \frac{1}{1 - \beta(1 + \varepsilon)} e^{\beta C_\varepsilon} e^{-T_0}. \end{aligned} \tag{1.11}$$

On the other hand,
$$\int_{T_0}^{\infty} |w(t)|^N e^{-t}\,dt \geq \int_{T_0}^{\infty} e^{-t}\,dt = e^{-T_0}. \tag{1.12}$$
Therefore it follows from (1.11) and (1.12) that
$$\int_{T_0}^{\infty} \Phi_N\left(\beta w(t)^{\frac{N}{N-1}}\right) e^{-t}\,dt \leq \frac{e^{\beta C_\varepsilon}}{1-\beta(1+\varepsilon)} \int_{T_0}^{\infty} |w(t)|^N e^{-t}\,dt. \tag{1.13}$$

Thus, setting $C_\beta = \max\{m_N, \frac{e^{\beta C_\varepsilon}}{1-\beta(1+\varepsilon)}\}$, we obtain (1.8). ∎

2. Proof of Theorem 0.2

It suffices to show Theorem 0.2 for $\alpha = \alpha_N$. We use the idea of Moser again. Repeating the argument of the previous section, it suffices to find a sequence of functions $w_k(t) : \mathbf{R} \to \mathbf{R}$ which satisfies (1.1)–(1.4), (1.9) and
$$\int_{-\infty}^{\infty} |w_k(t)|^N e^{-t}\,dt \to 0 \quad \text{as } k \to \infty, \tag{2.1}$$
$$\int_{-\infty}^{\infty} \Phi_N\left(w_k(t)^{\frac{N}{N-1}}\right) e^{-t}\,dt \geq \frac{1}{2} \quad \text{for large } k. \tag{2.2}$$

If we define a sequence of functions $(u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbf{R}^N)$ from $(w_k(t))_{k=1}^{\infty}$ through the relation (1.1), it satisfies (0.6) and (0.7).

Here we give an example of $(w_k(t))_{k=1}^{\infty}$ explicitly:
$$w_k(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ k^{\frac{N-1}{N}} \dfrac{t}{k} & \text{for } 0 \leq t \leq k, \\ k^{\frac{N-1}{N}} & \text{for } k \leq t. \end{cases}$$

Such functions appeared in [4] to show that the integral in the left hand side of (0.1) can be made arbitrarily large for $\alpha > \alpha_N$. ∎

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