数理解析研究所講究録
1102巻 1999年 64-75

A generalization of Calderón-Vaillancourt's Theorem

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1 Introduction

In this paper we study L^2 boundedness of pseudodifferential operators with Weyl symbols. We give a generalization of Calderón-Vaillancourt's theorem.

For a symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ we associate a pseudodifferential operator L_σ. If $\sigma \in L^2(\mathbf{R}^{2d})$, then we can easily prove that L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$. On the other hand Calderón and Vaillancourt gave another condition for the L^2 boundedness of pseudodifferential operators([1], [2]). Their condition is about pseudodifferential operators with Kohn-Nirenberg symbols. Similar results hold for the Weyl symbol case([3], [10]). A generalization of their results is known([11]). This generalization does not contain the L^2 symbol case. In this paper we give a generalization of both results.

First we give the definition of pseudodifferential operators with Weyl symbols.

Let $W(f,g)$ be the Wigner transform of $f, g \in \mathcal{S}(\mathbf{R}^d)$, that is,

$$W(f,g)(x,\xi) = \int_{\mathbf{R}^d} e^{-2\pi i \xi \cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)}\, dp$$

for $x, \xi \in \mathbf{R}^d$.

For $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ and $f \in \mathcal{S}(\mathbf{R}^d)$, we define $L_\sigma f \in \mathcal{S}'(\mathbf{R}^d)$ as

$$(L_\sigma f, g) = (\sigma, W(g,f))$$

for all $g \in \mathcal{S}$, where we use the notation $(F, f) = \langle F, \overline{f} \rangle$ for $F \in \mathcal{S}', f \in \mathcal{S}$. It turns out that L_σ is a continuous linear operator from $\mathcal{S}$ to $\mathcal{S}'$. We call L_σ a pseudodifferential operator with Weyl symbol σ(cf.[10]).

In Folland [10] it is proved that if $\sigma \in C^{2d+1}(\mathbf{R}^{2d})$ and

$$\sum_{|\alpha|+|\beta|\leq 2d+1} \|\partial_x^\alpha \partial_\xi^\beta \sigma\|_\infty < \infty,$$

then L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$. This is the Calderón-Vaillancourt theorem for pseudodifferential operators with Weyl symbols.

In [11] Gröchenig and Heil proved a generalization of Calderón-Vaillancourt's theorem. If $\sigma \in M_{\infty,1}(\mathbf{R}^{2d})$, then L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$, where $M_{\infty,1}(\mathbf{R}^{2d})$ denotes the set of $F \in \mathcal{S}'(\mathbf{R}^{2d})$ satisfying

$$\int_{\mathbf{R}^{2d}} \sup_{x \in \mathbf{R}^{2d}} |(F, e^{i\xi \cdot - (\cdot - x)^2})| \, d\xi < \infty.$$

In this paper we give a generalization of these results.

Let

$$\varphi(x) = 2^{d/4} e^{-\pi|x|^2}, \quad x \in \mathbf{R}^d \tag{1}$$

and

$$\phi(x,y) = W(\varphi, \varphi)(x,y) = 2^d e^{-2\pi(|x|^2+|y|^2)}, \quad (x,y) \in \mathbf{R}^{2d}.$$

For $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ and $\alpha, \beta \in \mathbf{R}^{2d}$ we define

$$S_\phi(\sigma)(\alpha, \beta) = (\sigma, e^{2\pi i \beta \cdot} \phi(\cdot - \alpha)).$$

For $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{R}^d$, we set

$$N(\alpha, \beta) = (-\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_2 + \beta_2}{2}, \alpha_2 - \beta_2, \alpha_1 - \beta_1).$$

For $\xi, \eta \in \mathbf{R}^{2d}$ we set

$$F(\xi, \eta) = |S_\phi(\sigma)(N(\xi, \eta))|$$

and

$$k(\xi) = (1 + |\xi|)^{-2d-1}. \tag{2}$$

For each positive integer n and $a, b \in \mathbf{R}^{2d}$ we set

$$\begin{aligned} G_n(a,b) &= \Big[\int_{\mathbf{R}^{2d}} \cdots \int_{\mathbf{R}^{2d}} k(a-\eta_1)F(\xi_1,\eta_1)k(\xi_1-\xi_1')F(\xi_1',\eta_1')k(\eta_1'-\eta_2) \\ &\quad \times F(\xi_2,\eta_2)k(\xi_2-\xi_2')F(\xi_2',\eta_2')k(\eta_2'-\eta_3)\cdots \\ &\quad \times F(\xi_n,\eta_n)k(\xi_n-\xi_n')F(\xi_n',\eta_n')k(\eta_n'-b)\, d\xi_1 d\eta_1 \cdots d\xi_n' d\eta_n'\Big]^{1/n}. \end{aligned}$$

Theorem 1.1 *We assume*

$$G_n(a,b) < \infty$$

for every $n \in \mathbf{N}, a, b \in \mathbf{R}^{2d}$ *and*

$$\sup_{n\in\mathbf{N}} \sup_{a\in\mathbf{R}^{2d}} G_n(a,a) < \infty.$$

Then L_σ *extends to a bounded operator on* $L^2(\mathbf{R}^d)$.

Corollary 1.1 *Let* $F(\xi,\eta)$ *be a function as above. For* $1 \le p \le \infty$ *and* $p^{-1} + p'^{-1} = 1$ *we assume*

$$\left\{\int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi,\eta)^p\, d\xi\right)^{p'/p} d\eta\right\}^{1/p'} < \infty$$

and

$$\left\{\int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi,\eta)^p\, d\eta\right)^{p'/p} d\xi\right\}^{1/p'} < \infty.$$

Then L_σ *extends to a bounded operator on* $L^2(\mathbf{R}^d)$.

Remark 1.1 *The case* $p = 1$ *or* $p = \infty$ *is mentioned in Gröchenig and Heil [11]. They used this fact to prove their result on* L^2 *boundedness of pseudodifferential operators. When* $p = 2$, *the conditions in the corollary is equivalent to saying* $\sigma \in L^2(\mathbf{R}^{2d})$. *When* $1 < p < 2$ *or* $2 < p < \infty$, *our corollary gives a new result.*

Remark 1.2 *We can prove similar results about the boundedness of pseudodifferential operators on Sobolev spaces.*

2 Proof of Theorem 1.1

First we recall the definition of the Weyl-Heisenberg frame. For $a = (x, y) \in \mathbf{R}^d \times \mathbf{R}^d$ and $f \in L^2(\mathbf{R}^d)$ we set

$$\rho(a)f(t) = \rho(x,y)f(t) = e^{\pi i x\cdot y}e^{2\pi i y\cdot t}f(t+x),$$

where $t \in \mathbf{R}^d$.

Let $\varphi(x)$ be the function defined by (1). Let $I = \mathbf{Z}^d \times \frac{1}{2\pi}\mathbf{Z}^d$. Then $\{\rho(a)\varphi\}_{a\in I}$ is a frame of $L^2(\mathbf{R}^d)$, that is, there exist positive constants C_1 and C_2 such that

$$C_1\|f\|_2^2 \le \sum_{a\in I}|(f,\rho(a)\varphi)|^2 \le C_2\|f\|_2^2$$

for all $f \in L^2(\mathbf{R}^d)$. The dual frame of $\{\rho(a)\varphi\}_{a\in I}$ is given by $\{\rho(a)\tilde{\varphi}\}_{a\in I}$, where $\tilde{\varphi}$ is the function in $\mathcal{S}(\mathbf{R}^d)$ which is constructed by φ. Furthermore we have

$$C_2^{-1}\|f\|_2^2 \le \sum_{a\in I}|(f,\rho(a)\tilde{\varphi})|^2 \le C_1^{-1}\|f\|_2^2$$

for all $f \in L^2(\mathbf{R}^d)$(cf.[5]).

By the frame theory we have the following proposition.

Proposition 2.1 **(i)** *For every $f \in L^2(\mathbf{R}^d)$ we have*

$$f = \sum_{a\in I}(f,\rho(a)\varphi)\rho(a)\tilde{\varphi} \tag{3}$$

$$= \sum_{a\in I}(f,\rho(a)\tilde{\varphi})\rho(a)\varphi \tag{4}$$

which converge in $L^2(\mathbf{R}^d)$.

(ii) *There exists a* $K > 0$ *such that*

$$\left\| \sum_{a \in I} c_a \rho(a)\varphi \right\|_2 \le K \left(\sum_{a \in I} |c_a|^2 \right)^{1/2}$$

for all $\{c_a\} \in \ell^2(I)$.

(iii) *For every* $f \in \mathcal{S}(\mathbf{R}^d)$ *we have the expansions (3) and (4) in* $\mathcal{S}$.

(iv) *For every* $f \in \mathcal{S}'(\mathbf{R}^d)$ *we have the expansions (3) and (4) in* $\mathcal{S}'$.

The proofs of (i) and (ii) is in [5]. The properties (iii) and (iv) are consequences of Feichtinger and Gröchenig's result([7], [8], [9], [14]).

Let $f \in \mathcal{S}(\mathbf{R}^d)$. By (iii) and (iv) of Proposition 2.1 we have

$$L_\sigma f = \sum_{a \in I} (L_\sigma f, \rho(a)\tilde{\varphi})\rho(a)\varphi$$

in $\mathcal{S}'$.

If we show

$$\sum_{a \in I} |(L_\sigma f, \rho(a)\tilde{\varphi})|^2 < \infty,$$

then we conclude $L_\sigma f \in L^2(\mathbf{R}^d)$ and

$$\|L_\sigma f\|_2^2 \le C \sum_{a \in I} |(L_\sigma f, \rho(a)\tilde{\varphi})|^2,$$

where we used (ii) of Proposition 2.1.

Here we have

$$\sum_{a \in I} |(L_\sigma f, \rho(a)\tilde{\varphi})|^2 = \sum_{a \in I} |\sum_{b \in I} (f, \rho(b)\tilde{\varphi})(L_\sigma \rho(b)\varphi, \rho(a)\tilde{\varphi})|^2.$$

If the infinite matrix $\{(L_\sigma \rho(b)\varphi, \rho(a)\tilde{\varphi})\}_{a,b \in I}$ is bounded on $\ell^2(I)$, then we conclude that

$$\|L_\sigma f\|_2^2 \le C \sum_{a \in I} |(f, \rho(a)\tilde{\varphi})|^2 \le C' \|f\|_2^2 < \infty.$$

Therefore L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$.

Now we use the following lemma to show the boundedness of an infinite matrix on ℓ^2.

Lemma 2.1 *([4]) Let $A = (a_{ij})$ be an infinite matrix which acts on the sequence space $\ell^2(\mathbf{N})$. Then the boundedness of A from $\ell^2(\mathbf{N})$ to $\ell^2(\mathbf{N})$ is equivalent to the following two conditions.*

(a) *For every $n \in \mathbf{N}$, $(A^*A)^n$ is well defined.*

(b)

$$\sup_{n\in\mathbf{N}} \sup_{i\in\mathbf{N}} |[(A^*A)^n]_{ii}|^{1/n} < \infty,$$

where $[(A^*A)^n]_{ii}$ is the (i,i) component of $(A^*A)^n$.

Remark 2.1 *In [4] Crone gave an additional condition. In [13] Maddox and Wickstead showed that the condition is unnecessary(cf.[12]).*

We shall show that the infinite matrix $\{(L_\sigma\rho(b)\varphi, \rho(a)\tilde{\varphi})\}_{a,b\in I}$ satisfies the conditions of Lemma 2.1. In order to prove this we use the following lemma by Gröchenig and Heil [11].

Lemma 2.2 *For $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ and $f, g \in \mathcal{S}(\mathbf{R}^d)$, we have*

$$(L_\sigma f, g) = \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} S_\phi(\sigma)(N(\alpha,\beta))e^{-2\pi i[\alpha,\beta]}(f, \rho(\beta)\varphi)(\rho(\alpha)\varphi, g)\, d\alpha d\beta,$$

where $[\alpha,\beta] = \frac{1}{2}(\alpha_2 \cdot \beta_1 - \alpha_1 \cdot \beta_2)$ for $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbf{R}^d \times \mathbf{R}^d$.

First we show the condition (a) of the Lemma 2.1.

Let $A = \{(L_\sigma\rho(b)\varphi, \rho(a)\tilde{\varphi})\}_{a,b\in I}$, $n \in \mathbf{N}$ and $c, d \in I$. The (c,d) component of the infinite matrix $(A^*A)^n$ is given by

$$\sum_{a_1,\ldots,a_n,b_1,\ldots,b_{n-1}\in I} \overline{(L_\sigma\rho(c)\varphi, \rho(a_1)\tilde{\varphi})}(L_\sigma\rho(b_1)\varphi, \rho(a_1)\tilde{\varphi})\overline{(L_\sigma\rho(b_1)\varphi, \rho(a_2)\tilde{\varphi})}$$
$$\times(L_\sigma\rho(b_2)\varphi, \rho(a_2)\tilde{\varphi})\cdots\overline{(L_\sigma\rho(b_{n-1})\varphi, \rho(a_n)\tilde{\varphi})}(L_\sigma\rho(d)\varphi, \rho(a_n)\tilde{\varphi}),$$

where we will show that the series in the above sum absolutely converges.

By Lemma 2.2 we have

$$\begin{aligned}
&\overline{(L_\sigma\rho(c)\varphi,\rho(a_1)\tilde{\varphi})}(L_\sigma\rho(b_1)\varphi,\rho(a_1)\tilde{\varphi})\overline{(L_\sigma\rho(b_1)\varphi,\rho(a_2)\tilde{\varphi})}\\
&\qquad\times(L_\sigma\rho(b_2)\varphi,\rho(a_2)\tilde{\varphi})\cdots\overline{(L_\sigma\rho(b_{n-1})\varphi,\rho(a_n)\tilde{\varphi})}(L_\sigma\rho(d)\varphi,\rho(a_n)\tilde{\varphi})\\
=&\int\int\overline{S_\phi(\sigma)(N(\xi_1,\eta_1))}e^{2\pi i[\xi_1,\eta_1]}\overline{(\rho(c)\varphi,\rho(\eta_1)\varphi)(\rho(\xi_1)\varphi,\rho(a_1)\tilde{\varphi})}\,d\xi_1 d\eta_1\\
&\cdot\times\int\int S_\phi(\sigma)(N(\xi_1',\eta_1'))e^{-2\pi i[\xi_1',\eta_1']}(\rho(b_1)\varphi,\rho(\eta_1')\varphi)(\rho(\xi_1')\varphi,\rho(a_1)\tilde{\varphi})\,d\xi_1' d\eta_1'\cdots\\
&\times\int\int\overline{S_\phi(\sigma)(N(\xi_n,\eta_n))}e^{2\pi i[\xi_n,\eta_n]}\overline{(\rho(b_{n-1})\varphi,\rho(\eta_n)\varphi)(\rho(\xi_n)\varphi,\rho(a_n)\tilde{\varphi})}\,d\xi_n d\eta_n\\
&\times\int\int S_\phi(\sigma)(N(\xi_n',\eta_n'))e^{-2\pi i[\xi_n',\eta_n']}(\rho(d)\varphi,\rho(\eta_n')\varphi)(\rho(\xi_n')\varphi,\rho(a_n)\tilde{\varphi})\,d\xi_n' d\eta_n'.
\end{aligned}$$

Let $k(\xi)$ be the function defined by (2). Since

$$|(\rho(b)\varphi,\rho(\eta)\varphi)|\le|(\rho(b-\eta)\varphi,\varphi)|\le ck(b-\eta)$$

and

$$|(\rho(\xi)\varphi,\rho(a)\tilde{\varphi})|\le|(\rho(\xi-a)\varphi,\tilde{\varphi})|\le ck(\xi-a)$$

for all $a,b\in I$ and $\xi,\eta\in\mathbf{R}^{2d}$, we have

$$\begin{aligned}
&\sum_{a_1,\ldots,a_n,b_1,\ldots,b_{n-1}\in I}\int\cdots\int|S_\phi(\sigma)(N(\xi_1,\eta_1))||(\rho(c)\varphi,\rho(\eta_1)\varphi)||(\rho(\xi_1)\varphi,\rho(a_1)\tilde{\varphi})|\\
&\times|S_\phi(\sigma)(N(\xi_1',\eta_1'))||(\rho(b_1)\varphi,\rho(\eta_1')\varphi)||(\rho(\xi_1')\varphi,\rho(a_1)\tilde{\varphi})|\cdots\\
&\times|S_\phi(\sigma)(N(\xi_n,\eta_n))||(\rho(b_{n-1})\varphi,\rho(\eta_n)\varphi)||(\rho(\xi_n)\varphi,\rho(a_n)\tilde{\varphi})||S_\phi(\sigma)(N(\xi_n',\eta_n'))|\\
&\times|(\rho(d)\varphi,\rho(\eta_n')\varphi)||(\rho(\xi_n')\varphi,\rho(a_n)\tilde{\varphi})|\,d\xi_1 d\eta_1 d\xi_1' d\eta_1'\cdots d\xi_n d\eta_n d\xi_n' d\eta_n'\\
\le\;& C^n\sum_{a_1,\ldots,a_n,b_1,\ldots,b_{n-1}\in I}\int\cdots\int F(\xi_1,\eta_1)k(c-\eta_1)k(\xi_1-a_1)F(\xi_1',\eta_1')k(b_1-\eta_1')\\
&\times k(\xi_1'-a_1)F(\xi_2,\eta_2)\cdots k(\xi_n-a_n)F(\xi_n',\eta_n')k(d-\eta_n')k(\xi_n'-a_n)\,d\xi_1\cdots d\eta_n'.
\end{aligned}$$

Since

$$\sum_{b\in I}k(\xi-b)k(\xi'-b)\le ck(\xi-\xi')$$

for all $\xi, \xi' \in \mathbf{R}^d$, the above quantity is bounded by

$$\begin{aligned} & C^n \int \cdots \int k(c-\eta_1)F(\xi_1,\eta_1)k(\xi_1-\xi_1')F(\xi_1',\eta_1')k(\eta_1'-\eta_2)F(\xi_2,\eta_2)\cdots \\ & \quad \times k(\xi_n-\xi_n')F(\xi_n',\eta_n')k(d-\eta_n')\,d\xi_1\cdots d\eta_n' \\ = \ & C^n G_n(c,d)^n < \infty. \end{aligned}$$

Hence we conclude that the (c,d) component of the infinite matrix $(A^*A)^n$ is well defined.

Next we check the condition (b). By similar calculations we have

$$\begin{aligned} & \Bigg| \sum_{a_1,\dots,a_n,b_1,\dots,b_{n-1}\in I} \overline{(L_\sigma\rho(c)\varphi,\rho(a_1)\tilde{\varphi})}(L_\sigma\rho(b_1)\varphi,\rho(a_1)\tilde{\varphi})\overline{(L_\sigma\rho(b_1)\varphi,\rho(a_2)\tilde{\varphi})} \\ & \quad \times (L_\sigma\rho(b_2)\varphi,\rho(a_2)\tilde{\varphi})\cdots\overline{(L_\sigma\rho(b_{n-1})\varphi,\rho(a_n)\tilde{\varphi})}(L_\sigma\rho(d)\varphi,\rho(a_n)\tilde{\varphi})\Bigg| \\ \leq \ & C^n G_n(a,a)^n. \end{aligned}$$

Hence n-th root of the quantity of the left hand side is bounded by $CG_n(a,a)$. By the assumption we get the condition (b).

3 Proof of Corollary 1.1

We shall prove Corollary 1.1 for $1 < p < \infty$. The proof for $p = 1$ or $p = \infty$ is similar. We set

$$\left\{ \int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi,\eta)^p \, d\xi \right)^{p'/p} d\eta \right\}^{1/p'} = K_1,$$

$$\left\{ \int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi,\eta)^p \, d\eta \right)^{p'/p} d\xi \right\}^{1/p'} = K_2$$

and $K = \max\{K_1, K_2\}$. We shall estimate

(5)
$$\begin{aligned} & \int \cdots \int k(a-\eta_1)F(\xi_1,\eta_1)k(\xi_1-\xi_1')F(\xi_1',\eta_1')k(\eta_1'-\eta_2)F(\xi_2,\eta_2) \\ & \quad \times k(\xi_2-\xi_2')\cdots F(\xi_n,\eta_n)k(\xi_n-\xi_n')F(\xi_n',\eta_n')k(\eta_n'-b)\,d\xi_1 d\eta_1\cdots d\xi_n' d\eta_n' \end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned}
&\int k(a-\eta_1)F(\xi_1,\eta_1)\,d\eta_1 \\
\leq\ &\left(\int k(a-\eta_1)^{p'}\,d\eta_1\right)^{1/p'}\left(\int F(\xi_1,\eta_1)^p\,d\eta_1\right)^{1/p} \\
\leq\ &C_1\left(\int F(\xi_1,\eta_1)^p\,d\eta_1\right)^{1/p},
\end{aligned}$$

where we set

$$C_1=\left(\int k(a-\eta_1)^{p'}\,d\eta_1\right)^{1/p'}.$$

Hence (5) is bounded by

$$C_1\int\cdots\int\left(\int F(\xi_1,\eta_1)^p\,d\eta_1\right)^{1/p}k(\xi_1-\xi_1')F(\xi_1',\eta_1')k(\eta_1'-\eta_2)F(\xi_2,\eta_2)k(\xi_2-\xi_2')$$

$$\cdots F(\xi_n,\eta_n)k(\xi_n-\xi_n')F(\xi_n',\eta_n')k(\eta_n'-b)\,d\xi_1d\xi_1'\cdots d\xi_n'd\eta_n'.$$

Now we have

$$\begin{aligned}
&\int\left(\int F(\xi_1,\eta_1)^p\,d\eta_1\right)^{1/p}k(\xi_1-\xi_1')F(\xi_1',\eta_1')\,d\xi_1d\xi_1' \\
\leq&\left\{\int\int\left(\int F(\xi_1,\eta_1)^p\,d\eta_1\right)^{p'/p}k(\xi_1-\xi_1')\,d\xi_1d\xi_1'\right\}^{1/p'} \\
&\times\left(\int\int k(\xi_1-\xi_1')F(\xi_1',\eta_1')^p\,d\xi_1d\xi_1'\right)^{1/p} \\
&\leq C_2K\left(\int F(\xi_1',\eta_1')^p\,d\xi_1'\right)^{1/p},
\end{aligned}$$

where

$$C_2=\int k(t)\,dt.$$

Hence (5) is bounded by

$$C_1C_2K\int\cdots\int\left(\int F(\xi_1',\eta_1')^p\,d\xi_1'\right)^{1/p}k(\eta_1'-\eta_2)F(\xi_2,\eta_2)k(\xi_2-\xi_2')$$

$$\cdots F(\xi_n,\eta_n)k(\xi_n-\xi_n')F(\xi_n',\eta_n')k(\eta_n'-b)\,d\eta_1'd\xi_2\cdots d\xi_n'd\eta_n'.$$

By repeating this calculation, we can estimate (5) by

$$
\cdot C_1C_2^{2n-1}K^{2n-1}\int\left(\int F(\xi_n',\eta_n')^p\,d\xi_n'\right)^{1/p}k(\eta_n'-b)\,d\eta_n'
$$
$$
\leq C_1C_2^{2n-1}K^{2n-1}\left\{\int\left(\int F(\xi_n',\eta_n')^p\,d\xi_n'\right)^{p'/p}d\eta_n'\right\}^{1/p'}\left(\int k(\eta_n'-b)^p\,d\eta_n'\right)^{1/p}
$$
$$
\leq C_1C_2^{2n-1}C_3K^{2n}<\infty,
$$

where

$$
C_3=\left(\int k(\eta_n'-b)^p\,d\eta_n'\right)^{1/p}.
$$

Hence the condition (a) of Theorem 1.1 is satisfied. Furthermore we have

$$
\sup_{n\in\mathbf{N}}\sup_{a\in\mathbf{R}^{2d}}G_n(a,a)\leq CK^2.
$$

Therefore we proved the L^2 boundedness by Theorem 1.1.

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