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Chauchy problem of nonlinear wave equations with small and smooth initial data

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In this note, we consider Chauchy problem for

$$\Box_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(u, \partial u, \partial^2 u) \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{0.1}$$

$$u^i(x,0) = \varepsilon f^i(x), \quad \partial_t u^i(x,0) = \varepsilon g^i(x) \quad \text{in } \mathbf{R}^n, \tag{0.2}$$

where $i = 1, \cdots, m$, $u^i(x,t)$ is a real valued unknown function, $c_i > 0$ and $\varepsilon > 0$. Besides, $F^i \in C^\infty(\mathbf{R}^{(n+1)m} \times \mathbf{R}^{(n+1)^2 m})$ and f^i, $g^i \in C_0^\infty(\mathbf{R}^n)$. We also denoted $u = (u^1, \cdots, u^m)$ and ∂u, $\partial^2 u$ stand for the first and second derivatives with respect to $\partial_t = \partial/\partial t$ $(\equiv \partial_0)$, $\partial_j = \partial/\partial x_j$ $(j = 1, \cdots, n)$. Roughly speaking, we would like to compare the behavior of the solution $u(x,t)$ to the problem (0.1) and (0.2) with that of $u_0(x,t) = (u_0^1(x,t), \cdots, u_0^m(x,t))$, which is the solution to the homogeneous wave equation

$$\Box_i u^i(x,t) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{0.3}$$

satisfying (0.2), provided the parameter ε is sufficiently small. If $n = 2$ or $n = 3$, the L^∞-norm of $u_0^i(x,t)$ can be controled as follows:

$$|u_0^i(x,t)| \le C\varepsilon(1 + r + t)^{-\frac{n-1}{2}}(1 + |c_i t - r|)^{-\frac{n-1}{2}}, \tag{0.4}$$

provided f^i, $g^i \in C_0^\infty(\mathbf{R}^n)$. In the following, we shall only consider the case where $n = 2$ or $n = 3$. To my knowledge, the imprtance of the factor $(1 + |c_i t - r|)$ is firstly pointed out by Professor F. John in [17]. And the factor also plays an essential role in our analisis. We divide our argument into two parts. First one is concerned with the quasilinear case. This part is a collabolation work with Professor A. Hosiga in [15]. While the second part, which is a joint work with Professor M. Ohta in [30], is concerned with the semilinear case. The author wishes to be thankful to Professor R. Agemi and Professor K. Kubota for their valuable comment.

Key words. A unique global smooth solution, Critical exponent, Small Data Global Existence, Small Data Blow-up, Null condition, Different propagation speeds

1 Quasilinear Case

In this section we study the quasilinear case. Namely, we assume

$$F^i = F^i(\partial u, \partial^2 u) = \sum_{l=1}^{m} \sum_{\gamma,\delta=0}^{n} H_{il}^{\gamma\delta}(\partial u)\partial_\gamma \partial_\delta u^l + K_i(\partial u), \tag{1.1}$$

where $H_{il}^{\gamma\delta}$, $K_i \in C^\infty(\mathbf{R}^{(n+1)m})$ satisfy

$$H_{il}^{\gamma\delta}(\partial u) = O(|\partial u|^{p-1}), \quad K_i(\partial u) = O(|\partial u|^p) \quad \text{near } \partial u = 0. \tag{1.2}$$

Here p is an integer with $p > 1$. In order to derive an energy estimate, we need to assume

$$H_{il}^{\gamma\delta}(\partial u) = H_{li}^{\gamma\delta}(\partial u) = H_{il}^{\delta\gamma}(\partial u). \tag{1.3}$$

Since the existence and the uniqueness of the local smooth solution of (0.1) and (0.2) are already known (see e.g. S. Klainerman [23]), we aim at the global solvbility of the problem. For this purpose, we shall establish a uniform *a priori* estimate of a suitable weighted L^∞-norm $[\partial u(t)]_N$ for some large integer N. More precisely, we wish to prove that if $[\partial u(t)]_N \le 3M_N\varepsilon$ for $0 \le t < T$, then we must have $[\partial u(t)]_N \le 2M_N\varepsilon$ for $0 \le t < T$, provided ε is sufficiently small. Here T is a given positive number and M_N is a constant depending only on the initial data f^i, g^i and the functions $H_{il}^{\gamma\delta}$, K_i. Once we could obtain the above proposition, we get a global solution due to the blow-up criterion. (See e.g. [34]). Althogh our basic strategy to get a uniform *a priori* estimate is based on the method developed by Professor S. Klainerman in [26], we need some modification to handle the case where the system (0.1) has different propagation speeds.

Notations. We introduce the following vector fields:

$$\Gamma = (\Gamma_0, \cdots, \Gamma_{N_0}) = (\partial, \Omega, S) \tag{1.4}$$

where $N_0 = n + \frac{n(n-1)}{2} + 1$ and

$$\begin{gathered}\partial = (\partial_0, \cdots, \partial_n), \quad \Omega = (\Omega_{i,j}), \quad S = t\partial_t + r\partial_r \\ \partial_0 = \partial_t, \quad \partial_j = \partial/\partial_{x_j}\ (1 \le j \le n), \quad \Omega_{i,j} = x_i\partial_j - x_j\partial_i\ (1 \le i < j \le n).\end{gathered}$$

Remark. In our analysis, Ω and S will play a crucial role. Indded, S will be used in (1.39) efeectively and Ω will be used in (1.32), (1.37) and (1.38).

For a vector valued function $u(x,t) = (u^1(x,t), \cdots, u^m(x,t))$, we set

$$\begin{aligned}\|u(t)\|_k &= \sum_{|a|\le k}\sum_{l=1}^m \|\Gamma^a u^l(\cdot,t)\|_{L^2}, \\ |u(x,t)|_k &= \sum_{|a|\le k}\sum_{l=1}^m |\Gamma^a u^l(x,t)|,\end{aligned}$$

where k is a nonnegative integer, $a = (a_0, \cdots, a_{N_0})$ is a multi-index, $\Gamma^a = \Gamma_0^{a_1} \cdots \Gamma_{N_0}^{a_{N_0}}$ and $|a| = a_0 + \cdots + a_{N_0}$. Moreover, we shall use the following weighted L^∞-norm:

$$[u(t)]_k = \sum_{|a|\le k}\sum_{l=1}^m \|w_l(|\cdot|,t)\Gamma^a u^l(\cdot,t)\|_{L^\infty}, \tag{1.5}$$

where w_l is a weight function associated with the l-th component of u defined by

$$w_l(r,t) = (1+r)^{\frac{n-1}{2}-\gamma}(1+t+r)^\gamma(1+|c_l t - r|)^{\frac{n-1}{2}} \quad \text{for } r \ge 0, t \ge 0,$$

where $0 < \gamma < (n-1)/2$. Moreover, we also use

$$\|u\|_{k,T} = \sup_{0<t<T} \|u(t)\|_k, \quad [u]_{k,T} = \sup_{0<t<T} [u(t)]_k.$$

Energy Estimates I. In order to assure the hyperbolicity of the system (0.1), we will consider the following assumption on the solution $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$:

$$\sup_{x\in\mathbf{R}^n} |\partial u(x,t)|_k \le \delta \quad \text{for } 0 \le t < T, \tag{1.6}$$

where k a nonnegative integer and δ $(0 < \delta < 1)$ is a real number. Note that the following commutator relations hold:

$$[\Gamma_\sigma, \Box_i] = -2\delta_{4\sigma}\Box_i \quad \text{for} \quad \sigma = 0,\cdots,N_0,\ i = 1,\cdots,m. \tag{1.7}$$

Here [,] denotes the usual commutator of linear operators and $\delta_{\alpha\beta}$ is Kronecker's delta. Therefore, if u^i satisfies $\Box_i u^i = F^i$, then we have

$$\Box_i \Gamma^a u^i = \sum_{b\le a} C_b \Gamma^b F^i \tag{1.8}$$

for some suitable constant C_b. Then following a standard argument, we obtain the energy estimate as follows. (For the details, see e.g. the proof of Proposition 5.1 in [15]).

Proposition 1. 1 *Let $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ be a solution of* (0.1) *and* (0.2). *Suppose that* (1.1)–(1.3) *hold. Then there is a sufficiently small positive number δ such that if* (1.6) *with $k = [(N+1)/2]$ holds, then we have for $0 \le t < T$*

$$\|\partial u(t)\|_N \le C_N \|\partial u(0)\|_N \exp\left[C_N \int_0^t \{\sup_{x\in\mathbf{R}^n} |\partial u(x,s)|_{[\frac{N+1}{2}]}\}^{p-1} ds\right]. \tag{1.9}$$

Following a formal argument for the moment, we will introduce a "critical exponent" of the nonlinerlity $F^i(\partial u, \partial^2 u)$. Having (0.4) in mind, we suppose that

$$|\partial u(x,t)|_{[\frac{N+1}{2}]} \le C(1+t+r)^{-\frac{n-1}{2}} \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,\infty) \tag{1.10}$$

holds. Then we have

$$\int_0^t \{\sup_{x\in\mathbf{R}^n} |\partial u(x,s)|_{[\frac{N+1}{2}]}\}^{p-1} ds \le C \int_0^t (1+s)^{-\frac{n-1}{2}(p-1)} ds.$$

Therefore, if $(n-1)(p-1)/2 > 1$, i.e., $p > 1 + 2/(n-1)$, we get a bound of $\|\partial u(t)\|_N$ from (1.9). Such a bound indicates the existence of a global solution to the problem (0.1) and (0.2). Hence it is interesting to consider whether or not a global solution to the problem exists when $p = p_c$. Here p_c is a critical exponent defined by

$$p_c := 1 + \frac{2}{n-1} = \begin{cases} 2 & (n=3), \\ 3 & (n=2). \end{cases} \tag{1.11}$$

Known results. The following results assure that the number p_c defined by (1.11) has real meaning. Let $m = 1$ or the system (0.1) has common propagation speeds, i.e., $c_1 = \cdots = c_m$. Suppose that (1.1)–(1.3) hold.

If $p > p_c$, then [Small Data Global Existence] holds. Namely, for any f^i, $g^i \in C^\infty(\mathbf{R}^n)$, the problem (0.1) and (0.2) has a smooth global solution for sufficiently small ε. Moreover, if $p = p_c$, then [Small Data "Almost" Global Existence] holds. Namely, for any f^i, $g^i \in C^\infty(\mathbf{R}^n)$, the problem (0.1) and (0.2) has a smooth local solution whose life span is estimated by $\exp(C/\varepsilon^{p_c-1})$ from below for sufficiently small ε. (See F. John and S. Klainerman [21], S. Klainerman [25] and M. Kovalyov [27], for instance).

On the other hand, if $1 < p \leq p_c$, then [Small Data Blou-up] holds. Namely, [Small Data Global Existence] dose NOT holds. (See R. Agemi [1], S. Alinhac [4], L. Hörmander [12], A. Hoshiga [14] and F. John [18]).

Furthermore, in the critical case $p = p_c$, the following interesting results are known. If the nonlinearity has a special form, a global solution of (0.1) and (0.2) exists, instead of an almost global solution. (See D. Christodoulou [6], P. Godin [11], A. Hoshiga [13], F. John [19], S. Katayama [22] and S. Klainerman [26], for instance). We shall call the restriction on the nonlinearlities null condition, according to S. Klainerman [24]. We shall give its definition for the 2-dimensional case in (1.43) below.

We now turn our attention to the case where $p = p_c$, $m \geq 2$ and the propagation speeds are different from each other. This case was studied by M. Kovalyov in [30] and R. Agemi and K. Yokoyama in [3]. They found some nonlinearities, which do not satisfy null condition (1.43), and proved the existence of the global solution to (0.1) and (0.2). In the following, we shall investigate to find a wider class of nonlinearity which garantees the global solvability of the problem (0.1) and (0.2). (See also (1.42) below.)

Basic estimate. Since one can treat the case where the system (0.1) has common propgation speeds less hard, we shall consider the case where the speeds are disitinct. Namely, we assume

$$c_1 > c_2 > \cdots > c_m. \tag{1.12}$$

Under this situation, we prepear basic estimates of the solution to the inhomogeneous wave equations as in Proposition 1. 2 below, so that we will be able to get a variant of (1.10). In what follows, we restict ourselves to the case where $n = 2$. (For the 3-dimensional case, see K. Yokoyama [41]).

We strat with splitting the region $(0, \infty) \times (0, \infty)$ for each i $(i = 1, \cdots, m)$ as follows:

$$\Lambda_i = \{(r,t) \in (0,\infty) \times (0,\infty) : \frac{1}{3}(2 + \frac{c_i}{c_{i-1}})r \leq c_i t \leq \frac{1}{3}(2 + \frac{c_i}{c_{i+1}})r \quad \text{and} \quad r \geq 1\}$$

and $\Lambda_i^c = (0,\infty) \times (0,\infty) \setminus \Lambda_i$, where we have set $c_0 = 4c_1$ and $c_{m+1} = c_m/4$. Because of (1.12), this definition is meaningful. In particular, we have

$$\Lambda_i \cap \Lambda_l = \emptyset \quad \text{if } i \neq l. \tag{1.13}$$

Notice that $1+r$ is equivalent to $1+t+r$ for $(r,t) \in \Lambda_i$, while so is $1+|c_i t - r|$ for $(r,t) \in \Lambda_i^c$.

Next we introduce the following weight functions:

$$z^{(i)}(\lambda,s)=\begin{cases}(1+\lambda+s)(1+|\lambda-c_js|) & \text{if}\quad (\lambda,s)\in\Lambda_j,\ j\neq i,\\ (1+\lambda+s)^{1+\mu}(1+|\lambda-c_is|)^{1-\mu} & \text{if}\quad (\lambda,s)\in\Lambda_i,\\ (1+\lambda)^{1-2\gamma}(1+s+\lambda)^{1+2\gamma} & \text{if}\quad (\lambda,s)\in(0,\infty)\times(0,\infty)\setminus\cup_{j=1}^m\Lambda_j\end{cases}$$

and

$$z(\lambda,s)=\begin{cases}(1+\lambda+s)(1+|\lambda-c_js|) & \text{if}\quad (\lambda,s)\in\Lambda_j,\\ (1+\lambda)^{1-2\gamma}(1+s+\lambda)^{1+2\gamma} & \text{if}\quad (\lambda,s)\in(0,\infty)\times(0,\infty)\setminus\cup_{j=1}^m\Lambda_j,\end{cases}$$

where $0<\mu<1$ and γ is the number in the definition of $w_l(r,t)$. Then we have

Proposition 1. 2 *Let $u(x,t)$ be a solution to $\Box_i u=\partial^b F$ with the zero initial data, where b is a multi-index with $|b|=1$.*
(i) Let $(|x|,t)\in\Lambda_i^c$ and $t<T$. Then we have

$$w_i(r,t)|u(x,t)|\le CM_1(F)(r,t), \tag{1.14}$$

where $w_i(r,t)$ is definded in (1.5) *and we have set for a nonnegative integer k*

$$M_k(F)(r,t)=\sum_{|a|\le k}\sup_{0<s<t}\sup_{y\in\mathbf{R}^2}||y|^{\frac{1}{2}}z(|y|,s)\Gamma^a F^i(y,s)|.$$

(ii) Let $(x,t)\in\mathbf{R}^2\times[0,T)$. Then we have

$$w_i(r,t)|u(x,t)|\le CM_1^{(i)}(F)(r,t), \tag{1.15}$$

where we have set for a positive integer k

$$M_k^{(i)}(F)(r,t)\ =\ \sum_{|a|\le k}\sup_{0<s<t}\sup_{y\in\mathbf{R}^2}||y|^{\frac{1}{2}}z^{(i)}(|y|,s)\Gamma^a F^i(y,s)|.$$

Outline of the proof: Without loss of generality, we may assume $c_i=1$. Then if $F\in C^\infty(\mathbf{R}^2\times[0,T))$, we have

$$u(x,t)=\frac{1}{2\pi}\int_{|x-y|\le t}\frac{\partial^b F(y,s)}{\sqrt{t^2-|x-y|^2}}dy. \tag{1.16}$$

Switching to polar coordinates as $x=(r\cos\theta,r\sin\theta)$ and $y=(\lambda\cos(\theta+\psi),\lambda\sin(\theta+\psi))$ as in Section 2 in [27], we have

$$\begin{aligned}u(x,t)\ &=\ \frac{1}{2\pi}\iint_{D'}\lambda d\lambda ds\int_{-\varphi}^{\varphi}\partial^b F(\lambda\xi,s)K_1 d\psi\\ &+\frac{1}{2\pi}H(t-r)\iint_{D''}\lambda d\lambda ds\int_{-\pi}^{\pi}\partial^b F(\lambda\xi,s)K_1 d\psi,\end{aligned} \tag{1.17}$$

where H is the Heviside function and we have set

$$\begin{aligned} \xi &= (\cos(\theta+\psi), \sin(\theta+\psi)), \\ K_1 &= K_1(\lambda, s, \psi; r, t) = \{(t-s)^2 - r^2 - \lambda^2 + 2r\lambda\cos\psi\}^{-\frac{1}{2}}, \\ \varphi &= \varphi(\lambda, s; r, t) = \arccos\left[\frac{r^2+\lambda^2-(t-s)^2}{2r\lambda}\right] \quad \text{for } (\lambda, s) \in D'. \end{aligned}$$

Moreover, the domains D' and D'' are defined as follows:

$$\begin{aligned} D' &= \{(\lambda, s) \in (0,\infty)\times(0,\infty) :\ 0 < s < t,\ \lambda_- < \lambda < \lambda_+\}, \\ D'' &= \{(\lambda, s) \in (0,\infty)\times(0,\infty) :\ 0 < s < t-r,\ 0 < \lambda < \lambda_-\}, \end{aligned}$$

where

$$\lambda_- = |t-s-r|, \quad \lambda_+ = t-s+r. \tag{1.18}$$

The key point to get such estimates as in Proposition 1. 2 is to integrate by parts with respect to λ and s. Following [27] and [3], we shall sketch this process briefly. To begin with, we split the regions of integration D' and D'' into subregions as follows:

$$\begin{aligned} &D' = \mathit{blue} \cup \mathit{white}, \quad D'' = \mathit{black} \cup \mathit{red} \\ &\mathit{blue} = \{(s,\lambda) \in D' :\ \lambda_- < \lambda \le \lambda_- + \delta_1 \ \text{ or } \ \lambda_+ - \delta_1 \le \lambda < \lambda_+\}, \\ &\mathit{black} = \{(s,\lambda) \in D' :\ \lambda_- - \delta_2 \le \lambda < \lambda_- \ \text{ or } \ 0 < \lambda \le \delta_2\}, \end{aligned} \tag{1.19}$$

where we have set $\delta_1 = \min\{r, 1/2\}$ and $\delta_2 = \min\{(t-r)/2, 1/2\}$. Notice thta *white* is empty, if $0 < r \le 1/2$ and that *red* is empty, if $0 < t-r \le 1$.

Let $\partial^b = \partial_\alpha$ $(\alpha = 0,1,2)$ in (1.17). Then, according to the above decompositions, we have

$$\begin{aligned} 2\pi u(x,t) &= \iint_{\mathit{blue}} \lambda d\lambda ds \int_{-\varphi}^{\varphi} (\partial_\alpha F)(\lambda\xi, s) K_1 d\psi \\ &\quad + H(r-\tfrac{1}{2}) \sum_{j=0}^{1} \iint_{\mathit{white}} \lambda d\lambda ds \int_0^1 (\partial_\alpha F)(\lambda\Xi_j, s) K_2 d\tau \\ &\quad + H(t-r) \iint_{\mathit{black}} \lambda d\lambda ds \int_{-\pi}^{\pi} (\partial_\alpha F)(\lambda\xi, s) K_1 d\psi \\ &\quad + H(t-r-1) \iint_{\mathit{red}} \lambda d\lambda ds \int_{-\pi}^{\pi} (\partial_\alpha F)(\lambda\xi, s) K_1 d\psi, \end{aligned} \tag{1.20}$$

where we have changed the variable as $\psi = \Psi$ in the second term and set

$$\begin{aligned} \Psi &= \Psi(\lambda, s, \tau; r, t) = \arccos[1 - (1-\cos\varphi)\tau], \\ \Xi_j &= \Xi_j(\lambda, s, \tau; r, t) = (\cos(\theta + (-1)^j\Psi), \sin(\theta + (-1)^j\Psi)), \\ K_2 &= K_2(\lambda, s, \tau; r, t) = \{2r\lambda\tau(1-\tau)(2-(1-\cos\varphi)\tau)\}^{-\frac{1}{2}}. \end{aligned}$$

Carrying out the integration by parts in the second and fourth terms, we get the following:

Lemma 1. 1 *Let $u(x,t)$ be the solution to $\square_i u = \partial_\alpha F$ with the zero initial data. If $F \in C^\infty(\mathbf{R}^2 \times [0,T))$, then $|u(t,x)|$ is dominated by the followings:*

$$\begin{aligned}
I_1(F)(x,t) &= \iint_{blue} \lambda d\lambda ds \int_{-\varphi}^{\varphi} |(\partial_\alpha F)(\lambda\xi, s)| K_1 d\psi, \\
I_2(F)(x,t) &= \int_{\partial(white)} \lambda d\sigma \int_0^1 |F(\lambda\Xi_j, s)| K_2 d\tau, \\
I_3(F)(x,t) &= \iint_{white} d\lambda ds \int_0^1 \{|F(\lambda\Xi_j, s)| + |(\Omega F)(\lambda\Xi_j, s)|\} K_2 d\tau, \\
I_4(F)(x,t) &= \iint_{white} \lambda d\lambda ds \int_0^1 |F(\lambda\Xi_j, s)| \{|\partial_s K_2| + |\partial_\lambda K_2|\} d\tau, \\
I_5(F)(x,t) &= \iint_{white} \lambda d\lambda ds \int_0^1 |(\Omega F)(\lambda\Xi_j, s)| K_2 \ \{|\partial_s \Psi| + |\partial_\lambda \Psi|\} d\tau, \\
J_1(F)(x,t) &= \iint_{black} \lambda d\lambda ds \int_{-\pi}^{\pi} |(\partial_\alpha F)(\lambda\xi, s)| K_1 d\psi, \\
J_2(F)(x,t) &= \int_{\partial(red)} \lambda d\sigma \int_{-\pi}^{\pi} |F(\lambda\xi, s)| K_1 d\psi, \\
J_3(F)(x,t) &= \iint_{red} d\lambda ds \int_{-\pi}^{\pi} \{|F(\lambda\xi, s)| + |(\Omega F)(\lambda\xi, s)|\} K_1 d\psi, \\
J_4(F)(x,t) &= \iint_{red} \lambda d\lambda ds \int_{-\pi}^{\pi} |F(\lambda\xi, s)| \{|\partial_s K_1| + |\partial_\lambda K_1|\} d\psi.
\end{aligned}$$

Proof: It is easy to see that the first and second terms in (1.20) are dominated by $I_1(F)$ and $J_1(F)$, respectively. Since

$$(\nabla F)(\lambda\xi, s) = \xi \partial_\lambda (F(\lambda\xi, s)) - \frac{\xi^\perp}{\lambda} (\Omega F)(\lambda\xi, s), \quad \xi^\perp = (\sin(\theta+\psi), -\cos(\theta+\psi)),$$

we find that the fourth term in (1.20) is dominated by $J_j(F)$ $(j = 2,3,4)$, by integration by parts.

To deal with the second term in (1.20), we use the following identities:

$$\begin{aligned}
&(\partial_s F)(\lambda\Xi_j, s) = \partial_s (F(\lambda\Xi_j, s)) - (-1)^j \partial_s \Psi (\Omega F)(\lambda\Xi_j, s), \\
&(\nabla F)(\lambda\Xi_j, s) = \Xi_j (\partial_\lambda (F(\lambda\Xi_j, s)) - (-1)^j \partial_\lambda \Psi (\Omega F)(\lambda\Xi_j, s)) - \frac{\Xi_j^\perp}{\lambda} (\Omega F)(\lambda\Xi_j, s),
\end{aligned}$$

where $\Xi_j^\perp = (\sin(\theta + (-1)^j\Psi), -\cos(\theta + (-1)^j\Psi))$. Again by integration by parts, we find that the second term is dominated by $I_j(F)$ $(j = 2, \cdots, 5)$. The proof is complete.

For the further details of the proof of Proposition 1. 2 , see the proof of Proposition 4.3 in [15]. □

Here we prepear the following proposition for the latter use. For the proof, see e.g. the proof of Proposition 4.4 in [15].

Proposition 1. 3 *Let $u(x,t)$ be the solution to $\square_i u = F$ with the zero initial data. Let $0 \le \mu < 1/2$. Then we have for $(|x|, t) \in \Lambda_i$ with $t < T$*

$$(1+t+r)^\mu |u(x,t)| \le C M_0(F)(r,t), \tag{1.21}$$

where $M_0(F)(r,t)$ is defined in (1.14).

Application of the Basic estimates. We now tern our attention to the original problem (0.1) and (0.2). Since $p_c = 3$ for $n = 2$ and the higher order terms in $F^i(\partial u, \partial^2 u)$ are harmless as long as we consider the small solution, we will assume that F^i is cubic with respect to ∂u and $\partial^2 u$, namely,

$$(1.22)\qquad F^i(\partial u, \partial^2 u) = \sum_{j,k,l=1}^{m} \sum_{\alpha,\beta,\gamma,\delta=0}^{2} A_{ijkl}^{\alpha\beta\gamma\delta} \partial_\alpha u^j \partial_\beta u^k \partial_\gamma \partial_\delta u^l + \sum_{j,k,l=1}^{m} \sum_{\alpha,\beta,\gamma=0}^{2} B_{ijkl}^{\alpha\beta\gamma} \partial_\alpha u^j \partial_\beta u^k \partial_\gamma u^l,$$

where $A_{ijkl}^{\alpha\beta\gamma\delta}$ and $B_{ijkl}^{\alpha\beta\gamma}$ are constants and $A_{ijkl}^{\alpha\beta\gamma\delta}$ satisfy

$$(1.23)\qquad A_{ijkl}^{\alpha\beta\gamma\delta} = A_{ljki}^{\alpha\beta\gamma\delta} = A_{ijkl}^{\alpha\beta\delta\gamma}.$$

As we have mentioned before, in order to get a global solution, we need to establish a uniform *a priori* estimate of $[\partial u(t)]_N$ for some large integer N. By (1.7) and (0.1), $\Gamma^a \partial^b u^i(x,t)$ satisfies

$$(1.24)\qquad \Box_i \Gamma^a \partial^b u^i(x,t) = \tilde{F}^i(\partial u, \partial^2 u) \quad \text{in } \mathbf{R}^2 \times (0,T),$$

where we have set $\tilde{F}^i(\partial u, \partial^2 u) = \sum_{d \le a} C_{a,b} \partial^b \Gamma^d F^i(\partial u, \partial^2 u)$ and a, b and d are multi-indices. Moreover, the initial values of $\Gamma^a \partial^b u^i(x,t)$ are determined by ε, f^j and g^j $(j = 1, \cdots, m)$ by using the equation (0.1). For instance, when $a = 0$ and $\partial^b = \partial_t$, we have

$$(\partial_t u^i)(x,0) = \varepsilon g^i(x), \quad (\partial_t^2 u^i)(x,0) = \varepsilon c_i^2 \Delta f^i(x) + F^i(\partial u, \partial^2 u)(x,0).$$

We can solve the second equation with respect to $(\partial_t^2 u^i)(x,0)$, if δ in (1.6) is sufficiently small.

Based on this, we decompose $\Gamma^a \partial^b u(x,t)$ as follows:

$$(1.25)\ \Gamma^a \partial^b u(x,t) = u_0(x,t) + u_1(x,t) \quad \text{with} \quad u_0 = (u_0^1, \cdots, u_0^m),\ u_1 = (u_1^1, \cdots, u_1^m),$$

where u_1^i is a solution to $\Box_i u_1^i = \tilde{F}^i(\partial u, \partial^2 u)$ with the zero initial data, while u_0^i is a solution to $\Box_i u_0^i = 0$ and $u_0^i(x,0) = (\Gamma^a \partial^b u)(x,0)$, $\partial_t u_0^i(x,0) = (\partial_t \Gamma^a \partial^b u)(x,0)$. Since $f^j(x)$, $g^j(x) \in C_0^\infty(\mathbf{R}^2)$, the initial values of u_0^i are also belongs to $C_0^\infty(\mathbf{R}^2)$. Therefore, when $|a| + |b| \le N$, we have

$$(1.26)\quad |u_0^i(x,t)| \le M_N \varepsilon (1+t+r)^{-\frac{1}{2}} (1 + |c_i t - r|)^{-\frac{1}{2}} \quad \text{for} \quad (x,t) \in \mathbf{R}^2 \times [0,\infty),$$

where M_N depends on L^1-norms of f^j , g^j and their finite times derivatives. (See Lemma 1 in R.T. Glassey [10], and also Lemma 4 in [27] and [32]).

Then we have the following L^∞-L^∞ estimates.

Proposition 1. 4 *Let $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ be a solution of* (0.1) *and* (0.2). *Suppose that* (1.22) *and* (1.23) *hold. Let M_N be the constant in* (1.26).

(*i*) *There is a sufficiently small positive number δ such that if* (1.6) *with $k = [(N+2)/2]$ holds, then we have for $(|x|,t) \in \Lambda_i^c$ with $t < T$ and $|a| \le N$*

$$(1.27)\qquad w_i(|x|,t) |\Gamma^a \partial u^i(x,t)| \le M_N \varepsilon + C_N [\partial u]^2_{[\frac{N+2}{2}],t} \|\partial u\|_{N+4,t}.$$

(*ii*) *Let* $0 \le \mu < 1/2$. *There is a sufficiently small positive number* δ *such that if* (1.6) *with* $k = [(N+1)/2]$ *holds, then we have for* $(|x|, t) \in \Lambda_i$ *with* $t < T$ *and* $|a| \le N$,

$$(1.28) \qquad (1+t+r)^{\mu}|\Gamma^a u^i(x,t)| \le M_N\varepsilon + C_N[\partial u]^2_{[\frac{N+1}{2}],t}\|\partial u\|_{N+3,t}.$$

Proof: First we shall show (1.27). Using the decomposition (1.25) with $|b| = 1$ and the estimates (1.26) and (1.14), we have

$$(1.29)\ w_i(r,t)|\Gamma^a \partial u^i(x,t)| \le M_N\varepsilon + C_N M_{|a|+1}(F^i)(r,t) \quad \text{for } (|x|,t) \in \Lambda_i^c \text{ with } t < T.$$

Therefore, it suffices to show

$$(1.30) \qquad M_{N+1}(F^i)(r,t) \le C_N[\partial u]^2_{[\frac{N+2}{2}],t}\|\partial u\|_{N+4,t}.$$

It follows that for $|a| \le N+1$

$$(1.31) \qquad |\Gamma^a F^i(y,s)| \le C \sum_{j,k,l=1}^{m} \frac{1}{(w_j w_k)(|y|,s)}[\partial u(s)]^2_{[\frac{N+2}{2}]} \sum_{|b|\le|a|+1} |\Gamma^b \partial u^l(y,s)|.$$

Employing the following imbedding theorem

$$(1.32) \qquad |x|^{\frac{1}{2}}|f(x)| \le \sum_{|a|\le 2} \|\Gamma^a f\|_{L^2} \quad \text{for } x \in \mathbf{R}^2$$

(for the proof, see e.g. Lemma 6 in [27]), we get (1.30).

Moreover, we can prove (1.28) in a similar fashion, if we use (1.21) instead of (1.14). The proof is complete. □

Now we are in a position to derive a L^∞-L^∞ estimate for any $(x,t) \in \mathbf{R}^2 \times [0,T)$. By (1.25) with $|b| = 1$, (1.26) and (1.15), it suffices to control $M^{(i)}_{N+1}(F^i)(r,t)$. When $(|y|,s) \in \Lambda_i^c$ with $s < t$, we have from (1.31) and (1.32)

$$(1.33) \qquad |y|^{\frac{1}{2}} z^{(i)}(|y|,s)|\Gamma^a F^i(y,s)| \le C_N[\partial u]^2_{[\frac{N+2}{2}],t}\|\partial u\|_{N+4,t}.$$

In order to deal with the case where $(|y|,s) \in \Lambda_i$ with $s < t$, we divide F^i into N^i and R^i as follows:

$$(1.34) \quad N^i(\partial u, \partial^2 u) = \sum_{\alpha,\beta,\gamma,\delta=0}^{2} A^{\alpha\beta\gamma\delta}_{iiii}\partial_\alpha u^i \partial_\beta u^i \partial_\gamma \partial_\delta u^i + \sum_{\alpha,\beta,\gamma=0}^{2} B^{\alpha\beta\gamma}_{iiii}\partial_\alpha u^i \partial_\beta u^i \partial_\gamma u^i,$$

$$(1.35) \quad R^i(\partial u, \partial^2 u) = F^i(\partial u, \partial^2 u) - N^i(\partial u, \partial^2 u).$$

Firstly, we shall show for $|a| \le N+1$

$$(1.36) \qquad |y|^{\frac{1}{2}} z^{(i)}(|y|,s)|\Gamma^a R^i(y,s)| \le C_N\delta^2 M_N\varepsilon + C_N[\partial u]^2_{[\frac{N+4}{2}],t}\|\partial u\|_{N+6,t},$$

provided (1.6) with $k = [(N+4)/2]$. Since there is at least one index among j, k and l which does not coincide with i in each term of R^i, we have

$$\begin{aligned} |\Gamma^a R^i(y,s)| \le\ & C \sum_{(j,k)\ne(i,i)} \sum_{l=1}^{m} \frac{1}{(w_j w_k)(|y|,s)}[\partial u(s)]^2_{[\frac{|a|+1}{2}]} \sum_{|b|\le|a|+1} |\Gamma^b \partial u^l(y,s)| \\ & + C \sum_{\substack{(j,k)=(i,i)\\ l\ne i}} \frac{1}{(w_i w_l)(|y|,s)} \frac{[\partial u(s)]^2_{[\frac{|a|+1}{2}]}}{w_i(|y|,s)} \sum_{|b|\le|a|+1} w_l(|y|,s)|\Gamma^b \partial u^l(y,s)|. \end{aligned}$$

By (1.32), (1.6) with $k = [(N+2)/2]$ and $w_i(|y|,s) \geq |y|^{1/2}$, we get for $|a| \leq N+1$

$$|y|^{\frac{1}{2}} z^{(i)}(|y|,s)|\Gamma^a R^i(y,s)| \leq C[\partial u(s)]^2_{[\frac{N+2}{2}]} \|\partial u(s)\|_{N+4} + C\delta^2 \sum_{|b|\leq N+2} w_l(|y|,s)|\Gamma^b \partial u^l(y,s)|.$$

Moreover, since $\Lambda_i \subset \Lambda_l^c$ by (1.13), we get from (1.27),

$$w_l(|y|,s)|\Gamma^b \partial u^l(y,s)| \leq M_N \varepsilon + C_N [\partial u]^2_{[\frac{N+4}{2}],t} \|\partial u\|_{N+6,t}$$

for $(|y|,s) \in \Lambda_i$ with $s < t$ and $|b| \leq N+2$, provided (1.6) with $k = [(N+4)/2]$. Hence, we get (1.36).

In order to treat N^i, we need a concept of "Null condition" below.

Null condition. The idea to deal with N^i is to decompose $\partial_\alpha u^i$ as follows:

$$\partial_1 u^i = \frac{x_1}{r}\partial_r u^i - \frac{x_2}{r^2}\Omega u^i, \tag{1.37}$$

$$\partial_2 u^i = \frac{x_2}{r}\partial_r u^i + \frac{x_1}{r^2}\Omega u^i \tag{1.38}$$

$$\partial_0 u^i = -c_i \partial_r u^i + \frac{c_i t - r}{t}\partial_r u^i + \frac{1}{t} S u^i, \tag{1.39}$$

where $r = |x|$, $\Omega = x_1\partial_2 - x_2\partial_1$, $S = t\partial_t + r\partial_r$ and $\partial_0 = \partial_t$.

First we handle the last trems in the right hand saide of (1.37), (1.38) and (1.39). If we set for a nonnegative integer k

$$\langle u(t)\rangle_k = \sum_{l=1}^{m} \sum_{|a|\leq k} \sup_{\{x\in\mathbf{R}^2:\ (x,t)\in\Lambda_i\}} |\Gamma^a u^l(x,t)|, \tag{1.40}$$

we see that those terms are dominated by

$$C(1+t+r)^{-1}\langle u(t)\rangle_1 \quad \text{for } (|x|,t) \in \Lambda_i.$$

Since $\langle u(t)\rangle_1$ can be estimated by using (1.28), we get an additional decaying factor $(1+t+r)^{-1}$.

Nest we consider the second trem in the right hand saide of (1.39). Since

$$\frac{|c_i t - r|}{t} \leq \frac{C(w_i(r,t))^2}{(1+t+r)^2} \quad \text{for } (|x|,t) \in \Lambda_i,$$

we get a good decaying factor $(1+t+r)^{-1}$ instead of $(1+|c_i t - r|)^{-1}$ from this term.

From the above consideration, we can approximate $\partial_\alpha u^i$ as follows:

$$\partial_\alpha u^i \sim \omega_\alpha \partial_r u^i, \tag{1.41}$$

where we have set $\omega_0 = -c_i$, $\omega_1 = x_1/r$ and $\omega_2 = x_2/r$. Substituting (1.41) into (1.34), we get

$$N^i(\partial u, \partial^2 u) \sim \sum_{\alpha,\beta,\gamma,\delta=0}^{2} A^{\alpha\beta\gamma\delta}_{iiii}\omega_\alpha\omega_\beta\omega_\gamma\omega_\delta(\partial_r u^i)^2 \partial_r^2 u^i + \sum_{\alpha,\beta,\gamma=0}^{2} B^{\alpha\beta\gamma}_{iiii}\omega_\alpha\omega_\beta\omega_\gamma(\partial_r u^i)^3.$$

In order to kill these terms, we naturally arrive at the following condition.

Definition. We fix i ($i = 1, \cdots, m$). When (1.12) holds, we say that F^i satisfies the "Null condition", if

$$(1.42)\qquad \sum_{\alpha,\beta,\gamma,\delta=0}^{2} A_{iiii}^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta = 0 \quad \text{and} \quad \sum_{\alpha,\beta,\gamma=0}^{2} B_{iiii}^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma = 0$$

hold on the hyper-surface $(X_0)^2 - c_i^2\{(X_1)^2 + (X_2)^2\} = 0$.

In addition, when $c_1 = \cdots = c_m$, we say that F^i satisfies the "Null condition", if for any j, k, l $(j, k, l = 1, \cdots, m)$

$$(1.43)\qquad \sum_{\alpha,\beta,\gamma,\delta=0}^{2} A_{ijkl}^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta = 0 \quad \text{and} \quad \sum_{\alpha,\beta,\gamma=0}^{2} B_{ijkl}^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma = 0$$

hold on the hyper-surface $(X_0)^2 - c_i^2\{(X_1)^2 + (X_2)^2\} = 0$.

By assuming that F^i satisfies the null condition (1.42), we obtain

$$\begin{aligned} |\Gamma^a N^i(\partial u^i, \partial^2 u^i)(x,t)| \le & \frac{C}{(1+t+r)^2}[\partial u(t)]^2_{[\frac{|a|+1}{2}]} \sum_{|b|\le|a|+1} |\partial\Gamma^b u^i(x,t)| \\ & + \frac{C}{(1+t+r)w_i(r,t)}[\partial u(t)]_{[\frac{|a|+1}{2}]}\langle u(t)\rangle_{|a|+2} \sum_{|b|\le|a|+1} |\partial\Gamma^b u^i(x,t)| \end{aligned}$$

for $(|x|,t) \in \Lambda_i$ with $t < T$. (See also the proof of Proposition 3.1 [15]). Using (1.32), we have

$$\begin{aligned} |x|^{\frac{1}{2}} z^{(i)}(|x|,t)|\Gamma^a N^i(\partial u^i, \partial^2 u^i)(x,t)| \le & C[\partial u(t)]^2_{[\frac{|a|+1}{2}]}\|\partial u(t)\|_{|a|+3} \\ & + C[\partial u(t)]_{[\frac{|a|+1}{2}]}\langle u(t)\rangle_{|a|+2}\|\partial u(t)\|_{|a|+3}, \end{aligned}$$

hence, by the aid of (1.28), we get

$$(1.44)\qquad |y|^{\frac{1}{2}} z^{(i)}(|y|,s)|\Gamma^a N^i(y,s)| \le C_N \delta M_N \varepsilon \|\partial u\|_{N+4,t} + C_N[\partial u]^2_{[\frac{N+4}{2}],t}\|\partial u\|^2_{N+6,t},$$

for $(|y|,s) \in \Lambda_i$ with $s < t$ and $|a| \le N+1$, provided (1.6) with $k = [(N+4)/2]$. As a conclusion, it follows from (1.33), (1.36) and (1.44) that

Proposition 1. 5 *Let $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ be a solution of* (0.1) *and* (0.2). *Suppose that* (1.22), (1.23) *and* (1.42) *hold. Let M_N be the constant in* (1.26). *Then there is a sufficiently small number $\delta = \delta(n) > 0$ such that if* (1.6) *with $k = [(N+4)/2]$ holds, then we have for $(x,t) \in \mathbf{R}^2 \times [0,T)$ and $|a| \le N$*

$$(1.45)\qquad w_i(|x|,t)|\Gamma^a \partial u^i(x,t)| \le M_N(\frac{3}{2} + \frac{1}{2}\|\partial u\|_{N+4,t})\varepsilon + C_N[\partial u]^2_{[\frac{N+4}{2}],t}\|\partial u\|^2_{N+6,t}.$$

Note that we have from Propositions 1.1 with $p = 3$

$$(1.46)\qquad \|\partial u\|_{N,t} \le C_N \varepsilon (1+t)^{C_N[\partial u]^2_{[\frac{N+1}{2}],t}}$$

for $0 \le t < T$ and the solution $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ to (0.1) and (0.2), provided (1.6) with $k = [(N+1)/2]$ holds. Combining this estimate with (1.45) and (1.28), we obtain the following.

Corollary 1. 1 *Let $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ be a solution of* (0.1) *and* (0.2). *Suppose that* (1.22), (1.23) *and* (1.42) *hold. Then there is a sufficiently small positive number δ such that if* (1.6) *with $k = [(N+7)/2]$ holds, then we have for $0 \le t < T$*

$$(1.47) \qquad [\partial u(t)]_N \le C_N\varepsilon(1+t)^{C_N[\partial u]^2_{[\frac{N+7}{2}],t}}.$$

and

$$(1.48) \qquad \langle u(t)\rangle_{N+1} \le C_N\varepsilon(1+t)^{C_N[\partial u]^2_{[\frac{N+5}{2}],t}}.$$

We thus obtain a upper bound of $[\partial u(t)]_N$. However, since the estimate is not uniform in t, we need to work a little bit.

Energy Estimates II. In oder to derive a uniform *a priori* estimate of $[\partial u(t)]_N$ with respect to t, we need the following another enrgy estimate. For the proof, see the proof of Proposition 5.2 in [15].

Proposition 1. 6 *Let $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ be a solution of* (0.1) *and* (0.2). *Suppose that* (1.22), (1.23) *and* (1.42) *hold. Then we have for $0 \le t < T$*

$$(1.49) \qquad \begin{aligned} \|\partial u(t)\|_N^2 \le\; & C_N^2\Big\{\|\partial u(0)\|_N^2 + \\ & + \int_0^t (1+s)^{-\frac{5}{4}}([\partial u(s)]^2_{N+1} + \langle u(s)\rangle^2_{N+2})\|\partial u(s)\|^2_{N+1}ds\Big\}. \end{aligned}$$

Now it follows from (1.49), Corollary 1. 1 and (1.46) that

Corollary 1. 2 *Let $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ be a solution of* (0.1) *and* (0.2) *and $0 < \varepsilon \le 1$. Suppose that* (1.22), (1.23) *and* (1.42) *hold. Then there is a sufficiently small positive number δ such that if* (1.6) *with $k = [(N+14)/2]$ holds, then we have for $0 \le t < T$*

$$(1.50) \qquad \|\partial u(t)\|^2_{N+6} \le C_N^2\varepsilon^2\Big\{1 + \int_0^t (1+s)^{-\frac{5}{4}+4C_N[\partial u]^2_{[\frac{N+14}{2}],s}}ds\Big\}.$$

Main result. Now we are in a position get a uniform *a priori* estimate of $[\partial u(t)]_N$. We fix an integer N satisfying $N \ge 13$, which guarantees $[(N+14)/2] \le N$. We take ε_0 to be

$$(1.51) \qquad 0 < \varepsilon_0 \le 1, \quad 3M_N\varepsilon_0 \le \delta, \quad 3C_N\varepsilon_0 \le 1 \quad \text{and} \quad 12C_NM_N\varepsilon_0 \le \frac{1}{8},$$

where M_N is the constant in (1.26), C_N is the constant in (1.50) and δ is the smallest number taken in Proposition 1.5 and Corollary 1.2. We will fix an ε in $[0, \varepsilon_0)$ in the following.

Suppose that the problem (0.1) and (0.2) has a solution $u^i \in C^\infty(\mathbf{R}^2 \times [0,T))$ satisfying $[\partial u]_{N,T} \le 3M_N\varepsilon$. Then we have

$$[\partial u]_{[\frac{N+14}{2}],T} \le [\partial u]_{N,T} \le \delta \ (<1).$$

Therefore, by (1.50) and (1.51), we have for $0 \le t < T$

$$\begin{aligned} \|\partial u(t)\|_{N+6} \le\; & C_N\varepsilon\left(1 + \int_0^t (1+s)^{-\frac{9}{8}}ds\right)^{\frac{1}{2}} \\ \le\; & 1. \end{aligned}$$

Substituting this into (1.45), we have

$$[\partial u]_{N,T} \le 2M_N\varepsilon + 3C_N M_N\varepsilon\ [\partial u]_{[\frac{N+4}{2}],T}.$$

Hence, by (1.51), we have

$$[\partial u]_{N,T_\varepsilon} \le 2M_N\varepsilon. \tag{1.52}$$

We thus obtain a uniform *a priori* estimate of $[\partial u]_{N,T}$. Now we state our main theorem.

Theorem 1. 1 ([*Hoshiga-K*]) *Let* $n = 2$ *and* $c_i \neq c_j$ *if* $i \neq j$. *Suppose that* (1.1), (1.2) *with* $p = 3$, (1.3) *and* (1.42) *hold. Then there exists a positive constant* ε_0 *such that the initial value problem* (0.1) *and* (0.2) *has a unique* C^∞*-solution in* $\mathbf{R}^2 \times [0,\infty)$ *for* $0 < \varepsilon \le \varepsilon_0$.

Remark. Recently, K. Yokoyama extended this result for 3-dimensional case in [41].

2 Semilinear Case

In this section we study the semilinear case, namely, we assume $F^i = F^i(u)$. We consider the following simple system of semilinear wave equations as a model case of the problem (0.1) and (0.2):

$$\partial_t^2 u - \Delta u = |v|^p \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{2.1}$$

$$\partial_t^2 v - \Delta v = |u|^q \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{2.2}$$

$$u(x,0) = \varepsilon f^1(x), \quad \partial_t u(x,0) = \varepsilon g^1(x) \quad \text{in } \mathbf{R}^n, \tag{2.3}$$

$$v(x,0) = \varepsilon f^2(x), \quad \partial_t v(x,0) = \varepsilon g^2(x) \quad \text{in } \mathbf{R}^n, \tag{2.4}$$

where $1 < p \le q$, $n = 2$, 3, f^j, $g^j \in C^\infty(\mathbf{R}^2)$ and $\varepsilon > 0$ is a small parameter. It is regarded as a natural extension of the following Cauchy problem:

$$\partial_t u - \Delta u = |u|^p \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{2.5}$$

$$u(x,0) = \varepsilon f(x), \quad \partial_t u(x,0) = \varepsilon g(x) \quad \text{in } \mathbf{R}^n, \tag{2.6}$$

where f, $g \in C_0^\infty(\mathbf{R}^2)$.

Known results. The problem (2.5)–(2.6) has been extensively studied by many authors (see, e.g., [1], [2], [5], [9], [10], [17], [31], [33], [35]–[40], [42], [43]). Set

$$\alpha_0 = pp^* - 1, \quad p^* = \frac{n-1}{2}p - \frac{n+1}{2}. \tag{2.7}$$

Let $p_0(n)$ be the positive root of the quadratic equation $\alpha_0 = 0$, namely, $p_0(3) = 1 + \sqrt{2}$ and $p_0(2) = (3 + \sqrt{17})/2$. Note that when $p > 1$, $p > p_0(n)$ and $1 < p < p_0(n)$ are equivalent to $\alpha_0 > 0$ and $\alpha_0 < 0$, resepectively. Then we have

Theorem 2. 1 *If* $\alpha_0 > 0$, *then* [*Small Data Global Existence*] *holds. While, if* $\alpha_0 < 0$, *then* [*Small Data Blow-up*] *holds.*

For the proof, see F. John [17], [20] for $n = 3$ and R.T. Glassey [9], [10] for $n = 2$. Here we give an observation why $\alpha_0 > 0$ implies [Small Data Global Existence] based on the following basic estimate.

Proposition 2. 1 *Let $u(x,t)$ be a solution to $\partial_t^2 u - \Delta u = F$ with the zero initial data. Then we have for $(x,t) \in \mathbf{R}^n \times (0,\infty)$*

$$
\begin{aligned}
(2.8) \qquad & |u(x,t)|(1+r+t)^{\frac{n-1}{2}}/\Phi_n(r,t;\nu) \\
& \le C \sup_{0<s<t} \sup_{y\in\mathbf{R}^n} \{|y|^{\frac{n-1}{2}}(1+s+|y|)^{1+\nu}(1+|s-|y||)^{1+\mu}|F(y,s)|\}
\end{aligned}
$$

for any $\mu > 0$ and $\nu > 0$. Here $r = |x|$ and we have set

$$(2.9) \qquad \Phi_3(r,t;\nu) = (1+|t-r|)^{-\nu},$$

and

$$(2.10) \qquad \Phi_2(r,t;\nu) = \begin{cases} (1+|t-r|)^{-\nu} & \text{if } \ 0<t\le r, \\ (1+t-r)^{-\frac{1}{2}}(1+t-r)^{[\frac{1}{2}-\nu]_+} & \text{if } \ r<t. \end{cases}$$

In addition, $[a]_+ = \max\{a,0\}$ and $A^{[0]_+} = 1+\log A$.

We introduce the norm of $u(x,t)$ as follows:

$$(2.11) \qquad |||u||| = \sup_{(x,t)\in\mathbf{R}^n\times(0,\infty)} \{|u(x,t)|(1+t+r)^{\frac{n-1}{2}}/\Phi_n(r,t;p^*)\}.$$

Then, if $\alpha_0 > 0$, we get

$$(2.12) \qquad \sup_{0<s<t} \sup_{y\in\mathbf{R}^n} \{|y|^{\frac{n-1}{2}}(1+s+|y|)^{1+p^*}(1+|s-|y||)^{1+\mu}|u(y,s)|^p\} \le C|||u|||^p.$$

Indeed, this estimate immidiately follows, if we choose μ such that $0<\mu<\alpha_0$. Having the estimate (2.8) with $\nu = p^*$ in mind, we need to assume $\alpha_0 > 0$ so that a *a priori* estimate holds.

Next we consider the critical case $\alpha_0 = 0$, namely, $p = p_0(n)$.

Theorem 2. 2 *If $\alpha_0 = 0$, then [Small Data Blow-up] holds. Moreover, as for the life span $T(\varepsilon)$ of classical solutions of (2.5)–(2.6), there exist positive constants A and B such that*

$$(2.13) \qquad \exp(A\varepsilon^{-p(p-1)}) \le T(\varepsilon) \le \exp(B\varepsilon^{-p(p-1)}).$$

For the proof, see J. Schaeffer [35], Y. Zhou [42] and [43], and also H. Takamura [36].

The p-q system. We turn our attention to the Cauchy problem (2.1)–(2.4). Set

$$
\begin{gathered}
\Gamma = \alpha + p\beta, \quad \alpha = pq^* - 1, \quad \beta = qp^* - 1, \\
p^* = \frac{n-1}{2}p - \frac{n+1}{2}, \quad q^* = \frac{n-1}{2}q - \frac{n+1}{2}.
\end{gathered}
$$

Then we have

Theorem 2. 3 *If $\Gamma > 0$ and $0 < p^* \le q^*$, then [Small Data Global Existence] holds. While, if $\Gamma < 0$, then then [Small Data Blow-up] holds.*

For the proof, see D. Del Santo, V. Georgiev and E. Mitidieri [7] for the case $\Gamma \ne 0$ and K. Deng [8] for the case $\Gamma < 0$.

In the following, we consider the critical case $\Gamma(p,q,n) = 0$. Here, given p, q, f_j and g_j, we define the life span $T^*(\varepsilon)$ as the supremum of all T such that a C^2-solution of (2.1)–(2.4) exists for all $x \in \mathbf{R}^2$ and $0 \le t < T$. Then our main result of this section is as follows.

Theorem 2. 4 ([*K-Ohta*]) *Let* $1 < p \le q$, $n = 2, 3$, $\Gamma(p,q,n) = 0$, $g_j \in C^\infty(\mathbf{R}^2)$ *and* $0 < \varepsilon \le 1$. *Assume that*

$$(2.14) \qquad f_j(x) = 0, \quad g_j(x) \ge 0 \quad (x \in \mathbf{R}^2, j = 1, 2), \quad g_2(0) > 0.$$

Then the classical solution of (2.1)–(2.4) *blows up in a finite time, and there exists a positive constant* C, *independent of* ε, *such that the life span* $T^*(\varepsilon)$ *of the classical solution of* (2.1)–(2.4) *satisfies*

$$(2.15) \qquad T^*(\varepsilon) \le \exp\left(C\varepsilon^{-p(pq-1)}\right) \quad (0 < \varepsilon \le 1) \qquad \text{if} \ \ p < q,$$

$$(2.16) \qquad T^*(\varepsilon) \le \exp\left(C\varepsilon^{-p(p-1)}\right) \quad (0 < \varepsilon \le 1) \qquad \text{if} \ \ p = q.$$

Remark. For 3-dimensional case, Y. Kurokawa and H. Takamura in [32] obtained the same estimates as in (2.15) and (2.16) and the lower bounds of the life span, independently.

Key lemma. The key point of the proof of Theorem 2. 4 is to reduce the blowup problem for (2.1)–(2.4) to that for a system of integral equations (2.17)–(2.18) below.

$$(2.17) \qquad \varphi(z) \ge 1 + \gamma\lambda^a \int_0^z \left(1 - e^{-\lambda(z-\zeta)}\right) e^{\alpha\lambda\zeta} |\phi(\zeta)|^p d\zeta \qquad (z \ge 0),$$

$$(2.18) \qquad \phi(z) \ge \gamma\lambda^b \int_0^z \left(1 - e^{-\lambda(z-\zeta)}\right) e^{\beta\lambda\zeta} |\varphi(\zeta)|^q d\zeta \qquad (z \ge 0),$$

where a, b, p, q, α, β, γ and λ be constants satisfying

$$(2.19) \qquad 1 < p \le q, \quad \beta \ge 0, \quad \gamma > 0, \quad \lambda \ge 1.$$

Then we have

Lemma 2. 1 *Let* $(\varphi(z), \phi(z))$ *be a solution of the system of integral inequalities* (2.17)–(2.18). *Assume that either*

$$(2.20) \qquad a + p(b-1) \ge 0, \quad \alpha + p\beta \ge 0,$$

or

$$(2.21) \qquad a + pb \ge 0, \quad \alpha \ge 0.$$

Then the life span of $(\varphi(z), \phi(z))$ *is bounded from above by a positive constant depending only on* p, q, β *and* γ.

Lemma 2. 1 is a key lemma to prove Theorem 2. 4. The terms $e^{\alpha\lambda\zeta}$ and $e^{\beta\lambda\zeta}$ with $\alpha < 0$ and $\beta > 0$ in (2.17)–(2.18), which never appear in the problem for the single equation (2.5)–(2.6), make the problem difficult. And our method is also applicable to the problem for the single equation and the assumptions (2.20) and (2.21) lead the different estimates of the life span (2.15) and (2.16) in the theorem. For the further details of the proof of the theorem, see [30].

Proof of the key lemma. Here we give the proof of Lemma 2.1. In what follows, we always assume that $(\varphi(z), \phi(z))$ is a solution of (2.17)–(2.18) under the condition (2.19).

Lemma 2.2 *Assume* (2.19). *Let* $A > 0$, $0 < h \leq 1$ *and* $Z \geq 0$. *Suppose that*

(2.22) $$\varphi(z) \geq A \qquad (z \geq Z).$$

Then there exists a positive constant C_4 *depending only on* β *and* γ *such that*

(2.23) $$\phi(z) \geq C_4 A^q h^2 \lambda^{b-1} e^{\beta\lambda z} \qquad (z \geq Z + h),$$

(2.24) $$\phi(z) \geq C_4 A^q h^2 \lambda^b \qquad (z \geq Z + h).$$

Proof: First we show (2.23). Since $\lambda \geq 1$ and $0 < h \leq 1$, it follows from (2.18) and (2.22) that for $z \geq Z + h$

(2.25) $$\begin{aligned} \phi(z) &\geq \gamma\lambda^b \int_{z-h/\lambda}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) e^{\beta\lambda\zeta} A^q d\zeta \\ &\geq \gamma\lambda^b e^{\beta\lambda(z-h/\lambda)} A^q \int_{z-h/\lambda}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta \\ &\geq \gamma e^{-\beta} A^q \lambda^b e^{\beta\lambda z} \int_{z-h/\lambda}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta. \end{aligned}$$

Thus (2.23) follows from (2.25) and the following fact:

$$\int_{z-h/\lambda}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta = \frac{e^{-h} - 1 + h}{\lambda} \geq \frac{h^2}{e\lambda} \qquad (0 < h \leq 1).$$

Next we show (2.24). Again from (2.18) and (2.22) we have for $z \geq Z$

(2.26) $$\begin{aligned} \phi(z) &\geq \gamma\lambda^b \int_{Z}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) e^{\beta\lambda\zeta} A^q d\zeta \\ &\geq \gamma\lambda^b A^q \int_{Z}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta. \end{aligned}$$

Thus (2.24) follows from (2.26) and the fact that for $z \geq Z + h$ we have

(2.27) $$\int_{Z}^{z} \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta \geq \int_{z-h}^{z} \left(1 - e^{-(z-\zeta)}\right) d\zeta = e^{-h} - 1 + h \geq \frac{h^2}{e} \quad (0 < h \leq 1).$$

This completes the proof. □

Lemma 2.3 *Let* $B > 0$, $0 < h \leq 1$ *and* $Z \geq 0$. *Suppose that either*

(2.28) $$(2.20) \quad \textit{and} \quad \phi(z) \geq B\lambda^{b-1} e^{\beta\lambda z} \qquad (z \geq Z),$$

or

(2.29) $$(2.21) \quad \textit{and} \quad \phi(z) \geq B\lambda^b \qquad (z \geq Z).$$

Then there exists a positive constant C_5 *depending only on* γ *such that*

(2.30) $$\varphi(z) \geq C_5 B^p (z - Z - 1) \qquad (z \geq Z + 1),$$

(2.31) $$\varphi(z) \geq C_5 B^p h^2 \qquad (z \geq Z + h).$$

Proof: Under the assumption (2.28) or (2.29), we show

$$\varphi(z) \geq \gamma B^p \int_Z^z \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta \quad (z \geq Z). \tag{2.32}$$

First we assume (2.28). It follows from (2.17) and (2.28) that for $z \geq Z$

$$\begin{aligned} \varphi(z) &\geq \gamma\lambda^a \int_Z^z \left(1 - e^{-\lambda(z-\zeta)}\right) e^{\alpha\lambda\zeta} B^p \lambda^{p(b-1)} e^{p\beta\lambda\zeta} d\zeta \\ &= \gamma B^p \lambda^{a+p(b-1)} \int_Z^z \left(1 - e^{-\lambda(z-\zeta)}\right) e^{(\alpha+p\beta)\lambda\zeta} d\zeta. \end{aligned} \tag{2.33}$$

By the assumption that $a + p(b-1) \geq 0$ and $\alpha + p\beta \geq 0$ in (2.20), we have (2.32). Next we assume (2.29). It follows from (2.17) and (2.29) that for $z \geq Z$

$$\begin{aligned} \varphi(z) &\geq \gamma\lambda^a \int_Z^z \left(1 - e^{-\lambda(z-\zeta)}\right) e^{\alpha\lambda\zeta} B^p \lambda^{pb} d\zeta \\ &= \gamma B^p \lambda^{a+pb} \int_Z^z \left(1 - e^{-\lambda(z-\zeta)}\right) e^{\alpha\lambda\zeta} d\zeta. \end{aligned} \tag{2.34}$$

By the assumption that $a + pb \geq 0$ and $\alpha \geq 0$ in (2.21), we have (2.32). Thus, (2.30) follows from (2.32) and the fact that for $z \geq Z + 1$ we have

$$\int_Z^z \left(1 - e^{-\lambda(z-\zeta)}\right) d\zeta = z - Z - \frac{1}{\lambda} + \frac{1}{\lambda} e^{-\lambda(z-Z)} \geq z - Z - 1.$$

While (2.31) follows from (2.32) and (2.27). □

From Lemmas 2. 2 and 2. 3 , we have the following lemma.

Lemma 2. 4 *Assume* (2.19) *and either* (2.20) *or* (2.21). *Let $A > 0$, $0 < h \leq 1$ and $Z \geq 0$. Suppose that*

$$\varphi(z) \geq A \quad (z \geq Z). \tag{2.35}$$

Then there exists a positive constant C_6 depending only on p, β and γ such that

$$\varphi(z) \geq C_6 A^{pq}(z - Z - 2) \quad (z \geq Z + 2), \tag{2.36}$$

$$\varphi(z) \geq C_6 A^{pq} h^{2p+2} \quad (z \geq Z + 2h). \tag{2.37}$$

Lemma 2. 5 *For any $L > 0$ there exists a constant $Z_0 = Z_0(L) > 0$ such that*

$$\varphi(z) \geq L \quad (z \geq Z_0). \tag{2.38}$$

Proof: From (2.17) we have $\varphi(z) \geq 1$ for all $z \geq 0$. Thus it follows from Lemma 2. 4 (we take $A = 1$ and $Z = 0$ in Lemma 2. 4) that

$$\varphi(z) \geq C_6(z - 2) \quad (z \geq 2). \tag{2.39}$$

Thus Lemma 2. 5 follows from (2.39). □

Lemma 2. 6 *Let j be a non-negative integer. Suppose that there exist constants A_j and Z_j such that*

$$\varphi(z) \geq A_j \qquad (z \geq Z_j). \tag{2.40}$$

Then there exists a constant $M > 1$ such that

$$\varphi(z) \geq A_{j+1} \qquad (z \geq Z_{j+1}), \tag{2.41}$$

where

$$A_{j+1} = \frac{A_j^{pq}}{M(j+1)^{4p+4}}, \quad Z_{j+1} = Z_j + \frac{2}{(j+1)^2}. \tag{2.42}$$

Proof: From (2.40) and Lemma 2. 4 (we take $A = A_j$, $h = 1/(j+1)^2$ and $Z = Z_j$ in Lemma 2. 4), we have

$$\varphi(z) \geq \frac{C_6 A_j^{pq}}{(j+1)^{4p+4}} \qquad \left(z \geq Z_j + \frac{2}{(j+1)^2}\right). \tag{2.43}$$

Thus we obtain (2.41) and (2.42). □

Lemma 2. 7 *Let $\{A_j\}_{j=0}^{\infty}$ be the sequence defined by* (2.42)*. If $A_0 > L_0 := M^{\nu} e^{(4p+4)m}$, then we have $\lim_{j\to\infty} A_j = \infty$. Here $\nu = 1/(pq-1)$ and $m = \sum_{k=2}^{\infty} (pq)^{-k} \log k$.*

Proof: From (2.42) we have

$$\begin{aligned}
\log A_j &= (pq)^j \log A_0 - \frac{(pq)^j - 1}{pq - 1} \log M \\
&\quad -(4p+4)\left\{\log j + (pq)\log(j-1) + (pq)^2 \log(j-2) + \cdots + (pq)^{j-2}\log 2\right\} \\
&\geq (pq)^j \left(\log A_0 - \frac{\log M}{pq-1} - (4p+4)\sum_{k=2}^{\infty}(pq)^{-k}\log k\right) \\
&= (pq)^j \left(\log \frac{A_0}{M^{\nu}} - (4p+4)m\right).
\end{aligned}$$

Since $pq > 1$, this completes the proof. □

We are now in a position to give the proof of Proposition 2. 1 .

Proof of Lemma 2. 1 : Put $A_0 = L_0 + 1$. If we take $Z_0 = Z_0(A_0)$ in Lemma 2. 5 , we have $\varphi(z) \geq A_0$ for $z \geq Z_0$. Moreover, it follows from (2.42) that $Z_j = Z_0 + \sum_{k=1}^{j} 2/k^2$ for $j \geq 1$. Thus if we put $Z^* := \sup_{j\geq 1} Z_j = Z_0 + \sum_{k=1}^{\infty} 2/k^2$, we have $Z^* < \infty$. From Lemma 2. 6 for any $j \geq 1$ we have $\varphi(z) \geq A_j$ for all $z \geq Z^*$. Therefore, from Lemma 2. 7 we see that the life span of $(\varphi(z), \phi(z))$ is less than or equal to Z^*. Since the positive constant Z^* depends only on p, q, β and γ, this completes the proof. □

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