数理解析研究所講究録
1102 巻 1999 年 112-127

An Estimate on the Heat Kernel of Magnetic Schrödinger Operators and Uniformly Elliptic Operators with Non-negative Potentials

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Abstract

In this paper we show an estimate of the heat kernel to the Schrödinger oprator with magnetic fields and to uniformly elliptic operators with non-negative potentials which belongs to the reverse Hölder class. We also give a weighted smoothing estimates for the semigroup generated by the operators above.

1 Introduction and Main Results

We consider the uniformly elliptic operator $L_E = -\nabla(A(x)\nabla)+V(x)$ with certain non-negative potential V and the Schrödinger operator $L_M = (i^{-1}\nabla - a(x))^2+V(x)$ with a magnetic field $a(x) = (a_1(x), \cdots, a_n(x)), n \geq 2$. We use the notation L_J for $J = E$ or $J = M$. The purpose of this paper is to give an estimate of the fundamental solution (or heat kernel) $\Gamma_J(x,t:y,s)$ to

$$(\partial_t + L_J)u(x,t) = 0, \quad (x,t) \in \mathbf{R}^n \times (0,\infty), \tag{1}$$

namely $\Gamma_J(x,t;y,s)$ satisfies

$$(\partial_t + L_J)\Gamma_J(x,t;y,s) = 0, \quad x \in \mathbf{R}^n,\ t > s, \tag{2}$$

$$\lim_{t\to s}\Gamma_J(x,t;y,s) = \delta(x-y). \tag{3}$$

For the elliptic operator L_E, we assume the following conditions for $A(x) = (a_{ij}(x))$.

ASSUMPTION (A.1): $a_{ij}(x)$ is a real-valued measurable function and satisfies $a_{ij}(x) = a_{ji}(x)$ for every $i, j = 1, \cdots, n$ and $x \in \mathbf{R}^n$.

ASSUMPTION (A.2): There exists a constant $\lambda > 0$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi^i\xi^j \leq \lambda^{-1}|\xi|^2, \quad \xi = (\xi^1, \cdots, \xi^n) \in \mathbf{R}^n. \tag{4}$$

To state our assumptions on V and a, we prepare some notations. We say $U \in (RH)_\infty$ if $U \in L^\infty_{loc}(\mathbf{R}^n)$ and satisfies

$$\sup_{y \in B(x,r)} |U(y)| \leq C\frac{1}{|B(x,r)|}\int_{B(x,r)} |U(y)|\, dy, \tag{5}$$

and say $U \in (RH)_q$ if $U \in L^q_{loc}(\mathbf{R}^n)$ and satisfies

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |U(y)|^q\, dy\right)^{1/q} \leq C\frac{1}{|B(x,r)|}\int_{B(x,r)} |U(y)|\, dy, \tag{6}$$

for some constant C and for every $x \in \mathbf{R}^n$ and $r > 0$, respectively. We can define the function $m(x, U)$ for $U \in (RH)_q$ with $q > n/2$ as follows:

$$\frac{1}{m(x,U)} = \sup\{r > 0; \frac{r^2}{|B(x,r)|}\int_{B(x,r)} U(y)\, dy \leq 1\}. \tag{7}$$

We note that if there exist positive constants K_1 and K_2 such that $K_1U_1(x) \leq U_2(x) \leq K_2U_1(x)$, then it is easy to see that there exist positive constants K_1' and K_2' such that

$$K_1'm(x, U_1) \leq m(x, U_2) \leq K_2'm(x, U_1).$$

When $n \geq 3$, since it is known $U \in (RH)_{n/2}$ actually belongs to $(RH)_{n/2+\epsilon}$ for some $\epsilon > 0$, $m(x, U)$ can be defined for $U \in (RH)_{n/2}$ ([Sh1]). For other properties of the class $(RH)_q$, see, e.g., [KS]. We denote by $B(x) = (B_{jk}(x))$ the magnetic field defined by $B_{jk}(x) = \partial_j a_k(x) - \partial_k a_j(x)$. We use the notation $m_J(x)$:

$$m_E(x) = m(x, V), \qquad m_M(x) = m(x, |B| + V)$$

for the operator L_J, $J = E$ or M, respectively. We assume the following conditions for $V(x)$ and $a(x) = (a_1(x), \cdots, a_n(x))$.

ASSUMPTION(V, a, B): For each $j = 1, \cdots, n$, $a_j(x)$ is a real-valued $C^1(\mathbf{R}^n)$-function, V is non-negative.

(i) For $n \geq 3$, we assume $V(x)$ and $a(x)$ satisfy

$$V + |B| \in (RH)_{n/2}, \qquad |\nabla B(x)| \leq Cm(x, V + |B|)^3.$$

(ii) For $n = 2$, we assume $V(x)$ and $a(x)$ satisfy

$$V + |B| \in (RH)_q, \qquad |\nabla B(x)| \leq Cm(x, V + |B|)^3$$

for some $q > 1$.

Remark 1 *For $n = 2$, we may assume the condition (ii') instead of (ii) by employing Lemma 1 (b).*

(ii') $V \in L^{\infty}_{loc}(\mathbf{R}^2), B(x) \geq 0$ and that $m_J(x)$ satisfies

$$C_1 \frac{m_J(x)}{(1 + |x - y| m_J(x))^{k_0/(k_0+1)}} \leq m_J(y) \leq C_2 (1 + |x - y| m_J(x))^{k_0} m_J(x) \qquad (8)$$

for some positive constants C_1, C_2, k_0 and for every $x, y \in \mathbf{R}^2$, where $m_E(x) = \sqrt{V(x)}$ and $m_M(x) = \sqrt{V(x) + B(x)}$.

We remark that it is known that $m_J(x)$ satisifies (8) under the assumption (V, a, B) for $n \geq 3$([Sh1]) and even for $n = 2$ in the same way. We also note that if $|B| + V \in (RH)_\infty$, then it is easy to see that $|B(x)| + V(x) \leq Cm(x, |B| + V)^2$ holds. For example, the condition $|B| + V \in (RH)_\infty$ is satisfied for any $a_j(x) = Q_j(x), V(x) = |P(x)|^\alpha$, where $P(x)$ and $Q_j(x), j = 1, \cdots, n$, are polymonials and α is a positive constant. In this case, under the assumption (V, a, B) (i) or (ii), we see that there exists a positive constant m_0 such that $m_J(x) \geq m_0$, although in general we cannot say $|B| + V$ is strictly positive for imhomogeneous polynomials. To state our main result, we introduce the notation:

$$\Gamma_{C_0}(x, t; y, s) = \frac{1}{(t - s)^{n/2}} \exp(-C_0 \frac{|x - y|^2}{t - s})$$

for some positive constant C_0.

Theorem 1 *(a) Suppose $A(x)$ and $V(x)$ satisfy the assumptions $(A.1), (A.2)$ and $(V, 0, 0)$. Then, there exist positive constants α_0 and C_j $(j = 0, 1, 2)$ such that*

$$(0 \leq)\Gamma_E(x,t;y,s) \leq C_1 \exp\Big(-C_2(1 + m_E(x)(t-s)^{1/2})^{\alpha_0/2}\Big)\Gamma_{C_0}(x,t;y,s) \tag{9}$$

for $x, y \in \mathbf{R}^n$ and $t > s > 0$.

(b) Suppose $V(x)$ and $a(x)$ satisfy the assumption (V, a, B). Then, there exist positive constants α_0 and C_j $(j = 0, 1, 2)$ such that

$$|\Gamma_M(x,t;y,s)| \leq C_1 \exp\Big(-C_2(1 + m_M(x)(t-s)^{1/2})^{\alpha_0/2}\Big)\Gamma_{C_0}(x,t;y,s) \tag{10}$$

for $x, y \in \mathbf{R}^n$ and $t > s > 0$.

The number α_0 is actually defined by $\alpha_0 = 2/(k_0 + 1)$, where k_0 is the constant in (8). The exponent $\alpha_0/2$ would not be sharp. If we restrict for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, the following sharp estimate is known ([Ma], [Er1,2] for $n \geq 3$ and [LT] for $n = 2$):

$$|\Gamma_M(x,t;y,s)| \leq D_1 \exp(-D_2 B_0 t)\Gamma_{D_0}(x,t;y,s).$$

More detail informations on the constants D_j $(j = 0, 1, 2)$ can be seen in those papers. By using the parabolic distance:

$$d_P((x,t),(y,s)) = \max(|x-y|, |t-s|^{1/2}),$$

we have the following decay estimate.

Corollary 1 *(a) Under the same assumptions as in Theorem 1, there exist positive constants C_j $(j = 1, 2)$ and C_0 such that*

$$|\Gamma_J(x,t;y,s)| \leq C_1 \exp\Big(-C_2(1 + m_J(x) d_P((x,t),(y,s)))^{2\alpha_0/(\alpha_0+4)}\Big)\Gamma_{C_0}(x,t;y,s)$$

for $J = E$ and M, for every $x, y \in \mathbf{R}^n$ and $t > s > 0$.

(b) Under the same assumptions as in Theorem 1, for each $k > 0$ there exist positive constants C_k and C_0 such that

$$|\Gamma_J(x,t;y,s)| \leq \frac{C_k}{(1 + m_J(x) d_P((x,t),(y,s)))^k}\Gamma_{C_0}(x,t;y,s)$$

for $J = E$ and M.

Remark 2 *Actually we can show the estimate in Theorem 1 for the operators* $L_E = -\nabla(A(x,t)\nabla) + V(x,t)$ *with time-dependent coefficients, if we assume the uniform ellipticity (4) of* $A(x,t)$ *and the existence of constants* $C_j, j = 1,2$, *such that* $C_1 U(x) \leq V(x,t) \leq C_2 U(x)$ *and* U *satisfies the condition* $(U,0,0)$. *For the magnetic Schrödinger operator* $L_M = (i^{-1}\nabla - a(x,t))^2 + V(x,t)$, *the estimate in Theorem 1 still holds, if there exists positive constants* $C_j, j = 1, \cdots, 5$, *such that* $C_1 U(x) \leq V(x,t) \leq C_2 U(x)$, $C_3|B'(x)| \leq |B(x,t)| \leq C_4|B'(x)|$, *and* $|\nabla B(x,t)| \leq C_5 m(x, |B'| + U)^3$, *where* $a(x,t)$ *is* C^1 *and* $B_{jk}(x,t) = \partial_j a(x,t) - \partial_k a_j(x,t)$ *and* $U(x)$ *and* $B'(x)$ *satisfy the* ASSUMPTION (U, a, B') *(except* $|\nabla B'(x)| \leq C m_J(x)^3 (= C m(x, |B'| + U|)^3)$*), and if the upper bound :*

$$|\Gamma_M(x,t;y,s)| \leq C\Gamma_{C_0}(x,t;y,s)$$

holds for some constants C *and* C_0.

Remark 3 *In particular, Corollary 1 (b) yields*

$$\begin{aligned} |\Gamma_J(x,t;y,s)| &\leq \frac{C_k}{(1+m_J(x)|x-y|)^k(1+m_J(x)|t-s|)^k}\Gamma_{C_0}(x,t;y,s) \\ &\leq \frac{C_k}{(1+m_J(x)|x-y|)^k}\Gamma_{C_0}(x,t;y,s) \end{aligned} \tag{11}$$

for $J = E$ *or* M. *Let* $n \geq 3$. *Then this implies*

$$|\Gamma_J(x,y) \equiv \int_s^{+\infty} \Gamma_J(x,t;y,s)\,dt| \leq \frac{C_k}{(1+m_J(x)|x-y|)^k|x-y|^{n-2}}$$

where $\Gamma_J(x,y)$ *is the fundamental solution to* $L_J u = 0$. *This estimate for the elliptic operator was proved by Shen [Sh1,2]. Thus, Corollary 1 (b) is a generalization of his estimate.*

Remark 4 *Recently we are informed by Z.Shen that he obtained the following shape estimate[Sh3] for the elliptic operators: under the assumption* $V \in (RH)_{n/2}$ *for* $n \geq 3$ *and* $V \in (RH)_q$ *with* $q > 1$ *for* $n = 2$,

$$C_1 \exp(-C_2 d(x,y))|x-y|^{2-n} \leq \Gamma_E(x,y) \leq C_3 \exp(-C_4 d(x,y))|x-y|^{2-n}$$

holds for some positive constants C_j $(j = 1,2,3,4)$, *where* $d(x,y)$ *is defined by*

$$d(x,y) = \inf_\gamma \int_0^1 m(\gamma(t), V)|\left(\frac{d\gamma}{dt}\right)(t)|\,dt.$$

Here the infimum is taken over all curves γ such that $\gamma(0) = x$ and $\gamma(1) = y$. Moreover, he gave the following estimate:

$$C_1(1+m(x)|x-y|)^{\alpha_0/2} \le d(x,y) \le C_2(1+m(x)|x-y|)^{\beta_0}$$

for some positive constants $C_j (j=1,2)$ and β_0. In particular, it follows

$$\Gamma_E(x,y) \le C_5 \exp(-C_6(1+m_E(x)|x-y|)^{\alpha_0/2})|x-y|^{2-n}$$

for some positive constants C_5 and C_6. We remark that this decay estimate also can be shown for the fundamental solution $\Gamma_M(x,y)$ to L_M in a similar way. On the other hand, it follows from Corollary 1 (a) a somewhat weaker decay estimate:

$$|\Gamma_J(x,y)| \le C \exp(-C(1+m_J(x)|x-y|)^{2\alpha_0/(\alpha_0+4)})|x-y|^{2-n}$$

for $J = E$ or M. We do not know whether his sharp estimate can be generalized to heat kernel estimates or not.

We denote by e^{-tL_J} the semigroup generated by L_J. Here we also denote by L_J the self-adjoint operator determined from the form associated with L_J(see, e.g., [Si], [LS]). We obtain the following weighted smoothing estimate by using Corollary 1 (b).

Theorem 2 *Assume the same assumptions as in Theorem 1. Let $J = E$ or M. Suppose $1 < p \le q \le +\infty$ and $1/p - 1/q < 1$ and put $\gamma = n(1/p - 1/q)$. Then for each $l \in [0,(n-\gamma)/2]$ there exists a constant C_l such that*

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^q(\mathbf{R}^n)} \le \frac{C_l}{t^{l+(\gamma/2)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0. \tag{12}$$

Corollary 2 *Suppose the additional condition $|B| + V \in (RH)_\infty$. Then we have the following estimates:*

$$\|(|B|+V)^l e^{-tL_J} f\|_{L^p(\mathbf{R}^n)} \le \frac{C_l}{t^l} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0 \tag{13}$$

holds for $1 < p < +\infty$ and $l \in [0, n/2]$, and

$$\|(|B|+V)^l e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \le \frac{C_l}{t^{l+(n/2p)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0 \tag{14}$$

holds for $1 \le p < +\infty$ and $l \in [0, n/(2p')]$. Here $1/p' = 1 - 1/p$ and C_l is a constant depending on l and p.

Corollary 2 is an easy consequence of Theorem 2 by using the inequality $(|B| + V)(x) \leq C m_J(x)^2$. Note that (14) for the case $l = 0$ is a classical result.

Theorem 1 yields a weighted smoothing estimate with an exponential decay in time.

Theorem 3 *Assume the same assumptions as in Theorem 1 and the additional assumption $m_J(x) \geq m_0 > 0$.*
(a) Let $1 \leq p < +\infty$ and $l \in [0, n/(2p')]$. Then we have

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \leq C \exp(-C(1 + m_0 t^{1/2})^{\frac{\alpha_0}{2}}) \frac{1}{t^{l+(n/2p)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0.$$

(b) Let $1 \leq p \leq 2$ and $l \in [0, n/(2p')]$. Then we have

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \leq C \exp(-C(1 + m_0^2 t)) \frac{1}{t^{l+(n/2p)}} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0.$$

Especially, for the case $CB_0 \geq |B(x)| \geq B_0 > 0$, Theorem 3 (b) yields an exponential decay estimate in time:

$$\|e^{-tL_M} f\|_{L^\infty(\mathbf{R}^n)} \leq \frac{C_1}{t^{n/2}} \exp(-C_2 B_0 t) \|f\|_{L^1(\mathbf{R}^n)}, \quad t > 0 \tag{15}$$

for some positive constant C_1 and C_2, which is known (see, e.g., [Ma], [Er1,2], [Ue], [LT]). Indeed, in this case $m_M(x) \sim \sqrt{B_0}$ holds. Note that Theorem 3 (a) gives weaker decay rate $e^{-C\sqrt{B_0}t}$, since $k_0 = 0$ and $\alpha_0 = 2$. We also emphasize that Theorem 3 can be applied to any polynomial like magnetic field $B(x)$ which may be zero somewhere.

Definition 1 *We say $u(x,t)$ is a complex-valued weak solution to*

$$(\partial_t + L_M)u = 0 \quad in \quad Q_r(x_0, t_0),$$

if $u \in L^\infty((t_0 - r^2, t_0); L^2(B(x_0, r); \mathbf{C})) \cap L^2((t_0 - r^2, t_0); H^1(B(x_0, r); \mathbf{C}))$ and satisfies

$$\begin{aligned} &\int_{B(x_0,r)} u(x,t)\overline{\phi(x,t)}\, dx - \int_{t_0-r^2}^{t} \int_{B(x_0,r)} u(x,s)\partial_s \overline{\phi}(x,s)\, dxds \\ &+ \int_{t_0-r^2}^{t} \int_{B(x_0,r)} \sum_{j=1}^{n} D_j^a u(x,s) \overline{D_j^a \phi(x,s)}\, dxds \\ &+ \int_{t_0-r^2}^{t} \int_{B(x_0,r)} V(x) u(x,s) \overline{\phi(x,s)}\, dxds = 0 \end{aligned} \tag{16}$$

for everu $\phi \in \mathcal{C} \equiv \{\phi \in L^2((t_0-r^2,t_0); H^1(B(x_0,r);\mathbf{C})); \partial_s\phi \in L^2((t_0-r^2,t_0); L^2(B(x_0,r);\mathbf{C})), \phi(x,t_0-r^2)=0\}$, *where* $\overline{\phi}$ *is the complex conjugate of* ϕ.

Here, we used the notation $D_j^a = i^{-1}\partial_{x_j} - a_j(x)$ and

$$Q_r(x_0,t_0) = \{(x,t) \in \mathbf{R}^n \times (0,+\infty); |x-x_0| < r, t_0 - r^2 < t < t_0\}.$$

A real-valued weak solution u to $(\partial_t + L_E)u = 0$ in $Q_r(x_0,t_0)$ can be defined in a similar way. Our proof of Theorem 1 is based on the following subsolution estimate.

Theorem 4 *Let* $u(x,t)$ *be a weak solution to* $\partial_t u + L_J u = 0$ *in* $Q_{2r}(x_0,t_0)$. *Then there exits positive constants* $C_j, j=1,2$, *such that*

$$\sup_{(x,t)\in Q_{r/2}(x_0,t_0)} |u(x,t)| \le C_1 \exp\Big(-C_2(1+rm_J(x_0))^{\alpha_0/2}\Big)\Big(\frac{1}{r^{n+2}}\int\int_{Q_r(x_0,t_0)} |u|^2\,dxdt\Big)^{1/2}. \tag{17}$$

Throughout this paper, we use the following notation: $D = i^{-1}\nabla - a$,

$$B(x_0,r) = \{y \in \mathbf{R}^n; |y-x_0| < r\}, \quad \langle A\nabla u, \nabla u\rangle = \sum_{j,k=1}^n a_{jk}\partial_{x_j}u\partial_{x_k}u,$$

$$Q_r(x_0,t_0) = \{(x,t) \in \mathbf{R}^n \times (0,+\infty); |x-x_0| < r, t_0 - r^2 < t < t_0\}.$$

2 Proof of Theorem 4

We use the following inequalities.

Lemma 1 *(a) ([Sh2]) Suppose* $n \ge 2$ *and* $V(x)$ *and* $a(x)$ *satisfy the condition* (V,a,B). *Then there exists a constant* C_0 *such that*

$$\int m(x,|B|+V)^2|u|^2\,dx \le C_0\int |(i^{-1}\nabla - a(x))u|^2 + V(x)|u|^2\,dx$$

for $u \in C_0^\infty(\mathbf{R}^n;\mathbf{C})$.

(b) ([AHS]) Suppose $n=2, V \ge 0, V \in L^\infty_{loc}(\mathbf{R}^2)$, $a \in C^1(\mathbf{R}^2)$, *and* $B(x) \ge 0$. *Then the inequality*

$$\int (B(x)+V(x))|u|^2\,dx \le \int |(i^{-1}\nabla - a(x))u|^2 + V(x)|u|^2\,dx$$

holds for $u \in C_0^\infty(\mathbf{R}^n;\mathbf{C})$.

We also prepare the following Caccioppoli-type inequality.

Lemma 2 *Let* $0 < \sigma < 1$. *Let* u *be a weak solution to* $(\partial_s + L_J)u = 0$ *in* $Q_{2r}(x_0, t_0)$ *for* $J = E$ *or* $J = M$. *Then there exists a constant* C *such that*

$$\sup_{t_0-(\sigma r)^2 \le t \le t_0} \int_{B(x_0,\sigma r)} |u(x,t)|^2\, dx \;+\; \int\int_{Q_{\sigma r}(x_0,t_0)} |(i^{-1}\nabla - a)u|^2 + V|u|^2\, dxds \le \frac{C}{(1-\sigma)^2 r^2} \int\int_{Q_r(x_0,t_0)} |u|^2\, dxdt.$$

PROOF: Although the proof is standard, we give it here for the sake of completeness. We show the estimate for a weak solution u to $(\partial_t + L_E)u = 0$ in $Q_{2r}(x_0, t_0)$. Since we can show the esimate for a weak solution to $(\partial_t + L_M)u = 0$ in the similar way, we just mention some modifications we need at the end of this proof. Take functions $\chi(x) \in C_0^\infty(B(x_0, r))$ and $\eta(t) \in C^\infty(\mathbf{R}^1)$ satisfying $0 \le \chi(x) \le 1$, $\chi(x) \equiv 1$ on $B(x_0, \sigma r)$ and $|\nabla\chi(x)| \le C/(1-\sigma)r$, and $0 \le \eta(t) \le 1$, $\eta(t) \equiv 1$ on $t \ge t_0 - (\sigma r)^2$, $\eta(t) \equiv 0$ on $t \le t_0 - r^2$, $|\partial_t \eta(t)| \le C/r^2(1-\sigma^2)$. For the sake of simplicity, we also assume $\partial_t u \in L^2(Q_{2r}(x_0, t_0))$. Actually, we can remove this additional assumption by using the argument as in [AS]. Fix $t \in [t_0 - (\sigma r)^2, t_0]$. Multiplying $\eta^2(t)\chi^2(x)u(x,t)$ to the equation and integrating over $B(x_0, r) \times [t_0 - r^2, t]$, we have

$$\begin{aligned} &\frac{1}{2}\int_{B(x_0,r)} u(x,t)^2 \chi(x)^2\, dx \\ &+ \int_{t_0-r^2}^{t} \int_{B(x_0,r)} \langle A(x)\nabla u(x,s), \nabla u(x,s)\rangle \eta(s)^2 \chi(x)^2\, dxds \\ &+ \int_{t_0-r^2}^{t} \int_{B(x_0,r)} V(x) u(x,s)^2 \eta(s)^2 \chi(x)^2\, dxds \\ &= \int_{t_0-r^2}^{t} \int_{B(x_0,r)} u(x,s)^2 \chi(x)^2 \eta(s) \partial_s \eta(s)\, dxds \\ &- \int_{t_0-r^2}^{t} \int_{B(x_0,r)} \langle A(x)\nabla u(x,s), \nabla(\chi^2(x))\rangle \eta(s)^2 u(x,s)\, dxds. \end{aligned} \tag{18}$$

Because of the ellipticity of $A(x)$ and the positivity of V, we obtain by (18)

$$\begin{aligned} &\sup_{t_0-(\sigma r)^2 \le t \le t_0} \int_{B(x_0,r)} u(x,t)^2 \chi(x)^2\, dx \\ &\le \int\int_{Q_r(x_0,t_0)} u^2 |\partial_s \eta|\, dxds \end{aligned} \tag{19}$$

$$
\begin{aligned}
&+ \iint_{Q_r(x_0,t_0)} |\nabla u||u|\eta^2\chi|\nabla\chi|\,dxds \\
&\le \frac{C}{(1-\sigma)}\Big\{\frac{1}{r^2}\iint_{Q_r(x_0,t_0)} u^2\,dxds + \iint_{Q_r(x_0,t_0)} \chi^2\eta^2|\nabla u|^2\,dxds\Big\}.
\end{aligned}
$$

By using (18) again, we have

$$
\begin{aligned}
&\lambda\iint_{Q_r(x_0,t_0)} |\nabla u|^2\chi^2\eta^2\,dxds + \iint_{Q_r(x_0,t_0)} Vu^2\chi^2\eta^2\,dxds \\
&\le \iint_{Q_r(x_0,t_0)} \langle A\nabla u,\nabla u\rangle\partial_k u\chi^2\eta^2\,dxds + \iint_{Q_r(x_0,t_0)} Vu^2\chi^2\eta^2\,dxds \\
&\le \frac{C}{(1-\sigma)r^2}\iint_{Q_r(x_0,t_0)} u^2\,dxds + \iint_{Q_r(x_0,t_0)} |\nabla u||\nabla\chi|\chi\eta^2|u|\,dxds \\
&\le \frac{C}{(1-\sigma)r^2}\iint_{Q_r(x_0,t_0)} u^2\,dxds + \frac{\lambda}{2}\iint_{Q_r(x_0,t_0)} |\nabla u|^2\chi^2\eta^2\,dxds. \qquad (20)
\end{aligned}
$$

It follows

$$
\begin{aligned}
&\frac{\lambda}{2}\iint_{Q_r(x_0,t_0)} |\nabla u|^2\chi^2\eta^2\,dxds + \iint_{Q_r(x_0,t_0)} Vu^2\chi^2\eta^2\,dxds \\
&\le \frac{C}{(1-\sigma)^2r^2}\iint_{Q_r(x_0,t_0)} u^2\,dxds. \qquad (21)
\end{aligned}
$$

(19) and (21) yield the desired result. For L_M, we can prove in a similar way by noting the following identities:

$$
D_j^a(u\chi) = (D_j^a u)\chi + u(i^{-1}\nabla\chi), \quad \int D_j^a u\overline{v}\,dx = \int u\overline{D_j^a v}\,dx.
$$

□

Proof of Theorem 3: Let $k \in \mathbf{N}$ and define p_j $(j = 1, 2, \cdots, k+1)$ by $p_j = 2/3 + ((j-1)/k)(1-(2/3))$. Let $\chi_j(x) \in C_0^\infty(B(x_0, p_j r))$ and $\eta_j(t) \in C^\infty(\mathbf{R})$ be the functions satisfying $0 \le \chi_j \le 1, \chi_j(x) \equiv 1$ on $B(x_0, p_{j-1}r)$, $|\nabla\chi_j(x)| \le Ck/r$, and $0 \le \eta_j \le 1, \eta_j(t) \equiv 1$ on $t \ge t_0 - (p_{j-1}r)^2$, $\eta_j(t) \equiv 0$ on $t \le t_0 - (p_j r)^2$, $|\nabla\eta_j(t)| \le Ck/r^2$. By Lemma 2 (see also (21)) , we have

$$
\begin{aligned}
&\iint_{Q_{p_{j+1}r}(x_0,t_0)} \Big(|(i^{-1}\nabla - a)u|^2\chi_{j+1}^2\eta_{j+1}^2 + V|u|^2\chi_{j+1}^2\eta_{j+1}^2\Big)\,dxds \\
&\le \frac{Ck^2}{r^2}\iint_{Q_{p_{j+1}r}(x_0,t_0)} |u|^2\,dxds.
\end{aligned}
$$

We write just $\chi = \chi_{j+1}$ and $\eta = \eta_{j+1}$, for simplicity. Since $|(i^{-1}\nabla - a)(u\eta\chi)|^2 \le 2|(i^{-1}\nabla - a)u|^2\chi^2\eta^2 + 2u^2|\nabla\chi|^2\eta^2$, it follows that

$$\int\int_{Q_{p_{j+1}r}(x_0,t_0)} \Big(|(i^{-1}\nabla - a)(\eta\chi u)|^2\chi^2\eta^2 \;+\; V|u|^2\chi^2\eta^2\Big)\,dxds \le \frac{Ck^2}{r^2}\int\int_{Q_{p_{j+1}r}(x_0,t_0)} |u|^2\,dxds$$

for $j = 1, \cdots, k$. By using Lemma 1, we obtain

$$\int_{t_0-(p_{j+1}r)^2}^{t_0}\Big(\int_{B(x_0,p_{j+1}r)} m_J(x)^2|\eta\chi u|^2\,dx\Big)dt \le \frac{Ck^2}{r^2}\int\int_{Q_{p_{j+1}r}(x_0,t_0)} |u|^2\,dxds.$$

By using $m_J(x) \ge C(1 + p_{j+1}rm_J(x_0))^{-k_0/(1+k_0)}m_J(x_0)$ on $|x - x_0| < p_{j+1}r$ and noting $2/3 \le p_{j+1} \le 1$ (see (8) and the remark after that), we have

$$\begin{aligned}\int\int_{Q_{p_jr}(x_0,t_0)} |u|^2\,dxdt &\le \int_{t_0-(p_{j+1}r)^2}^{t_0}\Big(\int_{B(x_0,p_{j+1}r)} |\eta\chi u|^2\Big)\,dxdt\\ &\le \frac{Ck^2}{r^2m_J(x_0)^2}(1 + rm_J(x_0))^{2k_0/(k_0+1)}\int\int_{Q_{p_{j+1}r}(x_0,t_0)} |u|^2\,dxdt.\\ &\le \frac{Ck^2}{(1 + rm_J(x_0))^{2/(k_0+1)}}\int\int_{Q_{p_{j+1}r}(x_0,t_0)} |u|^2\,dxdt \qquad (22)\end{aligned}$$

for each $j = 1, 2, \cdots, k$. Here we used a trival inequality $\int\int_{Q_{p_jr}(x_0,t_0)}(\cdots)\,dxdt \le \int\int_{Q_{p_{j+1}r}(x_0,t_0)}(\cdots)\,dxdt$ for the case $rm_J(x_0) \le 1$. By this proceedure, we can obtain the following: there exists a constant C such that for every $k \in \mathbf{N}$

$$\int\int_{Q_{2r/3}(x_0,t_0)} |u|^2\,dxdt \le \frac{C^k(k^2)^k}{(1 + rm_J(x_0))^{k\alpha_0}}\int\int_{Q_r(x_0,t_0)} |u|^2\,dxdt, \qquad (23)$$

where $\alpha_0 = 2/(k_0 + 1)$. Since $V(x) \ge 0$, the well-known subsolution estimate (see, e.g., [AS]) yields

$$\sup_{Q_{r/2}(x_0,t_0)} |u| \le C\Big(\frac{1}{r^{n+2}}\int\int_{Q_{2r/3}(x_0,t_0)} |u|^2\,dxdt\Big)^{1/2} \qquad (24)$$

for some constant C. For the magnetic Schrödinger operator case, we have used Kato's inequality. Combining (23) and (24), we arrive at

$$\sup_{Q_{r/2}(x_0,t_0)} |u| \le C\frac{C^{k/2}k^k}{(1 + rm_J(x_0))^{k\alpha_0/2}}\Big(\frac{1}{r^{n+2}}\int\int_{Q_r(x_0,t_0)} |u|^2\,dxdt\Big)^{1/2} \qquad (25)$$

for every $k \in \mathbf{N}$. Note that, by Stirling's formula $k^k \sim e^k k!(1/\sqrt{2\pi k})$ as $k \to \infty$, there exists a constant C_0 such that $k^k \leq C_0 e^k k!$ for $k \geq 1$. Multiplying $\epsilon^k/k!$ and taking the summation, we obtain

$$\begin{aligned}&(\sup_{Q_{r/2}(x_0,t_0)} |u|) \sum_{k=1}^{\infty} \frac{(\epsilon(1+rm_J(x_0))^{\alpha_0/2})^k}{k!} \\ \leq\ & CC_0 \sum_{k=1}^{\infty} (\epsilon e\sqrt{C})^k \Big(\frac{1}{r^{n+2}} \int\int_{Q_r(x_0,t_0)} |u|^2\, dx dt\Big)^{1/2}.\end{aligned}$$

Take $\epsilon > 0$ so that $\epsilon e\sqrt{C} < 1$. Then we have

$$\sup_{Q_{r/2}(x_0,t_0)} |u| \leq C \exp(-\epsilon(1+rm_J(x_0))^{\alpha_0/2}) \Big(\frac{1}{r^{n+2}} \int\int_{Q_r(x_0,t_0)} |u|^2\, dx dt\Big)^{1/2}.$$

This complete the proof. □

3 Proof of Theorem 1

To show Theorem 1 we prove the following proposition.

Proposition 1 *Under the assumptions as in Theorem 1, there exist positive constants C_1 and C_2 such that*

$$|\Gamma_J(x,t;y,s)| \leq C_1 \exp(-C_2(1+m_J(x)|t-s|^{1/2})^{\alpha_0/2}) \frac{1}{(t-s)^{n/2}} \tag{26}$$

for $x, y \in \mathbf{R}^n$ and $t > s > 0$.

PROOF: Assume $t-s \geq 2|y-x|^2$. Take $r^2 = |t-s|/8$. Then $u(z,u) = \Gamma_J(z,u;y,s)$ satisfies $(\partial_t + L_J)u(z,u) = 0$ in $Q_{2r}(x,t)$. Hence, by applying Theorem 4 to $u(z,u)$, we obtain

$$\begin{aligned}|\Gamma_J(x,t;y,s)| &\leq \sup_{Q_{r/2}(x,t)} |u| \\ &\leq C \exp(-C(1+m_J(x)|t-s|^{1/2})^{\alpha_0/2}) \Big(\frac{1}{r^{n+2}} \int\int_{Q_r(x,t)} |\Gamma(z,u;y,s)|^2\, dz du\Big)^{1/2}.\end{aligned}$$

By using the maximum principle for L_E and the diamagnetic inequality (see, e.g., [AS], [LS], [AHS]) for L_M, we have

$$|\Gamma_J(z,u;y,s)| \leq \frac{C}{(u-s)^{n/2}} \exp\Big(-C\frac{|z-y|^2}{(u-s)}\Big) \tag{27}$$

for some constant $C = C(n, \lambda)$. Since $t - s \geq u - s \geq 7r^2 \geq (7/8)(t - s)$ on $(z, u) \in Q_r(x, t)$, it is easy to see

$$\left(\frac{1}{r^{n+2}} \int\int_{Q_r(x,t)} |\Gamma_J(z, u; y, s)|^2 \, dz du\right)^{1/2} \leq \frac{C}{(t - s)^{n/2}}.$$

This yields the desired estimate. □

Proof of Theorem 1: The positivity of $\Gamma_E(x, t; y, s)$ is a consequence of $V \geq 0$ and the maximum principle. Hence Proposition 1 and (27) imply

$$|\Gamma_J(x, t; y, s)|^2 \leq C \exp(-C(1 + |t - s|^{1/2} m_J(x))^{\alpha_0/2}) \frac{1}{(t - s)^n} \exp\left(-C\frac{|y - x|^2}{(t - s)}\right)$$

for some constant C. This concludes the desired estimate. □

Proof of Corollary 1: Let $f(t) = (m_J(x)t^{1/2})^{\alpha_0/2} + |x - y|^2/t$ for $t > 0$. The, an easy computation shows that

$$\inf_{t>0} f(t) \geq C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0+4)}$$

for some positive constant C. Thus, we obtain

$$\begin{aligned} |\Gamma_J(x, t; y, s)| &\leq C\frac{1}{(t - s)^{n/2}} \exp(-Cf(t - s)) \exp(-\frac{C|x - y|^2}{t}) \\ &\times \exp(-C(m_J(x)(t - s)^{1/2})^{\alpha_0/2}) \\ &\leq C\Gamma_{C_0}(x, t; y, s) \exp(-C(m_J(x)|x - y|)^{2\alpha_0/(\alpha_0+4)}) \\ &\times \exp(-C(m_J(x)t^{1/2})^{\alpha_0/2}). \end{aligned}$$

This proves the part (a) since $2\alpha_0/(\alpha_0 + 4) \leq \alpha_0/2$. The part (b) is an easy consequence of the part (a). □

4 Proof of Theorem 2, 3

To show Theorem 2, we prove the following inequality.

Theorem 5 *Let $\gamma \in [0, n)$. Then there exists a constant C such that*

$$|m_J(x)^{2l}(e^{-tL_J} f)(x)| \leq \frac{C}{t^{l+(\gamma/2)}} (M_\gamma |f|)(x) \tag{28}$$

holds for every $0 < l \le (n-\gamma)/2$. *Here* $M_\gamma f$ *is the fractional maximal function defined by*

$$(M_\gamma f)(x) = \sup_{x\in B} \frac{1}{|B|^{1-\gamma/n}} \int_B |f|\, dy,$$

where the supremum is taken all balls B *containing* x.

Theorem 2 is a consequence of Theorem 5 and the following lemma (see, e.g., [St]).

Lemma 3 *Let* $0 \le \gamma < n$. *There exists a constant* C *such that*

$$\|M_\gamma f\|_q \le C\|f\|_p$$

for $1 < p \le q \le +\infty$ *and* $1/q = 1/p - \gamma/n$.

Proof of Theorem 5: Let $r = 1/m_J(x)$. By Corollary 1 (b) we have

$$\begin{aligned} & |m_J(x)^{2l}(e^{-tL_J} f)(x)| \\ \le\ & C m_J(x)^{2l} \int \frac{|f(y)|}{(1+m_J(x)|x-y|)^k t^{n/2}} \exp\Big(-\frac{C|x-y|^2}{t}\Big)\, dy \\ \le\ & \frac{C}{r^{2l}t^{n/2}} \sum_{j=-\infty}^{+\infty} \int_{\{2^{j-1}r<|x-y|\le 2^j r\}} \frac{|f(y)|}{(1+2^{j-1})^k} \exp\Big(-\frac{C(2^j r)^2}{t}\Big)\, dy. \end{aligned} \tag{29}$$

By the assumption on l, we take $\alpha \ge 0$ such that $2\alpha = n - \gamma - 2l$. Put $C_\alpha = \sup_{s>0} s^\alpha e^{-s} < +\infty$ for $\alpha \ge 0$. Then the right hand side of (29) is dominated by

$$\begin{aligned} & C_\alpha \frac{C}{t^{n/2}} \sum_{j=-\infty}^{+\infty} \int_{\{2^{j-1}r<|x-y|\le 2^j r\}} \frac{1}{r^{2l}(1+2^{j-1})^k} \Big(\frac{C(2^{j-1}r)^2}{t}\Big)^{-\alpha} |f(y)|\, dy \\ \le\ & \frac{C_\alpha C}{t^{n/2-\alpha}} \sum_{j=-\infty}^{+\infty} \frac{(2^j)^{n-\gamma}}{(1+2^{j-1})^k (2^{j-1})^{2\alpha}} \Big(\frac{1}{(2^j r)^{n-\gamma}} \int_{\{|x-y|\le 2^j r\}} |f(y)|\, dy\Big). \\ \le\ & \frac{C_\alpha C}{t^{n/2-\alpha}} \sum_{j=-\infty}^{+\infty} \frac{(2^j)^{n-\gamma}}{(1+2^{j-1})^k (2^{j-1})^{2\alpha}} (M_\gamma|f|)(x). \end{aligned} \tag{30}$$

Now, since $n - \gamma - 2\alpha = 2l > 0$, by taking $k > 2l$ we have

$$\sum_{j=1}^{+\infty} \frac{(2^j)^{n-\gamma}}{(1+2^{j-1})^k (2^{j-1})^{2\alpha}} \le \sum_{j=1}^{+\infty} \frac{C}{2^{j(k-2l)}} < +\infty,$$

and

$$\sum_{j=-\infty}^{0} \frac{(2^j)^{n-\gamma}}{(1+2^{j-1})^k (2^{j-1})^{2\alpha}} \leq \sum_{j=-\infty}^{0} C(2^j)^{2l} < +\infty.$$

Thus, we obtain the desired result. □

Proof of Theorem 3: First, the estimate for the case $l = 0$ and $p = 1$ is classical except the exponential factor in time. Under the assumption, by Corollary 1 (a) we have

$$\begin{aligned} |\Gamma_J(x,t;y,s)| &\leq C\Gamma_{C_0}(x,t;y,s)\exp(-C(1+m_J(x)|x-y|)^{2\alpha_0/(\alpha_0+4)}) \\ &\times \exp(-C(1+m_0 t^{1/2})^{\alpha_0/2}) \end{aligned} \tag{31}$$

for some positive constants C and C_0. Then by using this estimate we can prove the part (a) of Theorem 3 in a similar way as in the proof of Theorem 2. To show the part (b), we use the semigroup property and Theorem 2 and get

$$\|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} \leq \frac{C}{t^{l+(n/4)}} \|e^{-(2/3)tL_J} f\|_{L^2(\mathbf{R}^n)}$$

for some constant C. Note that under the assumption $m_J(x) \geq m_0$, Lemma 1 yields $\inf \sigma(L_J) \geq Cm_0^2$ for some positive constant C. Here $\sigma(L_J)$ is the spectrum of the operator L_J. So, we have

$$\|e^{-(1/3)tL_J} g\|_{L^2(\mathbf{R}^n)} \leq e^{-Cm_0^2 t} \|g\|_{L^2(\mathbf{R}^n)}.$$

Using this estimate, we obtain

$$\begin{aligned} \|m_J(x)^{2l} e^{-tL_J} f\|_{L^\infty(\mathbf{R}^n)} &\leq \frac{C}{t^{l+(n/4)}} e^{-Cm_0^2 t} \|e^{-(1/3)tL_J} f\|_{L^2(\mathbf{R}^n)} \\ &\leq \frac{C}{t^{l+(n/4)}} e^{-Cm_0^2 t} \frac{C}{t^{n/2(1/p-1/2)}} \|f\|_{L^p(\mathbf{R}^n)}. \end{aligned}$$

In the last inequality, we used $p \leq 2$ and Theorem 2.

□

REFERENCES

[AHS] Avron J.E., Herbst I., Simon B., Schrödinger operators with magnetic fields, I. General interactions, Duke Math. J., 45(1978), 847–883.

[AS] Aronson D.G., Serrin J., Local behaviour of solutions of quasilinear parabolic equations, Arch. Rat. Mech. Anal., 25(1967), 81–122.

[Er1] Erdös L., Estimates on stochastic oscillatory integrals and on the heat kernel of the magnetic Schrödinger operator, Duke Math. J., 76(1994), 541–566.

[Er2] Erdös L., Dia- and paramagnetism for nonhomogeneous magnetic fields, J.Math.Phys., 38(3)(1997), 1289–1317.

[LS] Leinfelder H., Simader C., Schrödinger operators with singular magnetic vector potentials, Math. Z., 176(1981), 1–19.

[LT] Loss M., Thaller B., Optimal heat kernel estimates for Schrödinger operator with magnetic fields in two dimensions, Comm. Math. Phys., 186(1997), 95–107.

[Ma] Malliavin P., Minoration de l'état fondamental de l'équation de Schrödinger du magnétisme et calcul des variations, C.R. Acad. Sci. Sér. I. Math. 302(1986), 481–486.

[KS] Kurata K., Sugano S., Fundamental Solution, eigenvalue asymptotics and eigenfunctions of degenerate elliptic operators with positive potentials, Preprint.

[Sh1] Shen Z., L^p estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier, 45(1995), 513–546.

[Sh2] Shen Z., Estimates in L^p for magnetic Schrödinger operators, Indiana Univ. Math. J., 45(1996),

[Sh3] Shen Z., by personal communication.

[Si] Simon B., Maximal and minimal Schrödinger forms, J. Opt. Theo., 1(1979), 37–47.

[St] Stein, E.M., Harmonic Analysis: Real variable methods, Orthogonality, and Oscillatory integrals, Princeton Univ. Press, 1993.

[Ue] Ueki N., Lower bounds on the spectra of Schrödinger operators with magnetic fields, J. Fun. Ana., 120(1994), 344–379.

AMS subject classification: Primary: 35K10, 35J15, 35J10
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