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Strichartz estimates for wave equations in the homogeneous Besov space

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1 Introduction

In this note the author describes his recent work on the linear estimates for wave equations in the homogeneous Besov space. We consider the inhomogeneous wave equations

$$\begin{aligned} &\partial_t^2 u(t,x) - \Delta u(t,x) = f(t,x), \quad t \in \mathbf{R}, \ x \in \mathbf{R}^n, \ n \geq 2, \\ &u(0,x) = \partial_t u(0,x) = 0, \end{aligned} \tag{1.1}$$

where n denotes the space dimension, f is a complex valued function on $\mathbf{R} \times \mathbf{R}^n$, and Δ denotes the Laplacian in space variables. We shall prove Strichartz estimates of the following type

$$\|u; L^q(I, \dot{B}^{\rho}_{r,2})\| \leq C \|f; L^{\tilde{q}}(I, \dot{B}^{\tilde{\rho}}_{\tilde{r},2})\|, \tag{1.2}$$

where I denotes an interval in $\mathbf{R}$, $\dot{B}^{\rho}_{r,2}$ denotes the homogeneous Besov space defined later and the constant C is independent of f and I. For any $1 \leq q \leq \infty$ and a Banach space X, we write the mixed norm of a function $g : I \to X$ by

$$\begin{aligned} &\|g; L^q(I,X)\| = \{\textstyle\int_{t \in I} \|g(t); X\|^q dt\}^{1/q} \quad \text{for} \ \ 1 \leq q < \infty, \\ &\|g; L^\infty(I,X)\| = \sup_{t \in I} \|g(t); X\|. \end{aligned} \tag{1.3}$$

On the estimate (1.2), Ginibre and Velo in [3] have shown some generalization of almost all Strichartz-type estimates obtained up to that point, in which one of conditions necessary for (1.2) is given by

$$\rho + \delta(r) - 1/q = 2 + \tilde{\rho} + \delta(\tilde{r}) - 1/\tilde{q}, \quad \rho, \tilde{\rho} \in \mathbf{R}, \tag{1.4}$$

where $\delta(r) = n(1/2 - 1/r)$ (see [3, Proposition 3.1]). On the other hand, Harmse in [5], Oberlin in [10], Bak, McMichael and Oberlin in [1] have already shown the "off duality" estimates, namely (1.2) for $(n+1)/2n - 2/(n+1) < 1/r < (n-1)/2n$ with $q = r$, $\tilde{q} = \tilde{r}$ and $\rho = \tilde{\rho} = 0$ in (1.4). The above two results meet only on the original Strichartz estimate [11], otherwise they are independent.

The author introduce Strichartz estimates which involve the above results and have new ones. Although the proofs are omitted, they are obtained by the abstract setting such as the unitarity of the operator $\exp(it\sqrt{-\Delta})$ $(i = \sqrt{-1})$, the duality argument, the Hardy-Littlewood-Sobolev inequality and the complex interpolation method. The key method is complex interpolation(see [2, Chapter 4] or Proposition 2. 1 below), by which we could loosen the conditions restricted by the Hardy-Littlewood-Sobolev inequality, therefore our results could involve [1, Theorem 6'], [5, Theorem 2.3] and [10, Theorem 3].

Our main result is Proposition 2. 2 . Recently Keel and Tao [7] have obtained the estimate at the "endpoint" by the real interpolation method. We used their methods to supplement our methods in the critical cases.

2 Notation and propositions

As usually done, we will rewrite (1.1) to the integral equation. For that purpose, we introduce some operators defined on the tempered distributions $\mathcal{S}'(\mathbf{R}^n)$ or $\mathcal{S}'(\mathbf{R}\times\mathbf{R}^n)$. We denote by ω^λ, $U(t)$ the operators on $\mathcal{S}'(\mathbf{R}^n)$ defined by $\omega^\lambda = (-\Delta)^{\lambda/2}$, $U(t) = \exp(it\sqrt{-\Delta})$, and by $G_0, G_\pm$ the integral operators defined by

$$G_0 f(t) = \int_0^t U(t-s)f(s)ds, \quad G_\pm f(t) = \int_{\pm\infty}^t U(t-s)f(s)ds, \tag{2.5}$$

for any function f in $\mathcal{S}'(\mathbf{R}^{n+1})$. We denote by G any of $G_0, G_\pm$, and by H the operator $\omega^{-1}G$. To show the required inequality (1.2), it suffices to show the boundedness of the operator H from $L^{\tilde{q}}(\mathbf{R}, \dot{B}^{\tilde{\rho}}_{\tilde{r},2}(\mathbf{R}^n))$ to $L^q(\mathbf{R}, \dot{B}^{\rho}_{r,2}(\mathbf{R}^n))$.

Here we shall introduce the homogeneous Besov space $\dot{B}^{\rho}_{r,s}(\mathbf{R}^n)$ for any $\rho \in \mathbf{R}$ and $1 \le r, s \le \infty$ (see also [2], [3] and [12]). For $1 \le q \le \infty$ and a normed space X, we denote by $\ell^q_j(X)$ the space of $\{a_j\}_{j\in\mathbf{Z}}$, $a_j \in X$, with the norm given by

$$\begin{aligned} \|a_j; \ell^q_j(X)\| &= \{\textstyle\sum_{j\in\mathbf{Z}} \|a_j; X\|^q\}^{1/q} \quad \text{for } 1 \le q < \infty, \\ \|a_j; \ell^\infty_j(X)\| &= \sup_{j\in\mathbf{Z}} \|a_j; X\|. \end{aligned} \tag{2.6}$$

We denote by F the Fourier transform in $\mathbf{R}^n$, and by $*$ the convolution in space. Let $\{\varphi_j\}_{j\in\mathbf{Z}} \subset C^\infty(\mathbf{R}^n)$ such that

$$\mathrm{supp}F\varphi_j \subset \{x|\ 2^{j-1} < |x| < 2^{j+1}\}, \quad \sum_{j\in\mathbf{Z}} F\varphi_j(x) = 1 \quad \text{for} \quad |x| \neq 0. \tag{2.7}$$

We denote by $\dot{B}^\rho_{r,s}(\mathbf{R}^n)$ the space given by

$$\{u \in \mathcal{S}'(\mathbf{R}^n)|\ \|u; \dot{B}^\rho_{r,s}(\mathbf{R}^n)\| \equiv \|2^{\rho j}\varphi_j * u; \ell^s_j(L^r(\mathbf{R}^n))\| < \infty\}. \tag{2.8}$$

We make abbreviation such as $\dot{B}^\rho_r = \dot{B}^\rho_{r,2}(\mathbf{R}^n)$ and $L^q\dot{B}^\rho_r = L^q(\mathbf{R}, \dot{B}^\rho_r)$.

The main tools are embeddings(see [2, Theorem 6.5.1])

$$\dot{B}^0_r \hookrightarrow L^r \quad \text{for} \quad 2 \le r < \infty, \quad L^r \hookrightarrow \dot{B}^0_r \quad \text{for} \quad 1 < r \le 2, \tag{2.9}$$

$$\dot{B}^\rho_r \hookrightarrow \dot{B}^{\rho_1}_{r_1} \quad \text{for} \quad \rho \ge \rho_1 \quad \text{with } \rho - n/r = \rho_1 - n/r_1, \tag{2.10}$$

and the following complex interpolation method (see [2, Th 5.1.2, Th 6.4.5]). Let μ be a positive measure on $\mathbf{R}$, and for any Banach space X, let $L^q(\mathbf{R}, \mu; X)$ be the space of a function $f : \mathbf{R} \to X$ with the norm

$$\begin{array}{ll} \{\int_{\mathbf{R}} \|f; X\|^q d\mu\}^{1/q} & \text{for } 1 \le q < \infty, \\ \sup_{t\in\mathbf{R}} \|f(t); X\| & \text{for } q = \infty. \end{array} \tag{2.11}$$

Proposition 2. 1 *Let $n \ge 1$. Let $1 \le s_0, s_1, r_0, r_1 \le \infty$, $1 \le q_0, q_1 < \infty$ and $\rho_0, \rho_1 \in \mathbf{R}$. Let K be an bounded operator from $L^{q_0}(\mathbf{R}, \mu; \dot{B}^{\rho_0}_{r_0})$ to $\dot{B}^0_{s_0}$, and from $L^{q_1}(\mathbf{R}, \mu; \dot{B}^{\rho_1}_{r_1})$ to $\dot{B}^0_{s_1}$. Then K is a bounded operator from $L^q(\mathbf{R}, \mu; \dot{B}^\rho_r)$ to $\dot{B}^0_s$, where s, r, q, ρ are given by*

$$\begin{array}{ll} 1/s = (1-\theta)/s_0 + \theta/s_1, & 1/r = (1-\theta)/r_0 + \theta/r_1, \\ 1/q = (1-\theta)/q_0 + \theta/q_1, & \rho = (1-\theta)\rho_0 + \theta\rho_1, \end{array} \tag{2.12}$$

for any $0 \le \theta \le 1$.

In order to describe our statement in concise form, following Kato [6], it is convenient to use the following geometric notation. We denote by $\square$ the closed unit square in $\mathbf{R}^2$, defined by $0 \le x, y \le 1$. In this note we denote by Q and $\tilde{Q}$ the points $(1/q, 1/r)$ and $(1/\tilde{q}, 1/\tilde{r})$ in $\square$ respectively, and we write $x(Q) = 1/q$, $y(Q) = 1/r$. For $P, Q \in \square$, $[PQ]$ and (PQ) represent the closed and open segment connecting P and Q respectively. And $[PQ)$ denotes $[PQ]\backslash\{Q\}$. We denote by q' the conjugate of q, namely $q' = q/(q-1)$ for $1 < q \le \infty$ and $q' = \infty$ for $q = 1$. And for $Q \in \square$, Q' denotes $(1/q', 1/r')$. We introduce some special points and sets in $\square$, by which it is convenient to state our propositions (see Figure 1,2,3).

$$\begin{aligned}
&O=(0,0),\quad A=(1,1),\quad B=(0,1/2),\quad C=(1/2,(n-3)/2(n-1)),\\
&(\ C=(1/4,0)\text{ if }n=2\ ),\quad D=(1/2,0),\\
&E=(1,(n-3)/2(n-1)),\quad F=(0,(n-3)/2(n-1)),\\
&(\ E=D,\ F=O\text{ if }n=2\ ),\\
&T_0=[OBCD]\ (T_0=[OBC]\text{ if }n=2,\ T_0=[OBC]\backslash\{C\}\text{ if }n=3\ ),\\
&T=\{B\}\cup(BEF),
\end{aligned}\tag{2.13}$$

where $[OBCD]$ denotes the closure of the square defined by O, B, C, D, and (BEF) denotes the interior domain of the triangle defined by B, E, F. For a set S in $\square$, we denote by S' the set of the point Q' with $Q \in S$.

If we introduce the linear functionals

$$\pi(Q)=1/r+2/(n-1)q,\quad \pi_1(Q)=1/r+1/(n-1)q,\tag{2.14}$$

for Q in $\square$, then B and C are on the line defined by $\pi(Q)=1/2$, B and E are on $\pi_1(Q)=1/2$, B' and C' are on $\pi(Q)=(n+3)/2(n-1)$, B' and E' are on $\pi_1(Q)=(n+1)/2(n-1)$. The pair $(Q,\tilde{Q})$ will be called a *conjugate* pair if Q and $\tilde{Q}$ in $\square$ satisfy

$$\pi(\tilde{Q})=\pi(Q)+2/(n-1).\tag{2.15}$$

In particular, for $Q\in[BC]$ and $\tilde{Q}\in[B'C']$, $(Q,\tilde{Q})$ is a *conjugate* pair. We now refer to the following two properties. Let (Q,Q') be a *conjugate* pair. If $x(Q)=0$ and $x(\tilde{Q})=1$, then $y(Q)=y(\tilde{Q})$. If Q is on $[BE]$ and $x(\tilde{Q})=1$, then $y(Q')=y(\tilde{Q})$.

We call the pair $(Q,\tilde{Q})$ *admissible* if the linear operator H is bounded from $L^{\tilde{q}}\dot{B}^{\tilde{\rho}}_{\tilde{r}}$ to $L^{q}\dot{B}^{\rho}_{r}$ for any ρ and $\tilde{\rho}$ in $\mathbf{R}$ such that

$$\rho+\delta(r)-1/q=2+\tilde{\rho}+\delta(\tilde{r})-1/\tilde{q}.\tag{2.16}$$

Since ω^{λ} ($\lambda\in\mathbf{R}$) is an isomorphism from $\dot{B}^{\rho}_{r}$ to $\dot{B}^{\rho-\lambda}_{r}$, if the linear operator G is bounded from $L^{\tilde{q}}\dot{B}^{\tilde{\rho}}_{\tilde{r}}$ to $L^{q}\dot{B}^{\rho}_{r}$ for any ρ and $\tilde{\rho}$ in $\mathbf{R}$ such that

$$\rho+\delta(r)-1/q=1+\tilde{\rho}+\delta(\tilde{r})-1/\tilde{q},\tag{2.17}$$

then $(Q,\tilde{Q})$ is admissible.

We are now in a position to state our main proposition.

Proposition 2. 2 (see Figure 4) *Let $n \geq 2$. Let $(Q, \tilde{Q})$ be a conjugate pair with $x(Q) < x(\tilde{Q})$. And let Q and $\tilde{Q}$ satisfy one of the following conditions.*

(1) $\tilde{Q} \in T'$, $(n-3)/(n-1)\tilde{r}' \leq 1/r$ ($(n-3)/(n-1)\tilde{r}' < 1/r$ *for* $n = 3$).
Moreover $\pi_1(Q) < 1/2$ *and* $0 < x(Q)$ *if* $\tilde{Q} \notin [B'C')$.

(2) $Q \in T$, $1/\tilde{r} \leq 1-(n-3)/(n-1)r$ ($1/\tilde{r} < 1-(n-3)/(n-1)r$ *for* $n = 3$).
Moreover $\pi_1(\tilde{Q}) > (n+1)/2(n-1)$ *and* $x(\tilde{Q}) < 1$ *if* $Q \notin [BC)$.

Then the pair $(Q, \tilde{Q})$ *is admissible.*

Remark 1. Let $(Q, \tilde{Q})$ be an admissible pair with $\tilde{q} \neq \infty$ and $\tilde{r} \neq \infty$. Then $(\tilde{Q}', Q')$ is also an admissible pair. Indeed, H', the dual operator of H, is a bounded operator from $L^{q'}\dot{B}^{-\rho}_{r'}$ to $L^{\tilde{q}'}\dot{B}^{-\tilde{\rho}}_{\tilde{r}'}$, and (2.16) could be written as

$$- \tilde{\rho} + \delta(\tilde{r}') - 1/\tilde{q}' = 2 - \rho + \delta(r') - 1/q'. \qquad (2.18)$$

Since H is written as a linear combination of H'_0, $H'_\pm$, therefore $(\tilde{Q}', Q')$ is also an admissible pair. In this sense, the proof for the case (2) in Proposition 2. 2 follows from that of (1) immediately.

Remark 2. In Proposition 2. 2 , applying the Sobolev embedding theorem, we could take Q and $\tilde{Q}$ in $\Box$ more widely. For example, let $(Q, \tilde{Q})$ be an admissible pair, then for any $r_1, \tilde{r}_1$ with $0 \leq 1/r_1 \leq 1/r$ and $1/\tilde{r} \leq 1/\tilde{r}_1 \leq 1$, $((1/q, 1/r_1), (1/\tilde{q}, 1/\tilde{r}_1))$ is also an admissible pair (note that the embeddings $\dot{B}^{\rho}_{r} \hookrightarrow \dot{B}^{\rho_1}_{r_1}$ and $\dot{B}^{\tilde{\rho}_1}_{\tilde{r}_1} \hookrightarrow \dot{B}^{\tilde{\rho}}_{\tilde{r}}$ imply $\rho + \delta(r) = \rho_1 + \delta(r_1)$ and $\tilde{\rho} + \delta(\tilde{r}) = \tilde{\rho}_1 + \delta(\tilde{r}_1)$ in (2.16) respectively).

To show some typical examples the Sobolev embedding theorem applied to Proposition 2. 2 , we introduce a set S in $\Box$. For $\tilde{Q} \in T'$, let ν be the supremum of $x(Q)$ with $(Q, \tilde{Q})$ in Proposition 2. 2 with (1). Let now S be a set given by

$$\begin{aligned} S \equiv\ & \{Q \in \Box \mid \pi(\tilde{Q}) \geq \pi(Q) + 2/(n-1),\ x(Q) < x(\tilde{Q}), \\ & \quad 0 < x(Q) \leq \nu\ (0 < x(Q) < \nu \ \text{ for } n = 3)\} \qquad \text{if } \tilde{Q} \in (B'C'E'), \\ S \equiv\ & B \cup \{Q \in \Box \mid \pi_1(Q) < 1/2, \pi(\tilde{Q}) \geq \pi(Q) + 2/(n-1), \\ & \quad x(Q) \leq \nu\ (x(Q) < \nu \ \text{ for } n = 3), x(Q) < x(\tilde{Q})\} \ \text{ if } \tilde{Q} \in T' \backslash (B'C'E'). \end{aligned} \qquad (2.19)$$

For $Q \in T$, let S be the set defined by Q' as above, and let S' be the set of the point Q_1' with $Q_1 \in S$.

Corollary 2. 1 (see Figure 5) *Let $\tilde{Q} \in T'$ and $Q \in S$. Or let $Q \in T$ and $\tilde{Q} \in S'$. Then $(Q, \tilde{Q})$ is admissible.*

Remark 3. The most familiar Strichartz-type estimates are the mixed space-time estimates in the Lebesgue space. If the conjugate pair $(Q, \tilde{Q})$ satisfies (2.16) with $\rho = \tilde{\rho} = 0$, then it holds

$$1/\tilde{r} - 1/r = 1/\tilde{q} - 1/q = 2/(n+1). \tag{2.20}$$

Therefore if Q and $\tilde{Q}$ satisfy (2.20) and (1) or (2) in Proposition 2. 2 , then we have

$$\|Hf; L^q L^r\| \le C\|f; L^{\tilde{q}} L^{\tilde{r}}\|, \tag{2.21}$$

for any $f \in L^{\tilde{q}} L^{\tilde{r}}$, where we have used the embedding (2.9). Especially for the diagonal case, namely $r = q$ and $\tilde{r} = \tilde{q}$, we obtain the estimate given by [1, theorem 6'], [5, Theorem 2.3], [10, Theorem 3]. Indeed for $(n+1)/2n - 2/(n+1) < 1/r \le (n-1)/2(n+1)$, the above Q and $\tilde{Q}$ satisfy (1) in Proposition 2. 2 , and for $(n-1)/2(n+1) < 1/r < (n-1)/2n$, (2) in Proposition 2. 2 . In the above argument, Q is uniquely determined by $\tilde{Q}$ as (2.20). But we should note that if $(Q, \tilde{Q})$ in Corollary 2. 1 satisfies (2.16) with $\rho = \tilde{\rho} = 0$, then (2.21) also holds.

In Proposition 2. 2 , we must assume $x(Q) < x(\tilde{Q})$ and $x(Q) > 0$, or $x(\tilde{Q}) < 1$. The following proposition could give some supplements for the cases $x(Q) = x(\tilde{Q}) = 1/2$, $x(Q) = 0$ and $x(\tilde{Q}) = 1$.

Proposition 2. 3 *Let $2 \le r \le \infty$, $1 \le \tilde{q} \le 2 \le q \le \infty$. Let $\tilde{r} = r'$, and let $\pi(\tilde{Q}) > \pi(Q) + 2/(n-1)$. Then $(Q, \tilde{Q})$ is admissible.*

The results in Proposition 2. 3 for the case $1 < \tilde{q} < 2 < q < \infty$ are also obtained by Corollary 2. 1 and Remark 2.

Let now $\tilde{Q}$ be fixed with $x(\tilde{Q}) = 1$, and let Q_c be the point such that

$$\pi(\tilde{Q}) = \pi(Q_c) + 2/(n-1) \quad \text{and} \quad \pi_1(Q_c) = 1/2. \tag{2.22}$$

And let T_1, S_1 be the sets given by

$$T_1 \equiv \{Q \in \square \mid \pi(Q) \le 1/2, x(Q) \le 1/2 \; (x(Q) < 1/2 \text{ for } n = 3)\}, \tag{2.23}$$

$$S_1 \equiv T_1 \cup \{Q \in \Box \mid \pi_1(Q) < 1/2, x(Q) < x(Q_c), x(Q) \leq 1/2\}. \quad (2.24)$$

For Q with $x(Q) = 0$, let S_1 be the set given by Q' as above, and let S_1' be the set of the point Q_1' with $Q_1 \in S_1$.

Corollary 2. 2 (see Figure 6) *Let $1 \leq \tilde{r} \leq 2$, $\tilde{Q} = (1, 1/\tilde{r})$ and $Q \in S_1$. Or let $2 \leq r \leq \infty$, $Q = (0, 1/r)$ and $\tilde{Q} \in S_1'$. Then $(Q, \tilde{Q})$ is admissible.*

Next we consider the case $q = \tilde{q} = 2$ with $n \geq 4$. In this case, applying Proposition 2. 3 and Remark 2, we were able to show the admissibility of $(Q, \tilde{Q})$ for any $Q \in (CD]$ and $\tilde{Q} \in (C'D']$. However the real interpolation method described in [7] could give some extension in this case. Namely with the proof in [7, section 6] slightly modified, we obtain the following lemma.

Lemma 2. 1 *Let $n \geq 4$. Let $(Q, \tilde{Q})$ be a conjugate pair. If $x(Q) = x(\tilde{Q}) = 1/2$ and*

$$(n-1)/2(n-2) < y(\tilde{Q}) < (n^2-5)/2(n-1)(n-2), \quad (2.25)$$

then $(Q, \tilde{Q})$ is admissible.

Since Lemma 2. 1 is obtained quite analogously to [7], we will use it without proof. To proceed our argument, it is convenient to introduce the linear functional

$$\pi_2(Q) = 1/r + 1/(n-2)q. \quad (2.26)$$

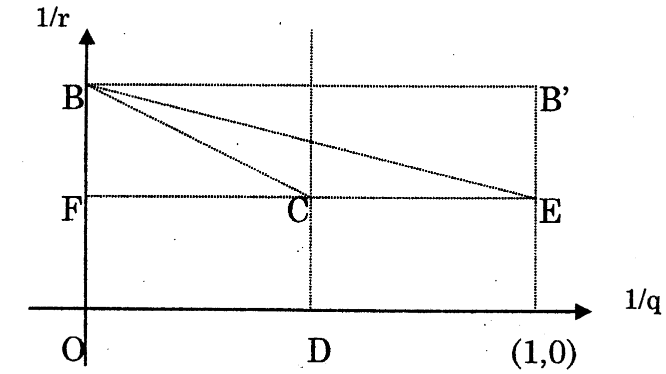
For $\tilde{Q}$ with $x(\tilde{Q}) = 1/2$, let T_2, S_2 be sets given by

$$T_2 \equiv \{Q \in \Box \mid \pi_2(Q) \leq 1/2, \pi(Q) \leq \pi(\tilde{Q}) + 2/(n-1), x(Q) \leq 1/2\}, \quad (2.27)$$

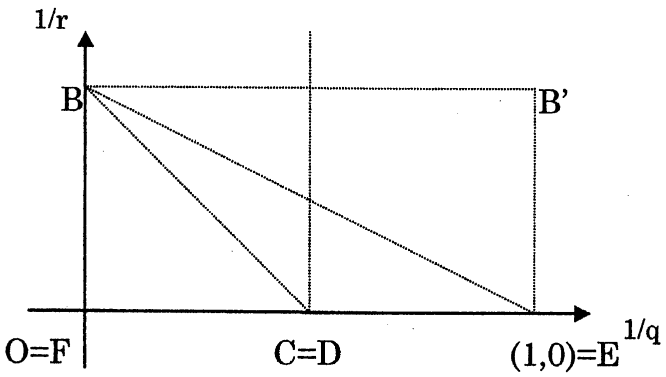
$$S_2 \equiv T_2 \text{ for } \tilde{Q} \in [C'D'], \quad S_2 \equiv T_2 \backslash [OB] \text{ for } \tilde{Q} \notin [C'D']. \quad (2.28)$$

For Q with $x(Q) = 1/2$, let S_2 be the set given by Q' as above, and let S_2' be the set of the point Q_1' with $Q_1 \in S_2$.

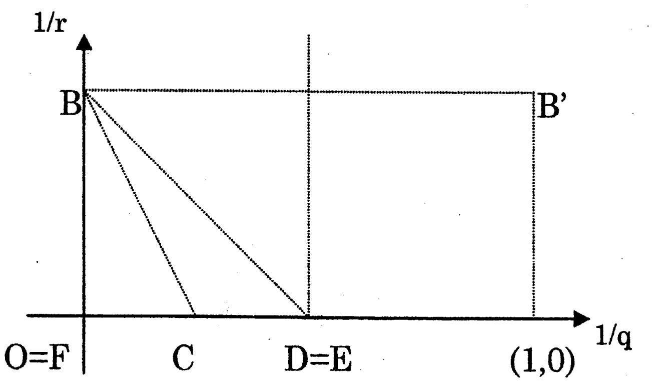
Corollary 2. 3 (see Figure 7) *Let $n \geq 4$. Let $\tilde{Q} \in \Box$ satisfy $x(\tilde{Q}) = 1/2$ and (2.25), and let $Q \in S_2$. Or let $Q \in \Box$ satisfy $x(Q) = 1/2$ and $(n-3)^2/2(n-1)(n-2) < y(Q) < (n-3)/2(n-2)$, and let $\tilde{Q} \in S_2'$. Then $(Q, \tilde{Q})$ is admissible.*



Fifure 1. n>3.



Fifure 2. n=3.



Fifure 3. n=2.

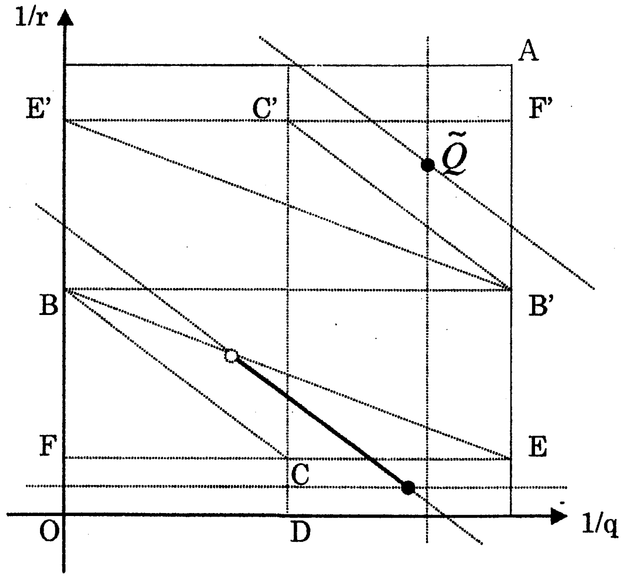


Figure 4. Proposition 2.2 with (1) (n>3).

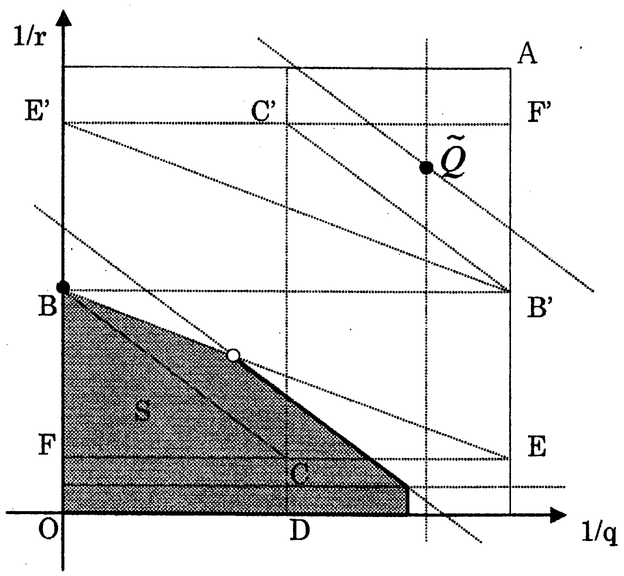


Figure 5. Corollary 2.1 (n>3).

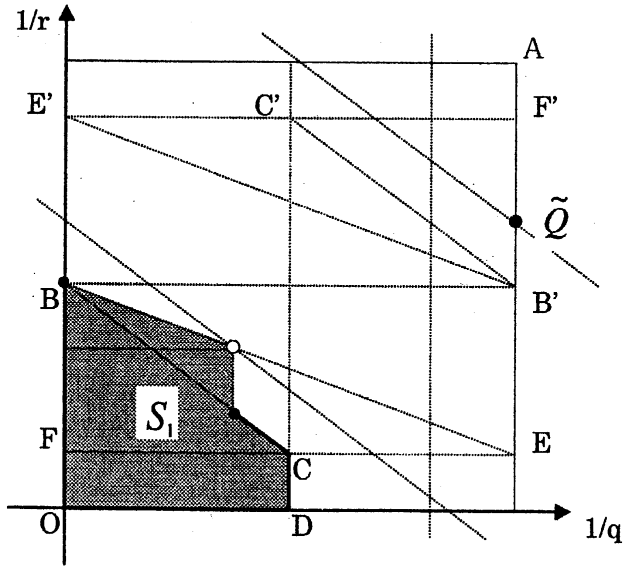


Figure 6. Corollary 2.2 (n>3).

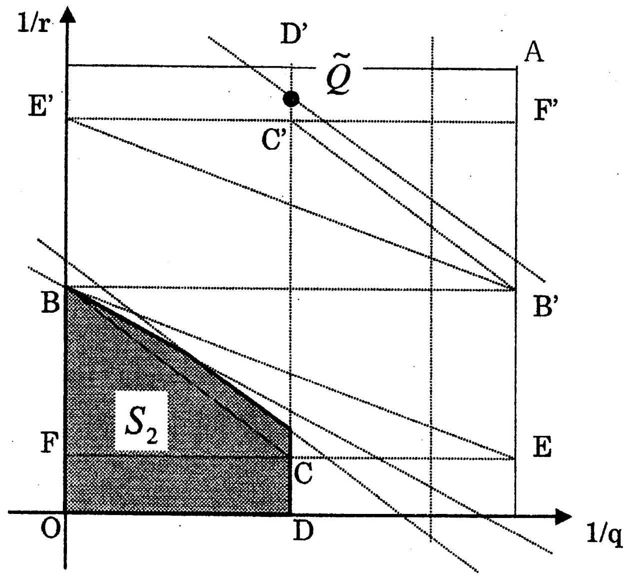


Figure 7. Corollary 2.3.

参考文献

[1] J.-G.Bak, D.McMichael, D.Oberlin, *L^p-L^q estimates off the line of duality*, J.Austral.Math.Soc. (Series A), **58**(1995), 154-166.

[2] J.Bergh, J.Löfström, "Interpolation Spaces," Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[3] J.Ginibre, G.Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal., **133**(1995), 50-68.

[4] G.Hardy, J.E.Littlewood, G.Pólya, "Inequalities (Second edition)," Cambridge Mathematical Library, 1952.

[5] J.Harmse, *On Lebesgue space estimates for the wave equation*, Indiana Univ.Math.J., **39**(1990), 229-248.

[6] T.Kato, *An $L^{q,r}$-Theory for nonlinear Scrödinger equations*, Advanced Studies in Pure Math., **23**(1994), 223-238.

[7] M.Keel, T.Tao, *Endpoint Strichartz estimates*, Amer.J.Math, **120**(1998), 955-980.

[8] H.Lindblad, *A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations*, Duke Math.J., **72**(1993), 503-539.

[9] M.Nakamura, T.Ozawa, *The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space*, Ann.Inst.Henri Poincaré, Physique théorique, (in press).

[10] D.Oberlin, *Convolution estimates for some distributions with singularities on the light cone*, Duke Math.J., **59**(1989), 747-757.

[11] R.S.Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math.J., **44**(1977), 705-714.

[12] H.Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, 1978.